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Research Article

Biquasilinear Functionals on Quasilinear Spaces and Some Related Results

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Abstract
In this paper, we will present the notion of the biquasilinear functional which is a new concept of quasilinear functional analysis. Just like bilinear functional, the notions of a biquasilinear functional and a quadratic form will not need to have the constitution of an inner product quasilinear space. We were able to define these functionals in any quasilinear space. After giving this new notion, we discuss some examples and prove some theorems for considerable exercises to the theory of biquasilinear functionals in Hilbert quasilinear spaces.

Keywords: Biquasilinear functional, quasilinear functional, quasilinear space, normed quasilinear space, inner product quasilinear space.

Quasilineer Uzaylarda Biquasilineer Fonksiyoneller ve Bazı Sonuçları

Öz

Bu çalışmada quasilineer fonksiyonel analizde yeni bir kavram olan biquasilineer fonksiyonel kavramını tanımladık. Bilineer fonksiyonel kavramında olduğu gibi biquasilineer fonksiyonel ve kuadratik form kavramlarında da bir iç çarpım quasilineer uzayına ihtiyaç duyulmadığını gördük. Bu fonksiyonelleri herhangi bir quasilineer uzayında tanımlayabildik. Çalışmamızda bu yeni kavramı verdikten sonra Hilbert quasilineer uzaylarda biquasilineer fonksiyoneller teorisi üzerine dikkate değer bazı örnekler verdik. Ve yine bu teori üzerine bazı teoremler ve ispatlarını çalışmamızda sunduk.

Anahtar Kelimeler: Biquasilineer fonksiyonel, quasilineer fonksiyonel, quasilineer uzay, normlu quasilineer uzay, iç çarpım quasilineer uzayı.

1. Introduction

The concept of quasilinear space presented by Aseev (Aseev, 1986), is a generalization of linear space. In the paper, Aseev defined normed quasilinear space and some reletad results which coherent counterpart of consequences in linear spaces. Additionaly, in the same study, he generalized the linear operators in linear spaces by giving the quasilinear operators in quasilinear spaces. Since then, several papers have deal with quasilinear functional analysis or the set-valued analysis (see, e.g., Rojas Medar et al., 2005; Alefeld and Mayer, 2000;Çakan and Yılmaz, 2015; Levent and Yılmaz, 2018b.; Bozkurt and Yılmaz, 2016a; Bozkurt and

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Yılmaz, 2018b; Bozkurt and Yılmaz, 2018b; Laksmikantham et al., 2006).

Recently, in Yılmaz et al., (2016), they have proposed a series of new concepts such as inner product quasilinear space, Hilbert quasilinear space and some results involved to the orthogonality. The important distinction between inner product spaces and inner product quasilinear spaces is that it is defined as a set-valued function. This difference has led to new analyzes in quasilinear functional analysis unlike linear functional analysis, for example, see (Yılmaz et al., 2016). In 2018, they have worked on bounded quasilinear interval-valued functions, especially, Hahn-Banach extension theorem for set-valued functions (Levent and Yılmaz, 2018a).

In this paper, we generalize the notion of bilinear functional by introducing the biquasilinear functional. Just as in the concept of quasilinear operator we used partial order relation while defining the notion of biquasilinear operator. With this new definition, we were able to give coherent counterparts of consequences linear functional analysis. Additionally, we realized that just like bilinear functional, the notions of a biquasilinear functional and a quadratic form will not need to have the construction of an inner product quasilinear space. These functionals can be described in any quasilinear spaces. After giving this new notion, we discuss some examples and prove some theorems for considerable implementations to the hypothesis of functionals in Hilbert quasilinear spaces.

2. Material and Methods

First, we recall some definitions from Aseev, (1986).

Definition 1. By a quasilinear space we mean a nonempty set M with the process:

$(m,n) \to m+n$ from $M \times M$ into M called addition,

$(\alpha,m) \to \alpha \cdot m$ from $\mathbb{R} \times M$ into M called multiplication by scalars, and with a partial order correlation "$\leq$", such that the consequent circumstances are satisfied for every $m,n,z,k \in M$ and every real numbers $\alpha,\beta \in \mathbb{R}$:

q1) $m \leq m$,

q2) $m \leq z$ if $m \leq n$ and $n \leq z$,

q3) $m = n$ if $m \leq n$ and $n \leq m$,

q4) $m+n = n+m$,

q5) $m+(n+z) = (m+n)+z$,

q6) there exists an element $\theta \in M$ such that $m+\theta = m$,

q7) $\alpha \cdot (\beta \cdot m) = (\alpha\beta) \cdot m$,

q8) $\alpha \cdot (m+n) = \alpha \cdot m + \beta \cdot n$,

q9) $1 \cdot m = m$,

q10) $0 \cdot m = \theta$,

q11) $(\alpha+\beta) \cdot m \leq \alpha \cdot m + \beta \cdot m$,

q12) $m+n \leq z+k$ if $m \leq z$ and $n \leq k$,

q13) $\alpha \cdot m \leq \alpha \cdot n$ if $m \leq n$.

Example 1. Let $\Omega(\mathbb{R})$ be space of whole nonempty closed and bounded subsets of a normed linear space over field $\mathbb{R}$ and $\Omega_C(\mathbb{R})$ be space of whole nonempty convex, compact subsets of a normed linear space over field $\mathbb{R}$. $\Omega_C(\mathbb{R})$ and $\Omega(\mathbb{R})$ are not linear spaces. They are quasilinear space with respect to the containment correlation " $\subseteq$ ", algebraic sum processing $A+B=\{a+b: a\in A, b\in B\}$ and the real-scalar multiplication $\alpha\cdot A=\{\alpha a: a\in A\}$ (Aseev, 1986).

If an element m in a quasilinear space M has an reverse, then m is called regular. Otherwise, we say that m is called singular. By M_r denote the set of overall regular elements of quasilinear space M and M_s denote the set of overall singular elements of quasilinear space M. Also, M_r and $M_s\cup\{0\}$ are subspace of M (Aseev, 1986).

Definition 2. A function $m\to\|m\|$ from a quasilinear space M into $\mathbb{R}$ is called a norm if it satisfied the succeding properties:

nq1) $\|m\|_M>0$ if $m\neq 0$,

nq2) $\|m+n\|_M\leq\|m\|_M+\|n\|_M$,

nq3) $\|\alpha\cdot m\|_M=\alpha\|m\|_M$,

nq4) if $m\leq n$, then $\|m\|_M\leq\|n\|_M$,

nq5) if for any $\varepsilon>0$ there exists an element $m_\varepsilon\in M$ such that $m\leq n+m_\varepsilon$ and $\|m_\varepsilon\|_M\leq\varepsilon$ then $m\leq n$ (Aseev, 1986).

A quasillinear space M with a norm is called a normed quasilinear space.

A norm on quasilinear space M describes a metric on X which is defined by

$$h_M(m,n)=inf\{\varepsilon\geq 0: m\leq n+a_1^\varepsilon,\ n\leq m+a_2^\varepsilon, \|a_i^\varepsilon\|\leq\varepsilon\}\qquad (m,n\in M)$$

and is called the Hausdorff metric or norm metric excited by the norm (Aseev, 1986).

Let M and N be quasilinear space. A mapping $Q: M\to N$ is called a quasilinear operator if it provides the following cases:

qo1) $Q(m+n)\leq Q(m)+Q(n)$,

qo2) $Q(\alpha\cdot m)=\alpha\cdot Q(m)$

qo3) if $m\leq n$, then $Q(m)\leq Q(n)$

for every $m,n\in M$ and $\alpha\in\mathbb{R}$.

Example 2. Let M be a Banach space. The function $\|A\|_{\Omega(M)}=sup_{a\in A}\|a\|_M$ is a norm on $\Omega(M)$. Also, $\Omega_C(M)$ is a normed quasilinear space with the identical norm. The Hausdorff metric on $\Omega_C(M)$ and $\Omega(M)$ is described by $h_M(A,B)=inf\{r\geq 0: A\subseteq B+S_r^\theta,\ B\subseteq A+S_r^\theta\}$ where S_r^θ states a closed ball of radius r about $\theta\in X$ (Aseev, 1986).

Definition 3. Let M be a quasilinear space and $m\in M$. The set of whole regular elements proceeding m is named floor of m and denoted by $F_m=\{n\in M_r: n\leq m\}$. The floor of every subset A of M is defined as F_A (Çakan and Yılmaz, 2015).

Definition 4. Let M be a quasilinear space. M is called solid floored if $m=sup\{n\in M_r: n\leq m\}$ for every $m\in M$. Other than this, M is named nonsolid floored quasilinear space (Çakan and Yılmaz, 2015).

Definition 5. Let M be a quasilinear space. Consolidation of floor of M is the minimum solid floored quasilinear space $\widehat{M}$ including M (Yılmaz et al., 2016).

Definition 6. Let M be quasilinear space. A function $\langle\cdot,\cdot\rangle: M\times M\to\Omega_C(\mathbb{R})$ is named an inner produt on M if for every $m,n,z,k\in M$ and $\alpha\in\mathbb{R}$ the next cases are provided:

ipq1) if $m,n\in M_r$ then $\langle m,n\rangle\in(\Omega_C(\mathbb{R}))_r\equiv\mathbb{R}$,

ipq2) $\langle m+n,z\rangle\subseteq\langle m,z\rangle+\langle n,z\rangle$, ipq3) $\langle \alpha \cdot m, n \rangle = \alpha \cdot \langle m, n \rangle$,

ipq4) $\langle m, n \rangle = \langle n, m \rangle$,

ipq5) $\langle m, m \rangle \geq 0$ for $m \in M_r$ and $\langle m, m \rangle = 0 \Leftrightarrow m = \theta$,

ipq6) $\|\langle m, n \rangle\|_{\Omega_C(\mathbb{R})} = sup\{\|\langle a, b \rangle\|_{\Omega_C(\mathbb{R})} : a \in F_m^{\widehat{M}}, b \in F_n^{\widehat{M}}\}$,

ipq7) if $m \leq n$ and $z \leq k$ then $\langle m, z \rangle \subseteq \langle n, k \rangle$,

ipq8) if for any $\varepsilon > 0$ there exists an element $m_\varepsilon \in M$ such that $m \leq n + m_\varepsilon$ and $\langle m_\varepsilon, m_\varepsilon \rangle \subseteq S_\varepsilon^\theta$ then $m \leq n$ (Bozkurt and Yılmaz, 2016a).

A quasilinear space with an inner product is named inner product quasilinear space. $\Omega_C(\mathbb{R})$ is an inner product quasilinear space with inner product described by $\langle A, B \rangle = \{ab : a \in A, b \in B\}$. For every two elements a and b of an inner product quasilinear space, we obtain $\|\langle a, b \rangle\|_{\Omega_C(\mathbb{R})} \leq \|a\|_M \|b\|_M$. An inner product on quasilinear space M defines a norm on M given by $\|m\| = \sqrt{\|\langle m, m \rangle\|_{\Omega_C(\mathbb{R})}}$ (Bozkurt and Yılmaz, 2016a).

Definition 7. A complete inner product quasilinear space is named a Hibert quasilinear space to the inner product norm (Bozkurt and Yılmaz, 2016a).

3. Resarch Findings

In this part, we will present the notion of biquasilinear functional which is a new concept of quasilinear functional analysis. After giving this new notion, we discuss some examples and prove some theorems.

Definition 8. Let M and N be two quasilinear spaces. A function $BQ: M \times M \to N$ is is said to be a biquasilinear operator on M, if BQ is satisfies the sequent circumstances:

bq1) $BQ(m + n, z) \leq BQ(m, z) + BQ(n, z)$,

bq2)$BQ(m, n + z) \leq BQ(m, n) + BQ(m, z)$,

bq3) $BQ(\alpha \cdot m, n) = \alpha \cdot BQ(m, n)$,

bq4) $BQ(m, \beta \cdot n) = \beta \cdot BQ(m, n)$,

bq5) if $(m, n) \leq (z, k)$, then $BQ(m, n) \leq BQ(z, k)$

for any scalars α and β and any $m, n, z, k \in M$.

If we take $N = \Omega_C(\mathbb{R})$, then the mapping BQ is called a biquasilinear functional on M.

Note that these are three of the properties defining a quasilinear inner product. Thus every inner product quasilinear is a biquasilinear functional. But, the reverse may not be true.

Example 3. Let A_1 and A_2 quasilinear operators on a inner product quasilinear spaces M. Then

$$\mathcal{B}Q_1(m, n) = \langle A_1 m, n \rangle,$$

$$\mathcal{B}Q_2(m, n) = \langle m, A_2 n \rangle$$

and

$$\mathcal{B}Q_3(m, n) = \langle A_1 m, A_2 n \rangle$$

are biquasilinear functionals.

We only show the first one. For all $m, n, z, k \in M$ and $\alpha, \beta \in \mathbb{R}$,

$\mathcal{B}Q_1(m + n, z) \leq \langle A_1(m + n), z \rangle$

$$\subseteq \langle A_1 m, z\rangle + \langle A_1 n, z\rangle$$

$$= \mathcal{B}Q_1(m,z) + \mathcal{B}Q_1(n,z).$$

$$\mathcal{B}Q_1(m,n+z) \leq \langle A_1 m, n+z\rangle$$

$$\subseteq \langle A_1 m, n\rangle + \langle A_1 m, z\rangle$$

$$= \mathcal{B}Q_1(m,n) + \mathcal{B}Q_1(m,z).$$

$$\mathcal{B}Q_1(\alpha \cdot m, n) = \langle A_1(\alpha \cdot m), n\rangle$$

$$= \alpha \cdot \mathcal{B}Q_1(m,n).$$

$$\mathcal{B}Q_1(m, \beta \cdot n) = \langle A_1 m, \beta \cdot n\rangle$$

$$= \beta \cdot \mathcal{B}Q_1(m,n).$$

If $(m,n) \leq (z,k)$, then $m \leq z$ and $n \leq k$. Since A_1 is a quasilinear operator, we have $A_1 m \leq A_1 z$. Then, we obtain $\langle A_1 m, n\rangle \leq \langle A_1 z, k\rangle$ from definition of inner product quasilinear space**.** Thus, we get $\mathcal{B}Q_1(m,n) \leq \mathcal{B}Q_1(z,k)$.

Now, let's give an another example of biquasilinear functionals.

Example 4. Let ϑ and μ be quasilinear functionals on a quasilinear space M. Then

$$\mathcal{B}Q(m,n) = \vartheta(m) \cdot \mu(n)$$

is a biquasilinear functional on M.

$$\mathcal{B}Q(m+n, z) = \vartheta(m+n) \cdot \mu(z)$$

$$\leq \vartheta(m) \cdot \mu(z) + \vartheta(n) \cdot \mu(z)$$

$$= \mathcal{B}Q(m,z) + \mathcal{B}Q(n,z).$$

$$\mathcal{B}Q(m, n+z) = \vartheta(m) \cdot \mu(n+z)$$

$$\leq \vartheta(m) \cdot \mu(n) + \vartheta(m) \cdot \mu(z)$$

$$= \mathcal{B}Q(m,n) + \mathcal{B}Q(m,z).$$

$$\mathcal{B}Q(\alpha \cdot m, n) = \vartheta(\alpha \cdot m) \cdot \mu(n)$$

$$= \alpha \cdot \mathcal{B}Q(m,n).$$

$$\mathcal{B}Q(m, \alpha \cdot n) = \vartheta(m) \cdot \mu(\alpha \cdot n)$$

$$= \alpha \cdot \mathcal{B}Q(m,n).$$

If $(m,n) \leq (z,k)$, then $m \leq z$ and $n \leq k$. Since ϑ and μ are two quasilinear functionals, we have $\vartheta m \leq \vartheta z$ and $\mu n \leq \mu k$. Then, we get $(\vartheta m) \cdot (\mu n) \leq (\vartheta z) \cdot (\mu k)$. Because, they are interval valued functions. This proves the $\mathcal{B}Q(m,n) \leq \mathcal{B}Q(z,k)$.

Let $\varphi(M^2, \Omega_C(\mathbb{R}))$ be the set of all biquasilinear functionals defined from $M \times M$ to $\Omega_C(\mathbb{R})$. Then $\varphi(M^2, \Omega_C(\mathbb{R}))$ becomes a quasilinear space if the addition, multiplication by scalars and the partial order relation "$\preccurlyeq$" defined in the following operations:

$$+: \varphi\left(M^2, \Omega_C(\mathbb{R})\right) \times \varphi\left(M^2, \Omega_C(\mathbb{R})\right) \to \varphi(M^2, \Omega_C(\mathbb{R}))$$

$$(\mathcal{B}Q_1 + \mathcal{B}Q_2)(m,m') = \mathcal{B}Q_1(m,m') + \mathcal{B}Q_2(n,m') \quad (1)$$

$$\cdot : \mathbb{R} \times \varphi(M^2, \Omega_C(\mathbb{R})) \to \varphi(M^2, \Omega_C(\mathbb{R}))$$

$$(\alpha \cdot \mathcal{B}Q)(m,m') = \alpha \cdot \mathcal{B}Q(m,m') \quad (2)$$

$$\mathcal{B}Q_1 \preccurlyeq \mathcal{B}Q_2 \Leftrightarrow \mathcal{B}Q_1(m,m') \leq \mathcal{B}Q_2(m,m') \quad (3)$$

for every $\mathcal{B}Q, \mathcal{B}Q_1, \mathcal{B}Q_2 \in \varphi(M^2, \Omega_C(\mathbb{R}))$ and $\alpha \in \mathbb{R}$.

A biquasilinear functional is quasilinear functional with respect to the first variable but the every quasilinear functional may not be biquasilinear.

Example 5. Let $\mathcal{B}Q$ be a functional defined from $\Omega_C(\mathbb{R}) \times \Omega_C(\mathbb{R})$ to $\Omega_C(\mathbb{R})$. Then

$\mathcal{B}Q(m,n) = m + 2n$ is a quasilinear functional but is not a biquasilinear functional. For all $(m,n),(z,k) \in \Omega_C(\mathbb{R}) \times \Omega_C(\mathbb{R})$ and $\alpha \in \mathbb{R}$:

$$\mathcal{B}\big(\alpha \cdot (m,n)\big) = \mathcal{B}(\alpha \cdot m, \alpha \cdot n)$$

$$= \alpha \cdot m + 2\alpha \cdot n$$

$$= \alpha \cdot \mathcal{B}(m,n),$$

$$\mathcal{B}\big((m,n) + (z,k)\big) = m + z + 2n + 2k$$

$$= \mathcal{B}(m,n) + \mathcal{B}(z,k).$$

If $(m,n) \leq (z,k)$, then $m \leq z$ and $n \leq k$ and $2n \leq 2k$. From properties of quasilinear space, we get $m + 2n \leq z + 2k$. This gives the desired. But, this functional is not biquasilinear since $\mathcal{B}(\alpha \cdot m, n) \neq \alpha \cdot \mathcal{B}(m,n)$ and $\mathcal{B}(m, \alpha \cdot n) \neq \alpha \cdot \mathcal{B}(m,n)$.

Definition 9. If M is a normed quasilinear space then $\mathcal{B}Q$ is called bounded if

$$\|\mathcal{B}Q(m,n)\|_{\Omega_C(\mathbb{R})} \leq L\|m\|_M\|n\|_M$$

for some $L > 0$ and every $m,n \in M$. The norm of the bounded biquasilinear functional described by

$$\|\mathcal{B}Q\| = sup_{\|m\|=\|n\|=1}\|\mathcal{B}Q(m,n)\|_{\Omega_C(\mathbb{R})} \quad (4)$$

Example 6. Let operators A_1 and A_2 in Example 3 be bounded on a inner product quasilinear spaces M. From the definition of bounded quasilinear operator, we get

$\|A_1(m)\| \leq l\|m\|_M$ and $\|A_2(m)\| \leq p\|m\|_M$

that for every $m \in M$. From the Schwartz inequality, we obtain

$$\|\mathcal{B}Q_1(m,n)\| = \|\langle A_1 m, n\rangle\| \leq l\|m\|_M\|n\|_M,$$

$$\|\mathcal{B}Q_2(m,n)\| = \|\langle m, A_2 n\rangle\| \leq p\|m\|_M\|n\|_M$$

and

$$\|\mathcal{B}Q_3(m,n)\| = \|\langle A_1 m, A_2 n\rangle\| \leq lp\|m\|_M\|n\|_M$$

for every $m,n \in M$. Thus, we have that $\mathcal{B}Q_1$ BQ_2 and $\mathcal{B}Q_3$ are bounded.

Theorem 1. The space $\varphi(M^2, \Omega_C(\mathbb{R}))$ is a normed quasilinear space with norm (4).

Proof. If $\mathcal{B}Q \neq \theta$ for a $\mathcal{B}Q \in \varphi(M^2, \Omega_C(\mathbb{R}))$, then we clearly know that $\|\mathcal{B}Q\| = sup_{\|m\|=\|n\|=1}\|\mathcal{B}Q(m,n)\|_{\Omega_C(\mathbb{R})} > 0$. For $\mathcal{B}Q_1$ $\mathcal{B}Q_2 \in \varphi(M^2, \Omega_C(\mathbb{R}))$, we find

$$\|\mathcal{B}Q_1 + \mathcal{B}Q_2\| = sup_{\|m\|=\|n\|=1}\|(\mathcal{B}Q_1 + \mathcal{B}Q_2)(m,n)\|_{\Omega_C(\mathbb{R})} \leq sup_{\|m\|=\|n\|=1}\|\mathcal{B}Q_1(m,n)\|_{\Omega_C(\mathbb{R})} + sup_{\|m\|=\|n\|=1}\|\mathcal{B}Q_2(m,n)\|_{\Omega_C(\mathbb{R})}$$

$$= \|\mathcal{B}Q_1\| + \|\mathcal{B}Q_2\|.$$

For any scalar α, we get

$$\|\alpha \cdot \mathcal{B}Q_1\| = sup_{\|m\|=\|n\|=1}\|(\alpha \cdot \mathcal{B}Q_1)(m,n)\|_{\Omega_C(\mathbb{R})}$$

$$= \alpha\big(sup_{\|m\|=\|n\|=1}\|\mathcal{B}Q_1(m,n)\|_{\Omega_C(\mathbb{R})}\big) = \alpha\|\mathcal{B}Q_1\|.$$

If $\mathcal{B}Q_1 \preccurlyeq \mathcal{B}Q_2$ for every $(m,n) \in M \times M$ and $\mathcal{B}Q_1, \mathcal{B}Q_2 \in \varphi(M^2, \Omega_C(\mathbb{R}))$, then $\mathcal{B}Q_1(m,n) \leq \mathcal{B}Q_2(m,n)$. Since the norm function is continuous on $\Omega_C(\mathbb{R})$, we find $\|\mathcal{B}Q_1(m,n)\|_{\Omega_C(\mathbb{R})} \leq \|\mathcal{B}Q_2(m,n)\|_{\Omega_C(\mathbb{R})}$. This gives us $\|\mathcal{B}Q_1\| \leq \|\mathcal{B}Q_2\|$.

Let $\mathcal{B}Q_1 \preccurlyeq \mathcal{B}Q_2 + \mathcal{B}Q_\varepsilon$ and $\|\mathcal{B}Q_\varepsilon\| \leq \varepsilon$ for every $\mathcal{B}Q_1, \mathcal{B}Q_2 \in \varphi(M^2, \Omega_C(\mathbb{R}))$. From (3) and (4), we obtain

$$\mathcal{B}Q_1(m,n) \leq \mathcal{B}Q_2(m,n) + \mathcal{B}Q_\varepsilon(m,n)$$

and $\|\mathcal{B}Q_{\varepsilon}(m,n)\|_{\Omega_C(\mathbb{R})} \leq \varepsilon$ for every $(m,n) \in M \times M$. Since $\Omega_C(\mathbb{R})$ is a normed quasilinear space with $\|A\|_{\Omega_C(\mathbb{R})} = sup_{a\in A}\|a\|_A$, we find $\mathcal{B}Q_1(m,n) \leq \mathcal{B}Q_2(m,n)$. Again from (3), we have $\mathcal{B}Q_1 \preccurlyeq \mathcal{B}Q_2$.

Theorem 2. $\varphi(M^2, \Omega_C(\mathbb{R}))$ normed quasilinear space is a Banach space with $\|\mathcal{B}Q\| = sup_{\|m\|=\|n\|=1}\|\mathcal{B}Q(m,n)\|_{\Omega_C(\mathbb{R})}$.

Proof . If $\mathcal{B}Q^i$ be any sequence in $\varphi(M^2, \Omega_C(\mathbb{R}))$. Then given every $\varepsilon > 0$, there is a N such that for every $i, j > N$ we get

$$\mathcal{B}Q^i \leq \mathcal{B}Q^j + \mathcal{B}Q^{1\varepsilon},$$

$$\mathcal{B}Q^j \leq \mathcal{B}Q^i + \mathcal{B}Q^{2\varepsilon}, \quad \|\mathcal{B}Q^{l\varepsilon}\| \leq \varepsilon, \quad l = 1,2.$$

Hence, for any $l = 1,2$, we obtain $\|\mathcal{B}Q^{l\varepsilon}(m,n)\|_{\Omega_C(\mathbb{R})} \leq \varepsilon$. Further, since $\varphi(M^2, \Omega_C(\mathbb{R}))$ quasilinear space, we get

$$\begin{aligned} \mathcal{B}Q^i(m,n) &\leq \mathcal{B}Q^j(m,n) + \mathcal{B}Q^{1\varepsilon}(m,n), \\ \mathcal{B}Q^j(m,n) &\leq \mathcal{B}Q^i(m,n) + \mathcal{B}Q^{2\varepsilon}(m,n). \end{aligned} \quad (5)$$

This shows that $\mathcal{B}Q^i(m,n)$ is Cauchy sequence in $\Omega_C(\mathbb{R})$. Since $\Omega_C(\mathbb{R})$ is a complete, there is a $\mathcal{B}Q(m,n) \in \Omega_C(\mathbb{R})$ such that $\mathcal{B}Q^i(m,n) \to \mathcal{B}Q(m,n)$ for all $(m,n) \in M \times M$. From (5), by letting $j \to \infty$, we find

$$\mathcal{B}Q^i(m,n) \leq \mathcal{B}Q(m,n) + \mathcal{B}Q^{1\varepsilon}(m,n),$$

$$\mathcal{B}Q(m,n) \leq \mathcal{B}Q^i(m,n) + \mathcal{B}Q^{2\varepsilon}(m,n),$$

$$\|\mathcal{B}Q^{l\varepsilon}(m,n)\|_{\Omega_C(\mathbb{R})} \leq \varepsilon, \qquad l = 1,2$$

for all $i > N$. Thus, we find $\|\mathcal{B}Q^{l\varepsilon}\| \leq \varepsilon$ for $l = 1,2$ and $\mathcal{B}Q^i \leq \mathcal{B}Q + \mathcal{B}Q^{1\varepsilon}$, $\mathcal{B}Q \leq \mathcal{B}Q^i + \mathcal{B}Q^{2\varepsilon}$. It means that the sequence $\mathcal{B}Q^n$ convergent to $\mathcal{B}Q$. Now, let's show that $\mathcal{B}Q \in \varphi(M^2, \Omega_C(\mathbb{R}))$:

Since $\mathcal{B}Q^n$ is a biquasilinear functional for every $i \in N$, we get

$$\begin{aligned} \mathcal{B}Q(m+n,z) &\leq \mathcal{B}Q^i(m+n,z) \\ &\leq \mathcal{B}Q^i(m,z) + \mathcal{B}Q^i(n,z) \\ &\leq \mathcal{B}Q(m,z) + \mathcal{B}Q(n,z) \end{aligned}$$

and

$$\mathcal{B}Q(m,n+z) \leq \mathcal{B}Q^i(m,n+z)$$

$$\leq \mathcal{B}Q(m,n) + \mathcal{B}Q(m,z).$$

Again, since $\mathcal{B}Q^i$ is a biquasilinear functional for every $\alpha \in \mathbb{R}$, we get

$$\mathcal{B}Q(\alpha \cdot m,n) \leq \mathcal{B}Q^i(\alpha \cdot m,n)$$

$$\leq \alpha \cdot \mathcal{B}Q(m,n)$$

And

$$\mathcal{B}Q(m,\alpha \cdot n) \leq \mathcal{B}Q^i(m,\alpha \cdot n)$$

$$\leq \alpha \cdot \mathcal{B}Q(m,n).$$

If $(m,n) \leq (z,k)$ for every $(m,n),(z,k) \in M \times M$, then $\mathcal{B}Q^i(m,n) \leq \mathcal{B}Q^i(z,k)$. Since $\mathcal{B}Q(m,n) \leq \mathcal{B}Q^n(m,n)$ and $\mathcal{B}Q^i(z,k) \leq \mathcal{B}Q(z,k)$, we find $\mathcal{B}Q(m,n) \leq \mathcal{B}Q(z,k)$. This finalizes the proof.

Further, in the above theorem, if $\mathcal{B}Q^i$ is bounded, then $\mathcal{B}Q$ is a bounded since

$$\|\mathcal{B}Q(m,n)\| \leq \left\|\mathcal{B}Q^i(m,n) + \mathcal{B}Q^{2\varepsilon}(m,n)\right\|$$

$$\leq \left(\left\|\mathcal{B}Q^i\right\| + \varepsilon\right)\|m\|\|n\|.$$

Definition 10. Let $\mathcal{B}Q$ be a biquasilinear functional on M.

$\mathcal{B}Q$ is called symmetric if $\mathcal{B}Q(m,n) = \mathcal{B}Q(n,m)$ for all $m,n \in M$.

$\mathcal{B}Q$ is called positive if $\mathcal{B}Q(m,n) \geq 0$ for every $m,n \in M$.

Example 7. Let T_1 and T_2 be two functions defined from $\mathbb{R}$ to $\Omega_C(\mathbb{R})$ such that

$$: m \to T_1(m) = T_2(m) = \begin{cases} [-m,m], & x \geq 0 \\ [m,-m], & x < 0 \end{cases}.$$

We know from [12] that these functions are quasilinear functionals. Now let's define a new functions using these quasilinear functional. Assume that $\sigma(m,n)$ be function defined from $\mathbb{R} \times \mathbb{R}$ to $\Omega_C(\mathbb{R})$ such that $\sigma(m,n) = T_1(m) \cdot T_2(n)$ for every $(m,n) \in \mathbb{R} \times \mathbb{R}$. This new function is biquasilinear functional. We know that $T_1(m) \cdot T_2(n) = [-m,n] \cdot [-m,n]$=$[minS, maxS]$ where $S = \{-m.-n, -m.n, m.-n, m.n\}$ from interval analysis. Also the function σ symmetric and positive functional.

Result 1. If we take $\vartheta = \mu$ in Example 4, then biquasilinear functional $\mathcal{B}Q$ is a symmetric and positive. Further, a inner product quasilinear is a symmetric and positive definite biquasilinear functional.

Theorem 3. Let A be a bounded quasilinear operator on a Hilbert quasilinear space H. Then the biquasilinear functional described by $\mathcal{B}Q(m,n) = \langle A_1 m, n\rangle$ is bounded and $\|\mathcal{B}Q\| = \|A_1\|$.

Proof. We have already shown its boundedness in Example 6. Let's just show the $\|\mathcal{B}Q\| = \|A_1\|$. Since $\|\mathcal{B}Q(m,n)\| = \|\langle A_1 m, n\rangle\| \leq \|A_1\|\|m\|\|n\|$ for all $m,n \in H$, we have $\|\mathcal{B}Q\| \leq \|A_1\|$. Conversely, we obtain

$$\|A_1 m\|^2 = \|\langle A_1 m, A_1 m\rangle\|$$

$$= \|\mathcal{B}Q(m, A_1 m)\|$$

$$\leq \|\mathcal{B}Q\|\|m\|\|A_1 m\|$$

from H is a Hilbert quasilinear spaces and $\mathcal{B}Q$ is a bounded. Thus, for $A_1 m \neq 0$, we get

$\|A_1 m\| \leq \|\mathcal{B}Q\|\|m\|$ and $\|A_1\| \leq \|\mathcal{B}Q\|$.

Because, the inequation is clearly provided if $A_1 m = 0$.

Definition 11. A function $\omega: M \to \Omega_C(\mathbb{R})$ is a quadratic form if there exists a biquasilinear form $\mathcal{B}Q: M \times M \to \Omega_C(\mathbb{R})$ such that $\omega(m) = \mathcal{B}(m,m)$ for every $m \in M$.

Example 8. Let $\mathcal{B}Q$ be a function identified from $I\mathbb{R}^2 \times I\mathbb{R}^2$ to $\Omega_C(\mathbb{R})$ such that

$\mathcal{B}Q\big((m,n),(z,k)\big) = m \cdot z + 2 \cdot m \cdot k + n \cdot z + 2 \cdot n \cdot k.$

$\mathcal{B}Q$ is a biquasilinear functional.

$$\omega\big((m,n)\big) = m \cdot m + 3 \cdot m \cdot n + 2 \cdot n \cdot n$$

$$= \mathcal{B}\big((m,n),(m,n)\big)$$

is a quadratic form. Consider, $m \cdot m$ may not always come up to to m^2 for all $m \in \Omega_C(\mathbb{R})$. Also, $\omega\big((m,n)\big) = \mathcal{B}Q_1\big((m,n),(m,n)\big)$ for the symmetric biquasilinear form $\mathcal{B}Q_1\big((m,n),(z,k)\big) = m \cdot z + 2 \cdot m \cdot k + n \cdot z + 2 \cdot n \cdot k$.

Theorem 4. Let M be a quasilinear space. For all quadratic form ω, there exists a symmetric biquasilinear form $\mathcal{B}Q$ such that $\omega(m) = \mathcal{B}Q(m,m)$ for every $m \in M$.

Proof. Since ω is quadratic, there exists a biquasilinear form $\mathcal{B}Q_0$ exists such that $\omega(m) = \mathcal{B}Q_0(m,m)$ for every $m \in M$. Let

$\mathcal{B}Q(m,n) = \frac{1}{2}\left(\mathcal{B}Q_0(m,n) + \mathcal{B}Q_0(n,m)\right)$.

Then $\mathcal{B}Q$ is a symmetric biquasilinear functional and $\mathcal{B}Q(m,m) = \mathcal{B}Q_0(m,m)$ since $\Omega_C(\mathbb{R})$ is a quasilinear space.

Theorem 5. Every biquasilinear forms have quasilinear parts.

Proof. Regard as $\mathcal{B}Q: M \times M \to \Omega_C(\mathbb{R})$ is biquasilinear functional. Define two functional on M such that $\omega_n(m) = \mathcal{B}Q(n,m)$ and $\omega_m(n) = \mathcal{B}Q(m,n)$ for every $m,n \in M$. Then, for $\alpha \in \mathbb{R}$ and every $m,n \in M$, we get

$$\omega_n(m+z) = \mathcal{B}Q(n, m+z)$$

$$\leq \omega_n(m) + \omega_n(z),$$

$$\omega_n(\alpha \cdot m) = \mathcal{B}Q(\alpha \cdot n, m)$$

$$= \alpha \cdot \omega_n(m),$$

for every $m \leq z$, then $\mathcal{B}Q(n,m) \leq \mathcal{B}Q(n,z)$. This means that $\omega_n(m) \leq \omega_n(z)$. Thus, ω_n is quasilinear functional. A similar proof holds for the functional ω_m.

4. Results

In this study, the notion of biquasilinear functional is defined and some examples and theorems are given related to the this new concept. A partial order relation was used to give this definition, just like the description of quasilinear functional. Thanks to this new definition, we were able to have consistent results that contribute to the advance of quasilinear functional analysis. The set of all biquasilinear functionals $f(M^2, \Omega_C(\mathbb{R}))$ was shown to be quasilinear space with defined addition, multiplication by scalars and the partial order relation " $\preccurlyeq$ " . Finally, we have given the quadratic form. We have shown that there is a symmetric biquasilinear form for each every quadratic form φ.

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