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Representation of all maximally accretive differential operators for first order

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Abstract

In the present paper, we construct the minimal and maximal operators generated by special type linear differential-operator expression for first order in the weighted Hilbert space of vector-functions defined on right semi-axis with the use of standard technique. In this case, the minimal operator is accretive but not maximal. Our main goal in this paper is to describe the general form of all maximally accretive extensions of the minimal operator in the weighted Hilbert space of vector-functions. Using the Calkin-Gorbachuk method, the general representation of all maximally accretive extensions of this minimal operator in terms of boundary conditions is obtained. We also investigate the structure of the spectrum set such maximally accretive extensions of this type of minimal operator.

Keywords: *Accretive operator, differential operator, deficiency index, space of boundary values, spectrum.*

Birinci dereceden tüm maksimal akretif diferansiyel operatörlerin gösterimi

Öz

Bu çalışmada, standart teknik kullanılarak, sağ yarı eksende tanımlanan vektör-fonksiyonlarının ağırlıklı Hilbert uzayında birinci mertebeden özel tip lineer diferansiyel-operatör ifadesi tarafından üretilen minimal ve maksimal operatörleri yapılandırdık. Bu durumda, minimal operatör akretif olup maksimal değildir. Bu çalışmadaki asıl amacımız, vektör fonksiyonlarının ağırlıklı Hilbert uzayında, minimal

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operatörün tüm maksimal akretif genişlemelerinin genel formunu tanımlamaktır. Calkin-Gorbachuk metodu kullanılarak, bu minimal operatörün tüm maksimal akretif genişlemelerinin genel gösterimi sınır değerleri dilinde ifade edilmiştir. Ayrıca bu minimal operatörün maksimal akretif genişlemelerinin spektrum yapısı araştırılmıştır.

Anahtar kelimeler: *Akretif operatör, diferansiyel operatör, defekt sayıları, sınır değerler uzayı, spektrum.*

1. Introduction

Operator theory is important to understand the nature of the spectral properties of an operator associated with a boundary value problem acting on a Hilbert space. A linear closed densely defined operator $T: D(T) \subset X \to X$ in a Hilbert space X is called to be accretive (dissipative) if and only if

$$Re(T\psi, \psi)x \geq 0 \ \ (Im(T\psi, \psi)x \geq 0 \ ), \psi \in D(T),$$

where, $Re(\cdot\,,\,\cdot)$ $(Im(\cdot\,,\,\cdot))$ and $D(T)$ denote the real (imaginary) part of the inner product and the domain of the operator T, respectively (see [1, 2]). If an accretive (dissipative) operator has no any proper accretive (dissipative) extension, then it is called maximally accretive (dissipative) (see [1, 2]). The class of accretive operators is an important class of non-selfadjoint operators in the operator theory and maximally accretive operators play very efficient role in mathematics and physics. In physics, there are many interesting applications of the accretive operators in areas like hydrodynamic, laser and nuclear scattering theories. It is noteworthy to recall that the spectrum set of the accretive operators lies in right half-plane.

The maximally accretive extensions and their spectral analysis of the minimal operator generated by regular differential-operator expression in Hilbert space of vector-functions defined on a finite interval $(0, b)$ have been studied by Levchuk [3].

In the present study, in Section 3, using the Calkin-Gorbachuk method, the representation of all maximally accretive extensions of the minimal operator generated by the linear singular differential operator expression in the weighted Hilbert spaces of the vector functions defined at right semi-axis is obtained. In Section 4, the geometry of the spectrum of these type extensions is researched.

2. Statement of the problem

Let X be a separable Hilbert space and $a \in \mathbb{R}$. In the weighted Hilbert space $L^2_\varrho(X, (a, \infty))$ of X-valued vector-functions defined on the right semi-axis, consider the following linear differential operator expression for first order in the form

$$l(\nu) = \frac{\kappa(\tau)}{\varrho(\tau)}(\kappa\nu)'(\tau) + A\nu(\tau),$$

where:
(1) $\kappa, \varrho : (a, \infty) \to (0, \infty)$, $\kappa,\ \varrho \in C(a, \infty)$;

(2) $\frac{\varrho}{\kappa^2} \in L^1(a,\infty)$;
(3) $A: X \to X$ is a selfadjoint operator with condition $A \geq 0$.

The minimal L_0 and maximal L operators corresponding to differential expression $l(\cdot\,,\cdot)$ can be constructed by following the way in [4]. In this case, the minimal operator L_0 is accretive, but it is not maximal in $L_\varrho^2(X,(a,\infty))$.

The main goal of this work is to describe of all maximally accretive extensions of the minimal operator L_0 in terms of boundary condition in $L_\varrho^2(X,(a,\infty))$ and to investigate the geometry of the spectrum set of these extensions.

3. Description of maximally accretive extensions

The minimal operator L_0^+ generated by the operator expression

$$l^+(v) = -\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa v)'(\tau) + Av(\tau)$$

can be defined in $L_\varrho^2(X,(a,\infty))$ in a similar way following [4]. In this case, the operator $L^+ = (L_0)^*$ in $L_\varrho^2(X,(a,\infty))$ is called the maximal operator generated by $l^+(\cdot\,,\,\cdot)$. It is easy to see that $L_0 \subset L$ and $L_0^+ \subset L^+$.

If $\tilde{L}$ is any maximally accretive extension of the operator L_0 in $L_\varrho^2(X,(a,\infty))$ and $\widetilde{M}$ is corresponding extension of the minimal operator M_0 generated by the differential expression

$$m(v) = i\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa v)'(\tau)$$

in $L_\varrho^2(X,(a,\infty))$, then it is clear that

$$\begin{aligned}
\tilde{L}(v) &= \frac{\kappa(\tau)}{\varrho(\tau)}(\kappa v)'(\tau) + Av(\tau) \\
&= i\left(-i\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa v)'\right)(\tau) + Av(\tau) \\
&= i\big(-\widetilde{M}\big)v(\tau) + Av(\tau) \\
&= i\left(-\big(Re\widetilde{M} + iIm\widetilde{M}\big)\right)v(\tau) + Av(\tau) \\
&= \big(Im\widetilde{M}\big)v(\tau) - i\big(Re\widetilde{M}\big)v(\tau) + Av(\tau) \\
&= \big[\big(Im\widetilde{M}\big) + A\big]v(\tau) - i\big(Re\widetilde{M}\big)v(\tau).
\end{aligned}$$

Therefore,

$$\big(Re\tilde{L}\big) = \big(Im\widetilde{M}\big) + A.$$

Furthermore,

$$\big(Re\tilde{L}\big) = \big(Im\widetilde{M}\big) + A = Im\big(\widetilde{M} + A\big).$$

Hence, the necessary and sufficient condition for describing all accretive extension of the minimal operator L_0 in $L^2_\varrho(X,(a,\infty))$ is to describe all dissipative extensions of the minimal operator S_0 generated by the differential expression

$$s(\nu) = i\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa\nu)'(\tau) + A\nu(\tau)$$

in $L^2_\varrho(X,(a,\infty))$.

Note that the maximally dissipative operator generated by the differential expression $s(\cdot, \cdot)$ in $L^2_\varrho(X,(a,\infty))$ will be denoted by S.

In this chapter, using the Calkin-Gorbachuk method we will research the general representation of all maximally dissipative extensions of the operator S_0 in $L^2_\varrho(X,(a,\infty))$.

Firstly, let us define the deficiency indices of any symmetric operator in a Hilbert space.

Definition 1 [5]**.** Let T be a symmetric operator, λ be an arbitrary non-real number and X be a Hilbert space. We denote by $\mathcal{R}_{\bar{\lambda}}$ and $\mathcal{R}_\lambda$ the ranges of the operator $(T-\bar{\lambda}I)$ and $(T-\lambda I)$, respectively, where I is identity operator on X. Clearly, $\mathcal{R}_{\bar{\lambda}}$ and $\mathcal{R}_\lambda$ are subspaces of X, which need not necessarily be closed. We call $(X-\mathcal{R}_{\bar{\lambda}})$ and $(X-\mathcal{R}_\lambda)$, which are their orthogonal complements, the deficiency spaces of the operator T and we denote them by $\mathcal{N}_{\bar{\lambda}}$ and $\mathcal{N}_\lambda$ respectively: thus

$$\mathcal{N}_{\bar{\lambda}} = X-\mathcal{R}_{\bar{\lambda}},\ \mathcal{N}_\lambda = X-\mathcal{R}_\lambda.$$

The numbers

$$n_{\bar{\lambda}} = dim\mathcal{N}_{\bar{\lambda}},\ n_\lambda = dim\mathcal{N}_\lambda$$

are called deficiency indices of the operator T.

Let us prove the following auxiliary result which we will need for our main result.

Lemma 1. The deficiency indices S_0 has the following form

$$(n_+(S_0), n_-(S_0)) = (dimX, dimX).$$

Proof. Let $A=0$ for the simplicity of calculations then the general solutions of the differential equations

$$i\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa\nu_\pm)'(\tau) \pm i\nu_\pm(\tau) = 0\ ,\ \tau > a$$

are in the forms

$$\nu_\pm(\tau) = \frac{1}{\kappa(\tau)}\exp\left(\mp\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right)f, f\in X, \tau > a.$$

For any $f \in X$ we have

$$\begin{aligned}\|\nu_+\|^2_{L^2_\varrho(X,(a,\infty))} &= \int_a^\infty \varrho(\tau)\|\nu_+(\tau)\|_X^2 d\tau \\ &= \int_a^\infty \left\|\frac{1}{\kappa(\tau)}\exp\left(-\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right) f\right\|_X^2 \varrho(\tau)d\tau \\ &= \int_a^\infty \frac{\varrho(\tau)}{\kappa^2(\tau)}\exp\left(-2\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right) d\tau \, \|f\|_X^2 \\ &= \int_a^\infty \exp\left(-2\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right) d\left(\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right)\|f\|_X^2 \\ &= \frac{1}{2}\left(1-\exp\left(-2\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds\right)\right)\|f\|_X^2 < \infty.\end{aligned}$$

Consequently, $n_+(S_0) = \dim\ker(S+iE) = \dim X$.

Similarly, for any $f \in X$ we get

$$\begin{aligned}\|\nu_-\|^2_{L^2_\varrho(X,(a,\infty))} &= \int_a^\infty \varrho(\tau)\|\nu_-(\tau)\|_X^2 d\tau \\ &= \int_a^\infty \left\|\frac{1}{\kappa(\tau)}\exp\left(\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right) f\right\|_X^2 \varrho(\tau)d\tau \\ &= \int_a^\infty \frac{\varrho(\tau)}{\kappa^2(\tau)}\exp\left(2\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right) d\tau \, \|f\|_X^2 \\ &= \int_a^\infty \exp\left(2\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right) d\left(\int_a^\tau \frac{\varrho(s)}{\kappa^2(s)}ds\right)\|f\|_X^2 \\ &= \frac{1}{2}\left(\exp\left(2\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds\right)-1\right)\|f\|_X^2 < \infty.\end{aligned}$$

As a result, $n_-(S_0) = \dim\ker(S-iE) = \dim X$. This completes the proof of theorem.

Accordingly, the operator S_0 has a maximally dissipative extension (see [1]).

In order to describe all maximally dissipative extensions of S_0, it is necessary to construct a space of boundary values of it.

Definition 2 [1]**.** Let $\mathfrak{X}$ be any Hilbert spaces and $S: D(S) \subset \mathfrak{X} \to \mathfrak{X}$ be a closed densely defined symmetric operator on the Hilbert space having equal finite or infinite deficiency indices. A triplet $(\mathrm{X}, \beta_1, \beta_2)$, where X is a Hilbert space, β_1 and β_2 are linear mappings from $D(S^*)$ into X, is called a space of boundary values for the operator S, if for any $\eta, \kappa \in D(S^*)$

$$(S^*\eta,\kappa)_{\mathfrak{X}} - (\eta, S^*\kappa)_{\mathfrak{X}} = \big(\beta_1(\eta),\beta_2(\kappa)\big)_{\mathrm{X}} - \big(\beta_2(\eta),\beta_1(\kappa)\big)_{\mathrm{X}}$$

while for any $\mathcal{G}_1, \mathcal{G}_2 \in \mathrm{X}$, there exists an element $\eta \in D(S^*)$ such that $\beta_1(\eta) = \mathcal{G}_1$ and $\beta_2(\eta) = \mathcal{G}_2$.

Lemma 2. The triplet $(\mathrm{X}, \beta_1, \beta_2)$, where

$\beta_1: D(S) \to \mathrm{X}$, $\beta_1(\nu) = \frac{1}{\sqrt{2}}\big((\kappa\nu)(\infty) - (\kappa\nu)(a)\big)$ and

$\beta_2: D(S) \to \mathrm{X}$, $\beta_2(\nu) = \frac{1}{i\sqrt{2}}\big((\kappa\nu)(\infty) + (\kappa\nu)(a)\big)$, $\nu \in D(S)$

is a space of boundary values of the operator S_0 in $L^2_\varrho(X,(a,\infty)$.

Proof. For any $\nu,\vartheta \in D(S)$

$$
\begin{aligned}
&(S\nu,\vartheta)_{L^2_\varrho(X,(a,\infty))} - (\nu,S\vartheta)_{L^2_\varrho(X,(a,\infty))}\\
&\quad= \left(i\frac{\kappa}{\varrho}(\kappa\nu)' + A\nu,\vartheta\right)_{L^2_\varrho(X,(a,\infty))} - \left(\nu, i\frac{\kappa}{\varrho}(\kappa\vartheta)' + A\vartheta\right)_{L^2_\varrho(X,(a,\infty))}\\
&\quad= \left(i\frac{\kappa}{\varrho}(\kappa\nu)',\vartheta\right)_{L^2_\varrho(X,(a,\infty))} - \left(\nu, i\frac{\kappa}{\varrho}(\kappa\vartheta)'\right)_{L^2_\varrho(X,(a,\infty))}\\
&\quad= \int_a^\infty \left(i\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa\nu)'(\tau),\vartheta(\tau)\right)_{\mathrm{X}}\varrho(\tau)d\tau - \int_a^\infty \left(\nu(\tau), i\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa\vartheta)'(\tau)\right)_{\mathrm{X}}\varrho(\tau)d\tau\\
&\quad= i\left[\int_a^\infty((\kappa\nu)'(\tau),(\kappa\vartheta)(\tau))_{\mathrm{X}}d\tau + \int_a^\infty((\kappa\nu)(\tau),(\kappa\vartheta)'(\tau))_{\mathrm{X}}d\tau\right]\\
&\quad= i\int_a^\infty\big((\kappa\nu)(\tau),(\kappa\vartheta)(\tau)\big)'_{\mathrm{X}}d\tau\\
&\quad= i\left[\big((\kappa\nu)(\infty),(\kappa\vartheta)(\infty)\big)_{\mathrm{X}} - \big((\kappa\nu)(a),(\kappa\vartheta)(a)\big)_{\mathrm{X}}\right]\\
&\quad= (\beta_1(\nu),\beta_2(\vartheta))_{\mathrm{X}} - (\beta_2(\nu),\beta_1(\vartheta))_{\mathrm{X}}.
\end{aligned}
$$

Nom let $f,g \in \mathrm{X}$. Let us find the function $\nu \in D(S)$ such that

$\beta_1(\nu) = \frac{1}{\sqrt{2}}\big((\kappa\nu)(\infty) - (\kappa\nu)(a)\big) = f$ and $\beta_2(\nu) = \frac{1}{i\sqrt{2}}\big((\kappa\nu)(\infty) + (\kappa\nu)(a)\big) = g.$

Taking into account these equations, one can see

$(\kappa\nu)(\infty) = \frac{ig+f}{\sqrt{2}}$ and $(\kappa\nu)(a) = \frac{ig-f}{\sqrt{2}}$.

If we choose the functions $\nu(\cdot\,,\ \cdot)$ in the following form

$$\nu(\tau) = \frac{1}{\kappa(\tau)}(1-e^{a-\tau})\frac{ig+f}{\sqrt{2}} + \frac{1}{\kappa(\tau)}e^{a-\tau}\frac{ig-f}{\sqrt{2}},$$

then it is obvious that $\nu \in D(S)$ and $\beta_1(\nu) = f$, $\beta_2(\nu) = g$.

With the use of the Calkin-Gorbachuk method [1], we obtain the following:

Theorem 1. If $\tilde{S}$ is a maximally dissipative extension of the operator S_0 in $L^2_\varrho(X,(a,\infty))$, then it is generated by the differential-operator expression $s(\cdot)$ and the boundary condition

$$(\kappa\nu)(a) = \Gamma(\kappa\nu)(\infty),$$

where $\Gamma\colon \mathrm{X} \to \mathrm{X}$ is a contraction operator. Additionally, the contraction operator Γ in X is uniquely determined by the extension $\tilde{S}$, i.e. $\tilde{S} = S_\Gamma$ and vice versa.

Proof. Each maximally dissipative extension $\tilde{S}$ of the operator S_0 is described by differential-operator expression $s(\cdot)$with boundary condition

$$(U-E)\beta_1(\nu) + i(U+E)\beta_2(\nu) = 0$$

where $U: \mathrm{X} \rightarrow \mathrm{X}$ is a contraction operator. Therefore, from Lemma 2, we obtain

$$(U-E)\big((\kappa\nu)(\infty)-(\kappa\nu)(a)\big)+(U+E)\big((\kappa\nu)(\infty)+(\kappa\nu)(a)\big)=0, \nu \in D(\tilde{S}).$$

Hence, it is obtained that

$$(\kappa\nu)(a)=-\mathrm{U}(\kappa\nu)(\infty).$$

Choosing $\Gamma=-U$ in the last boundary condition we have

$$(\kappa\nu)(a)=\Gamma(\kappa\nu)(\infty).$$

Therefore considering this and Theorem 1 together, we can give the following result.

Theorem 2. Each maximally accretive extension $\tilde{L}$ of the operator L_0 generated by the linear singular differential expression $l(\cdot)$ and the boundary condition

$$(\kappa\nu)(a)=\Gamma(\kappa\nu)(\infty),$$

where $\Gamma: \mathrm{X} \rightarrow \mathrm{X}$ is a contraction operator such that this operator is uniquely determined by the extension $\tilde{L}$, i.e. $\tilde{L}=L_\Gamma$ and vice versa.

4. The spectrum of the maximally accretive extensions

In this section, we will research the geometry of the spectrum set of the maximally accretive extensions of the operator L_0 in $L_\varrho^2(X,(a,\infty))$.

Theorem 3. The spectrum of any maximally accretive extension L_Γ is in form

$$\sigma(L_\Gamma)=\Big\{\lambda \in \mathbb{C}: \lambda=\left(\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)} ds\right)^{-1}(\ln(|\mu|^{-1})+i\arg(\bar{\mu})+2n\pi i), \\ \mu \in \sigma\left(\Gamma \exp\left(-A\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)} ds\right)\right), \quad n \in \mathbb{Z}\Big\}.$$

Proof. Let us consider the following spectrum problem defined by

$$L_\Gamma(\nu)=\lambda\nu+f, \lambda \in \mathbb{C}, \lambda_r=Re\,\lambda \geq 0.$$

Then, we have

$$\frac{\kappa(\tau)}{\varrho(\tau)}(\kappa\nu)'(\tau)+A\nu(\tau)=\lambda\nu(\tau)+f(\tau), \tau>a,$$
$$(\kappa\nu)(a)=\Gamma(\kappa\nu)(\infty).$$

The general solution of the last differential equation

$$(\kappa\nu)'(\tau)=\frac{\varrho(\tau)}{\kappa^2(\tau)}(\lambda E-A)(\kappa\nu)(\tau)+\frac{\varrho(\tau)}{\kappa(\tau)}f(\tau), \tau>a$$

is in the following form

$$\nu(\tau;\lambda)=\frac{1}{\kappa(\tau)}\exp\left((\lambda E-A)\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda$$
$$-\frac{1}{\kappa(\tau)}\int_\tau^\infty\exp\left((\lambda E-A)\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\varrho(s)}{\kappa(s)}f(s)ds,\ f_\lambda\in \mathrm{X},\ \tau>a.$$

In this case

$$\left\|\frac{1}{\kappa(\tau)}\exp\left((\lambda E-A)\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda\right\|^2_{L^2_\varrho(X,(a,\infty))}$$
$$=\int_a^\infty\left\|\frac{1}{\kappa(\tau)}\exp\left((\lambda E-A)\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda\right\|^2_{\mathrm{X}}\varrho(\tau)d\tau$$
$$=\int_a^\infty\left(\frac{1}{\kappa(\tau)}\exp\left((\lambda E-A)\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda,\frac{1}{\kappa(\tau)}\exp\left((\lambda E-A)\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda\right)_{\mathrm{X}}\varrho(\tau)d\tau$$
$$=\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\exp\left(2\lambda_r\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)$$
$$\left(\exp\left(-A\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda,\exp\left(-A\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda\right)_{\mathrm{X}}d\tau$$
$$=\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\exp\left(2\lambda_r\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)\left\|\exp\left(-A\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)f_\lambda\right\|^2_{\mathrm{X}}d\tau$$
$$\leq\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\exp\left(2\lambda_r\int_a^\tau\frac{\varrho(s)}{\kappa^2(s)}ds\right)d\tau\|f_\lambda\|^2_{\mathrm{X}}$$
$$=\frac{1}{2\lambda_r}\left(\exp\left(2\lambda_r\int_a^\infty\frac{\varrho(s)}{\kappa^2(s)}ds\right)-1\right)\|f_\lambda\|^2_{\mathrm{X}}<\infty$$

and

$$\left\|-\frac{1}{\kappa(\tau)}\int_\tau^\infty\exp\left((\lambda E-A)\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\varrho(s)}{\kappa(s)}f(s)ds\right\|^2_{L^2_\varrho(X,(a,\infty))}$$
$$=\int_a^\infty\left\|\frac{1}{\kappa(\tau)}\int_\tau^\infty\exp\left((\lambda E-A)\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\varrho(s)}{\kappa(s)}f(s)ds\right\|^2_{\mathrm{X}}\varrho(\tau)d\tau$$
$$=\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\left\|\int_\tau^\infty\exp\left((\lambda E-A)\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\varrho(s)}{\kappa(s)}f(s)ds\right\|^2_{\mathrm{X}}d\tau$$
$$=\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\left\|\int_\tau^\infty\exp\left(\lambda E\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\left[\exp\left(-A\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\varrho(s)}{\kappa(s)}f(s)\right]ds\right\|^2_{\mathrm{X}}d\tau$$
$$=\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\left\|\int_\tau^\infty\exp\left((\lambda_r+i\lambda_i)E\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\left[\exp\left(-A\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\varrho(s)}{\kappa(s)}f(s)\right]ds\right\|^2_{\mathrm{X}}d\tau$$
$$\leq\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\left(\int_\tau^\infty\exp\left(\lambda_r E\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\sqrt{\varrho(s)}}{\kappa(s)}\left(\sqrt{\varrho(s)}\|f(s)\|_{\mathrm{X}}\right)ds\right)^2d\tau$$
$$\leq\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\left(\int_a^\infty\frac{\varrho(s)}{\kappa^2(s)}\exp\left(2\lambda_r E\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)ds\right)\left(\int_a^\infty\varrho(s)\|f(s)\|^2_{\mathrm{X}}ds\right)d\tau$$
$$\leq\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}\left(\int_a^\infty\frac{\varrho(s)}{\kappa^2(s)}\exp\left(2\lambda_r E\int_s^\tau\frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)ds\right)d\tau\,\|f\|^2_{L^2_\varrho(X,(a,\infty))}$$
$$=\exp\left(2\lambda_r E\int_a^\infty\frac{\varrho(\xi)}{\kappa(\xi)}d\xi\right)\left(\int_a^\infty\frac{\varrho(\tau)}{\kappa^2(\tau)}d\tau\right)^2\|f\|^2_{L^2_\varrho(X,(a,\infty))}<\infty.$$

Hence, $\nu(\cdot,\ \lambda)\in L^2_\varrho(X,(a,\infty))$ for $\lambda\in\mathbb{C}$, $Re\,\lambda\geq 0$.

Using this and boundary condition, we have

$$\left(E-\Gamma\exp\left((\lambda E-A)\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds\right)\right)f_\lambda=\int_a^\infty \exp\left((\lambda E-A)\int_s^a \frac{\varrho(\xi)}{\kappa^2(\xi)}d\xi\right)\frac{\varrho(s)}{\kappa(s)}f(s)ds.$$

One can see that the necessary and sufficient condition for $\lambda\in\sigma(L_\Gamma)$ is

$$\exp\left(-\lambda\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds\right)=\mu\in\sigma\left(\Gamma\exp\left(-A\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds\right)\right).$$

Therefore,

$$-\lambda\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds=\ln|\mu|+i\arg\mu+2m\pi i,\ \ m\in\mathbb{Z}.$$

Thus,

$$\lambda=\left(\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds\right)^{-1}(\ln(|\mu|^{-1})+i\arg(\bar{\mu})+2n\pi i),\quad \mu\in\sigma\left(\Gamma\exp\left(-A\int_a^\infty \frac{\varrho(s)}{\kappa^2(s)}ds\right)\right),$$
$n\in\mathbb{Z}$.

This completes the proof.

Now, we present an example as an application of our results.

Example. All maximally accretive extensions L_r of the minimal operator L_0 generated by the following first order linear symmetric singular differential expression

$l(v)=\tau^{\gamma-\alpha}(\tau^\gamma v(\tau))'+av(\tau),\ \gamma,\alpha,a\in\mathbb{R}$ and $2\gamma-\alpha-1>0$

in the Hilbert space $L^2_{\tau^\alpha}(1,\infty)$ are described by the boundary condition

$(\tau^\gamma v)(1)=r(\tau^\gamma v)(\infty)$,

where $r\in\mathbb{C}$ and $|r|\leq 1$.

Moreover, in this case that $r\neq 0$ the spectrum of maximally accretive extension L_r is of the form

$\sigma(L_r)=(1+\alpha-2\gamma)(\ln|r|+i\arg(r)+2n\pi i),\ n\in\mathbb{Z}$.

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