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Homotopy Analysis Method for the Time-Fractional Boussinesq Equation

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Abstract

In this paper, the exact and approximate analytical solutions to the time-fractional Boussinesq equation are constructed using the homotopy analysis method. Several examples about the fourth-order and sixth-order time-fractional Boussinesq equations show the flexibility and efficiency of the method. Furthermore, by choosing an appropriate value for the auxiliary parameter h, we can obtain the N-term approximate solution with improved accuracy.

1. Introduction

In recent years, the time-fractional differential equations have attracted a large amount of attention due to their broad applications in physics, biology, hydrology and engineering [9]. In particular, the time-fractional Boussinesq equation, as a generalization of the Boussinesq equation, can be used to describe the surface water waves with a long memory property. Roughly speaking, two types of methods have been used to solve the fractional differential equation, including analytical [5, 6, 12, 13, 16, 18–20] and numerical methods [1, 3, 7, 8, 11, 14, 17, 21–24]. As far as the analytical methods for the time-fractional Boussinesq equations are concerned, the authors in [5] applied the modified Kudryashov method to solve the nonlinear conformable time-fractional Boussinesq equations. The authors in [6] used the fractional Lie group method to solve the time-fractional Boussinesq equation. Xu et al. [16] also proposed an iterative method to construct the analytical solution. Recently, a Fourier spectral method [24] was developed to obtain the numerical solutions of the equation.

In this paper, we consider the time-fractional Boussinesq equation, which is defined by replacing the integer-order time derivatives with the fractional-order time derivatives. The fractional derivatives are in the Caputo's sense, so that the same initial conditions for the Boussinesq equation with integer-order derivatives can be imposed for the time-fractional equation. Furthermore, we apply the homotopy analysis method [10] to obtain the exact and approximate solution of the time-fractional Boussinesq equation. By solving various examples using the homotopy analysis method, we can show the flexibility and efficiency of the method. A key component of the method is the selection of the auxiliary parameter h. Numerical results provide some insights on how to choose h to obtain the N-term approximate solutions with improved accuracy.

The remaining of the paper is as follows: in section 2, we introduce the notations and basic properties of the fractional calculus, and describe the homotopy analysis method for the general nonlinear partial differential equations. In section 3, several examples about the fourth-order and sixth-order time-fractional Boussinesq equations are presented to demonstrate the performance of the homotopy analysis method. Numerical results about the N-term approximate solutions with various h and α are also discussed to provide the guideline of choosing the parameters.

2. Homotopy Analysis Method

In this section, we introduce the basic properties of fractional calculus and the homotopy analysis method for the general nonlinear partial differential equations.

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2.1. Preliminaries and Notations

In the literature, there are many definitions of fractional derivatives, including the Riemann-Liouville, Caputo, Grünwald-Letnikov, Caputo-Fabrizio and Atangana-Baleanu derivative [2,4,15]. Among all the aforementioned definitions, the Caputo derivative is one of the most widely used definitions for the time-fractional partial differential equations models. It is defined based on the Riemann-Liouville fractional integral. Let $\alpha > 0$ and n be the smallest integer that is greater than or equal to α, then for any locally integrable function f, its Riemann-Liouville (RL) fractional integral of order α is given by

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{2.1}$$

Here Γ is the Gamma function. The RL fractional integral (2.1) is a generalization of the n-fold integral of f. For $f(t) = t^{\beta}$ with $\beta \geq 0$, we can show that $J^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$. The α^{th}-order Caputo fractional derivative of smooth enough f can then be defined as ${}_0D_t^{\alpha} f(t) := J^{n-\alpha}(f^{(n)}(t))$, where $f^{(n)}(t)$ is the n^{th} order derivative of f with respect to t and $\alpha \in (n-1, n]$. That is,

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} f^{(n)}(s) ds. \tag{2.2}$$

The Caputo fractional derivative and the Riemann-Liouville fractional integral satisfy the following equality for smooth enough function f:

$$J^{\alpha}({}_0D_t^{\alpha} f(t)) = f(t) - \sum_{m=0}^{n-1} f^{(m)}(0+) \frac{t^m}{m!}, \quad t > 0, \tag{2.3}$$

where $f^{(m)}(0+)$ is the right-hand limit of $f^{(m)}(x)$ when x approaches zero from the right. Some exact solutions of the time-fractional differential equations can be represented using the Mittag-Leffler function [15], which is defined by the following series

$$E_{\alpha}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(1+m\alpha)}, \quad \text{for } \alpha > 0. \tag{2.4}$$

2.2. Homotopy Analysis Method

Suppose we consider the nonlinear partial differential equation

$$\mathscr{N}(u(x,t)) = 0, \tag{2.5}$$

where $\mathscr{N}$ in general is a nonlinear operator, and $u(x,t)$ is the exact solution to the equation (2.5). Let $\mathscr{L}$ be a linear operator that is a part of $\mathscr{N}$, then we consider the following equation

$$(1-q)\mathscr{L}(u(x,t) - u_0(x,t)) = qh\mathscr{N}(u(x,t)). \tag{2.6}$$

Here $q \in [0,1]$, h is a non-zero auxiliary parameter and $u_0(x,t)$ is an initial guess of the solution. We can see that when $q = 0$, equation (2.6) becomes $\mathscr{L}(u(x,t) - u_0(x,t)) = 0$. Therefore, $u_0(x,t)$ is the solution to (2.6) when $q = 0$, and $u(x,t)$ is the solution when $q = 1$. Since there is a solution to equation (2.6) for every given value of $q \in [0,1]$, we can replace $u(x,t)$ in (2.6) with $U(x,t;q)$. That is,

$$(1-q)\mathscr{L}(U(x,t;q) - u_0(x,t)) = qh\mathscr{N}(U(x,t;q)). \tag{2.7}$$

We further assume that $U(x,t;q)$ can be written as $U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m$, and we also assume that all the series are convergent. Thus (2.7) leads to

$$(1-q)\mathscr{L}\left(\sum_{m=1}^{\infty} u_m(x,t) q^m\right) = qh\mathscr{N}\left(u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m\right). \tag{2.8}$$

We then use the Taylor expansion of equation (2.8) and let the corresponding coefficients of q^m on both sides of the equation to be the same for all $m \geq 1$, and get

$$\mathscr{L}(u_1(x,t)) = h\mathscr{N}(u_0(x,t)), \tag{2.9}$$

$$\mathscr{L}(u_m(x,t) - u_{m-1}(x,t)) = h\frac{1}{(m-1)!} \frac{\partial^{m-1}\mathscr{N}(u_0(x,t))}{\partial q^{m-1}}\Big|_{q=0}, \quad m \geq 2. \tag{2.10}$$

If one can apply the inverse operator of $\mathscr{L}$ (denoted by $\mathscr{L}^{-1}$) on both sides of (2.9) and (2.10), then $u_m(x,t)$ can be calculated iteratively for $m \geq 1$. Finally, the analytical solution of equation (2.5) can be represented by

$$U(x,t;1) = \sum_{m=0}^{\infty} u_m(x,t), \tag{2.11}$$

assuming the series above converges.

3. Application to the Time-Fractional Boussinesq Equation

In this section, we construct the analytical and approximate analytical solutions of the time-fractional Boussinesq equation using the homotopy analysis method.

3.1. Example 1

We consider the following fourth-order time-fractional Boussinesq equation

$$\begin{cases} {}_0D_t^{\alpha}u = \beta u_{xxxx} + \gamma u_{xx} + \theta(u^2)_{xx} - 4\theta u^2, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = e^x, \quad u_t(x,0) = 0, \end{cases}$$

where $\alpha \in (1,2]$, u is a function of x and t, the coefficients $\beta > 0$, $\gamma > 0$ and θ are constants. To solve the initial value problem using the homotopy analysis method, we let $\mathscr{L}(u(x,t)) = {}_0D_t^{\alpha}u$ and $\mathscr{N}(u(x,t)) = {}_0D_t^{\alpha}u - \beta u_{xxxx} - \gamma u_{xx} - \theta(u^2)_{xx} + 4\theta u^2$ for sufficiently smooth $u(x,t)$. We apply the homotopy analysis method, and let $u_0 = e^x$. Thus (2.9) leads to

$${}_0D_t^{\alpha}u_1 = h\left({}_0D_t^{\alpha}u_0 - \beta(u_0)_{xxxx} - \gamma(u_0)_{xx} - \theta((u_0)^2)_{xx} + 4\theta(u_0)^2\right) = -h(\beta+\gamma)e^x. \tag{3.1}$$

We then apply J^{α} on both sides of (3.1) to get

$$u_1(x,t) = -\frac{\beta+\gamma}{\Gamma(1+\alpha)}he^x t^{\alpha}. \tag{3.2}$$

Similarly, (2.10) leads to

$${}_0D_t^{\alpha}(u_2 - u_1) = h({}_0D_t^{\alpha}u_1) - h\beta(u_1)_{xxxx} - h\gamma(u_1)_{xx} - h\theta\sum_{j=0}^{1}(u_j u_{1-j})_{xx} + 4h\theta\sum_{j=0}^{1}u_j u_{1-j} = h\,{}_0D_t^{\alpha}u_1 + \frac{(\beta+\gamma)^2}{\Gamma(1+\alpha)}h^2 e^x t^{\alpha}. \tag{3.3}$$

Again we apply J^{α} to (3.3) to get

$$u_2(x,t) = -\frac{\beta+\gamma}{\Gamma(1+\alpha)}h(1+h)e^x t^{\alpha} + \frac{(\beta+\gamma)^2}{\Gamma(1+2\alpha)}h^2 e^x t^{2\alpha}. \tag{3.4}$$

For general $m \geq 1$, we can obtain

$$u_m(x,t) = (1+h)u_{m-1} - h\beta J^{\alpha}((u_{m-1})_{xxxx}) - h\gamma J^{\alpha}((u_{m-1})_{xx}) - h\theta J^{\alpha}\left(\sum_{j=0}^{m-1}(u_j u_{m-1-j})_{xx}\right) + 4h\theta J^{\alpha}\left(\sum_{j=0}^{m-1}u_j u_{m-1-j}\right). \tag{3.5}$$

Using mathematical induction, we can show that

$$u_m(x,t) = e^x\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}\frac{h^{j+1}(1+h)^{m-1-j}}{\Gamma(1+\alpha+j\alpha)}t^{(j+1)\alpha}(\beta+\gamma)^{j+1}. \tag{3.6}$$

Therefore, the analytical solution can be represented by

$$u(x,t) = e^x + e^x\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}\frac{h^{j+1}(1+h)^{m-1-j}}{\Gamma(1+\alpha+j\alpha)}t^{(j+1)\alpha}(\beta+\gamma)^{j+1}. \tag{3.7}$$

When $h = -1$, equation (3.7) becomes $u(x,t) = e^x\sum_{m=0}^{\infty}\frac{t^{m\alpha}(\beta+\gamma)^m}{\Gamma(1+m\alpha)}$, which is convergent for all $\beta, \gamma > 0$, $\alpha \in (1,2]$, $x \in \mathbf{R}$ and $t \geq 0$. Other choices of the parameter h will determine the rate of convergence. The N^{th} order approximation using the homotopy analysis method is given by

$$U^N(x,t) = e^x + e^x\sum_{m=1}^{N}\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}\frac{h^{j+1}(1+h)^{m-1-j}}{\Gamma(1+\alpha+j\alpha)}t^{(j+1)\alpha}(\beta+\gamma)^{j+1}. \tag{3.8}$$

To investigate the convergence of the approximation solution for various values of h, we compute the L^{∞} error for $x \in [-1,1]$ at $t = 1$ and it is shown in Table 1.

N	$h=-1.2$	$h=-1.1053$	$h=-1$	$h=-0.8$	$h=-0.6$
4	1.0005e-02	1.4127e-06	6.7082e-03	1.7332e-01	7.0676e-01
8	1.4976e-05	9.8654e-08	6.2465e-08	1.7657e-03	5.2175e-02
12	1.1419e-08	2.3393e-11	4.2633e-14	1.2178e-05	3.1777e-03
16	8.0469e-11	1.0658e-14	0	6.7143e-08	1.7199e-04
20	6.9278e-14	3.5527e-15	0	3.1885e-10	8.5779e-06

Table 1: $\|U^N(\cdot,T) - u_{exact}(\cdot,T)\|_{\infty,x\in[-1,1]}$ at $T = 1$ for various approximation order N and auxiliary parameter h when $\alpha = 1.5$.

We can see from Table 1 that at $T = 1$, $h = -1.1053$ leads to the most accurate solution (with the L^{∞} error being 1.4127×10^{-6}) among all the 4-term approximations, which implies that the 4-term approximation with suitable choice of h can be very accurate. If we consider larger values of N, we can observe that the approximate solution with $h = -1$ becomes more accurate than the other values of h when $N \geq 8$. We then fix the values of N and h, i.e., $N = 10$ and $h = -1.1053$, and investigate the time evolution of the approximate analytical solutions for various t and α. The results are shown in Table 2. We observe that the 10-term approximate solutions with $h = -1.1053$ are very accurate.

T	$\alpha = 1.2$		$\alpha = 1.6$		$\alpha = 2.0$	
	Solution	Error	Solution	Error	Solution	Error
0.1	1.1200	2.6379e-12	1.0354	1.3352e-12	1.0100	1.3947e-12
0.2	1.2932	1.0809e-11	1.1096	7.7398e-12	1.0403	2.6135e-12
0.3	1.5111	3.6375e-11	1.2150	7.4181e-12	1.0914	2.4134e-12
0.4	1.7779	7.7804e-12	1.3516	2.8288e-11	1.1643	1.5979e-11
0.5	2.1011	1.9902e-10	1.5211	6.9401e-11	1.2606	2.9834e-11
0.6	2.4906	6.8443e-12	1.7268	2.4791e-11	1.3821	2.3717e-11
0.7	2.9584	9.4368e-10	1.9733	1.7407e-10	1.5313	2.6883e-11
0.8	3.5195	1.1823e-09	2.2661	4.4433e-10	1.7112	1.3323e-10
0.9	4.1916	2.1797e-09	2.6120	4.8104e-10	1.9254	2.7052e-10
1.0	4.9961	9.3082e-09	3.0193	1.4490e-10	2.1782	3.5943e-10

Table 2: $|U^N(x=0,T) - u_{exact}(x=0,T)|$ at $T = 0.1, 0.2, \ldots, 1$ and $\alpha = 1.2, 1.6$ and 2 when $N = 10$ and $h = -1.1053$.

3.2. Example 2

We then consider the fourth-order time-fractional Boussinesq equation in two dimensions.

$$\begin{cases} {}_0D_t^{\alpha} u = \beta_1 u_{xxxx} + \beta_2 u_{yyyy} + \gamma_1 u_{xx} + \gamma_2 u_{yy} + \theta_1 (u^2)_{xx} + \theta_2 (u^2)_{yy} - 4(\theta_1 + \theta_2) u^2, \quad -\infty < x, y < \infty,\ t > 0, \\ u(x,y,0) = e^{x+y}, \quad u_t(x,y,0) = 0. \end{cases}$$

Here the solution u is a function of x, y and t. We let $\mathscr{L}(u) = {}_0D_t^{\alpha} u$ and $\mathscr{N}(u) = {}_0D_t^{\alpha} u - \beta_1 u_{xxxx} - \beta_2 u_{yyyy} - \gamma_1 u_{xx} - \gamma_2 u_{yy} - \theta_1 (u^2)_{xx} - \theta_2 (u^2)_{yy} + 4(\theta_1 + \theta_2) u^2$. Therefore,

$$\begin{aligned} {}_0D_t^{\alpha} u_1 &= h \left({}_0D_t^{\alpha} u_0 - \beta_1 (u_0)_{xxxx} - \beta_2 (u_0)_{yyyy} - \gamma_1 (u_0)_{xx} - \gamma_2 (u_0)_{yy}\right) - h\left(-\theta_1 (u_0)^2_{xx} - \theta_2 (u_0)^2_{yy} + 4(\theta_1 + \theta_2)(u_0)^2\right) \\ &= -h(\beta_1 + \beta_2 + \gamma_1 + \gamma_2) e^{x+y}. \end{aligned} \tag{3.9}$$

Here we have chosen $u_0 = e^{x+y}$. Therefore, we can obtain u_1 by applying the operator J^{α} on both sides of (3.9). That is,

$$u_1(x,y,t) = -h \frac{(\beta_1 + \beta_2 + \gamma_1 + \gamma_2) t^{\alpha}}{\Gamma(1+\alpha)} e^{x+y}. \tag{3.10}$$

For $m \geq 2$, we get

$$\begin{aligned} {}_0D_t^{\alpha}(u_m - u_{m-1}) = h &\left[{}_0D_t^{\alpha} u_{m-1} - \beta_1 \frac{\partial^4 u_{m-1}}{\partial x^4} - \beta_2 \frac{\partial^4 u_{m-1}}{\partial y^4} - \gamma_1 \frac{\partial^2 u_{m-1}}{\partial x^2} - \gamma_2 \frac{\partial^2 u_{m-1}}{\partial y^2}\right] - h\theta_1 \sum_{j=0}^{m-1} \frac{\partial^2 (u_j u_{m-1-j})}{\partial x^2} \\ &- h\theta_2 \sum_{j=0}^{m-1} \frac{\partial^2 (u_j u_{m-1-j})}{\partial y^2} + 4h(\theta_1 + \theta_2) \sum_{j=0}^{m-1} u_j u_{m-1-j}. \end{aligned} \tag{3.11}$$

We can show that when $m = 2$, there is

$${}_0D_t^{\alpha}(u_2 - u_1) = h\, {}_0D_t^{\alpha} u_1 + h^2 \frac{(\beta_1 + \beta_2 + \gamma_1 + \gamma_2)^2 t^{\alpha}}{\Gamma(1+\alpha)} e^{x+y}. \tag{3.12}$$

Thus, (3.10) and (3.12) lead to

$$u_2(x,y,t) = -(1+h) h \frac{(\beta_1 + \beta_2 + \gamma_1 + \gamma_2) t^{\alpha}}{\Gamma(1+\alpha)} e^{x+y} + h^2 \frac{(\beta_1 + \beta_2 + \gamma_1 + \gamma_2)^2 t^{2\alpha}}{\Gamma(1+2\alpha)} e^{x+y}.$$

Similarly, we can further derive the following formulations of $u_m(x,y,t)$ for $m = 3$:

$$u_3(x,y,t) = -(1+h)^2 h C_1 t^{\alpha} e^{x+y} + 2(1+h) h^2 C_2 t^{2\alpha} e^{x+y} - h^3 C_3 t^{3\alpha} e^{x+y},$$

where

$$C_n = \frac{(\beta_1 + \beta_2 + \gamma_1 + \gamma_2)^n}{\Gamma(1+n\alpha)}, \quad n = 1, 2, \ldots$$

In general, we can obtain

$$u_m(x,y,t) = e^{x+y} \sum_{j=0}^{m-1} (-1)^{j+1} \binom{m-1}{j} C_{j+1} h^{j+1} (1+h)^{m-1-j} t^{(j+1)\alpha}, \tag{3.13}$$

for $m \geq 1$. Thus the analytical solution is given by $u(x,y,t) = \sum_{m=0}^{\infty} u_m(x,y,t)$ where $u_m(x,y,t)$ is defined by equation (3.13). The 10-term approximate solution and its point-wise absolute error are plotted in Figure 3.1. It shows that the maximum absolute error occurs at $x = y = 1$, and the overall error is at the magnitude of 10^{-8}, which indicates the accuracy of the method.

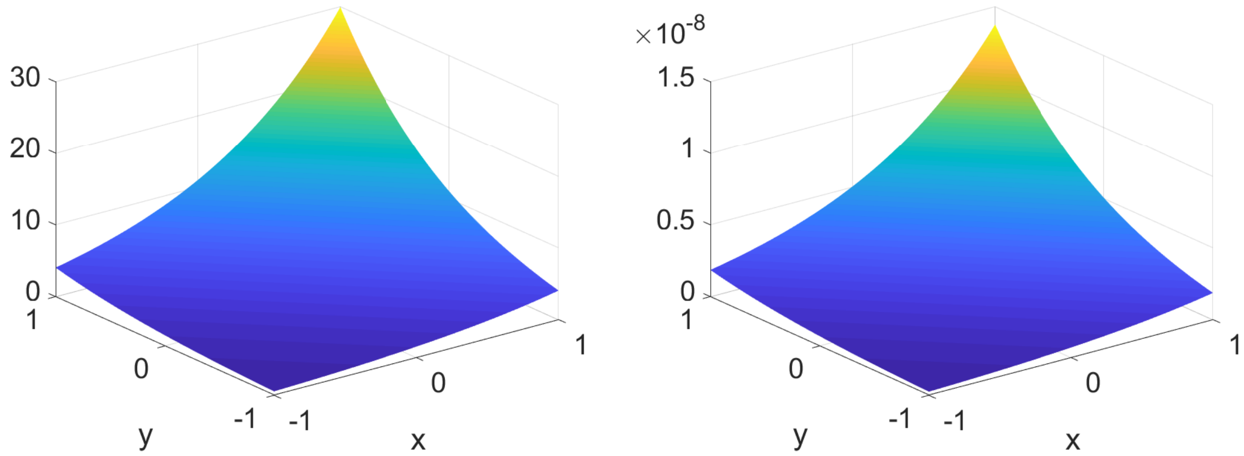

Figure 3.1: The 10-term approximate solution in example 2 and its absolute error. Here $\alpha = 1.7$, $\beta_1 = \gamma_2 = 1$, $\beta_2 = \gamma_1 = 0.5$ and $T = 1$. Left: the approximate solution at $T = 1$. Right: the absolute error of the approximate solution at $T = 1$.

3.3. Example 3

We then consider the time-fractional Boussinesq equation with the sixth-order spatial derivative:

$$\begin{cases} {}_0D_t^\alpha u = u_{xxxxxx} + u_{xxxx} + u_{xx} + \theta(u^2)_{xx} - 4\theta u^2, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = e^x, \quad u_t(x,0) = 0. \end{cases}$$

From the process of homotopy analysis method, we have

$$(1-q)\mathscr{L}(U(x,t;q) - u_0(x,t)) = qh\mathscr{N}(U(x,t;q)), \tag{3.14}$$

where $\mathscr{L}(U(x,t;q)) = {}_0D_t^\alpha U$, $\mathscr{N}(U(x,t;q)) = {}_0D_t^\alpha U - U_{xxxxxx} - U_{xxxx} - U_{xx} - \theta(U^2)_{xx} + 4\theta U^2$ and $u_0(x,t) = e^x$. We then differentiate (3.14) with respect to q for m times, and let $q = 0$ to get

$${}_0D_t^\alpha(u_1(x,t)) = h\mathscr{N}(u_0(x,t)), \tag{3.15}$$

$${}_0D_t^\alpha(u_m(x,t)) = {}_0D_t^\alpha(u_{m-1}(x,t)) + h\mathscr{R}(u_1, u_2, \ldots, u_{m-1}). \tag{3.16}$$

Here we have assumed that $U(x,t;q) = \sum_{m=0}^\infty u_m(x,t)q^m$ is convergent for $q \in [0,1]$. The operator $\mathscr{R}$ is given by

$$\mathscr{R}(u_1, u_2, \ldots, u_{m-1}) = {}_0D_t^\alpha u_{m-1} - (u_{m-1})_{xxxxxx} - (u_{m-1})_{xxxx} - (u_{m-1})_{xx} - \theta\sum_{j=0}^{m-1}(u_j u_{m-1-j})_{xx} + 4\theta\sum_{j=0}^{m-1}(u_j u_{m-1-j}). \tag{3.17}$$

If we apply J^α to both sides of (3.15) and (3.16), we can obtain

$$u_1(x,t) = hJ^\alpha\mathscr{N}(u_0(x,t)), \tag{3.18}$$

$$u_m(x,t) = u_{m-1}(x,t) + hJ^\alpha\mathscr{R}(u_1, u_2, \ldots, u_{m-1}), \quad m \geq 2. \tag{3.19}$$

We can calculate the next three terms, i.e., $m = 1, 2$ and 3, as follows

$$u_1(x,t) = -3h\frac{t^\alpha e^x}{\Gamma(1+\alpha)}, \tag{3.20}$$

$$u_2(x,t) = -3(1+h)h\frac{t^\alpha e^x}{\Gamma(1+\alpha)} + 9h^2\frac{t^{2\alpha} e^x}{\Gamma(1+2\alpha)}, \tag{3.21}$$

$$u_3(x,t) = -3(1+h)^2h\frac{t^\alpha e^x}{\Gamma(1+\alpha)} + 18(1+h)h^2\frac{t^{2\alpha} e^x}{\Gamma(1+2\alpha)} - 27h^3\frac{t^{3\alpha} e^x}{\Gamma(1+3\alpha)}. \tag{3.22}$$

To derive the general formulation of $u_m(x,t)$, we first let $u_m(x,t)$ be

$$u_m(x,t) = \sum_{j=1}^m C_{m,j}h^j(1+h)^{m-j}\frac{t^{j\alpha} e^x}{\Gamma(1+j\alpha)}, \tag{3.23}$$

where $C_{m,j}$ is the undetermined coefficient in $u_m(x,t)$. From (3.20)-(3.22), we know that $C_{1,1} = C_{2,1} = C_{3,1} = -3$, $C_{2,2} = 9$, $C_{3,2} = 18$ and $C_{3,3} = -27$. Thus we only need to calculate $C_{m,j}$ for $m \geq 4$ and $j \in [1,m]$. Using equation (3.19), one can show that

$$u_m(x,t) = (1+h)u_{m-1}(x,t) - 3hJ^\alpha u_{m-1}(x,t). \tag{3.24}$$

We further apply (3.23) to (3.24) to get

$$\begin{aligned}u_m(x,t) &= \sum_{j=1}^{m-1} C_{m-1,j} h^j (1+h)^{m-j} \frac{t^{j\alpha} e^x}{\Gamma(1+j\alpha)} - \sum_{j=1}^{m-1} 3C_{m-1,j} h^{j+1} (1+h)^{m-1-j} \frac{t^{(j+1)\alpha} e^x}{\Gamma(1+(j+1)\alpha)} \\ &= C_{m-1,1} h (1+h)^{m-1} \frac{t^\alpha e^x}{\Gamma(1+\alpha)} + \sum_{j=2}^{m-1} (C_{m-1,j} - 3C_{m-1,j-1}) h^j (1+h)^{m-j} \frac{t^{j\alpha} e^x}{\Gamma(1+j\alpha)} - 3C_{m-1,m-1} h^{m-1} (1+h) \frac{t^{m\alpha} e^x}{\Gamma(1+m\alpha)}.\end{aligned}$$

Therefore, we can derive the following difference equations for $m \geq 3$:

$$\begin{cases} C_{m,1} = C_{m-1,1}, \\ C_{m,j} = C_{m-1,j} - 3C_{m-1,j-1}, \quad \text{for } j \in [2, m-1], \\ C_{m,m} = -3C_{m-1,m-1}. \end{cases} \tag{3.25}$$

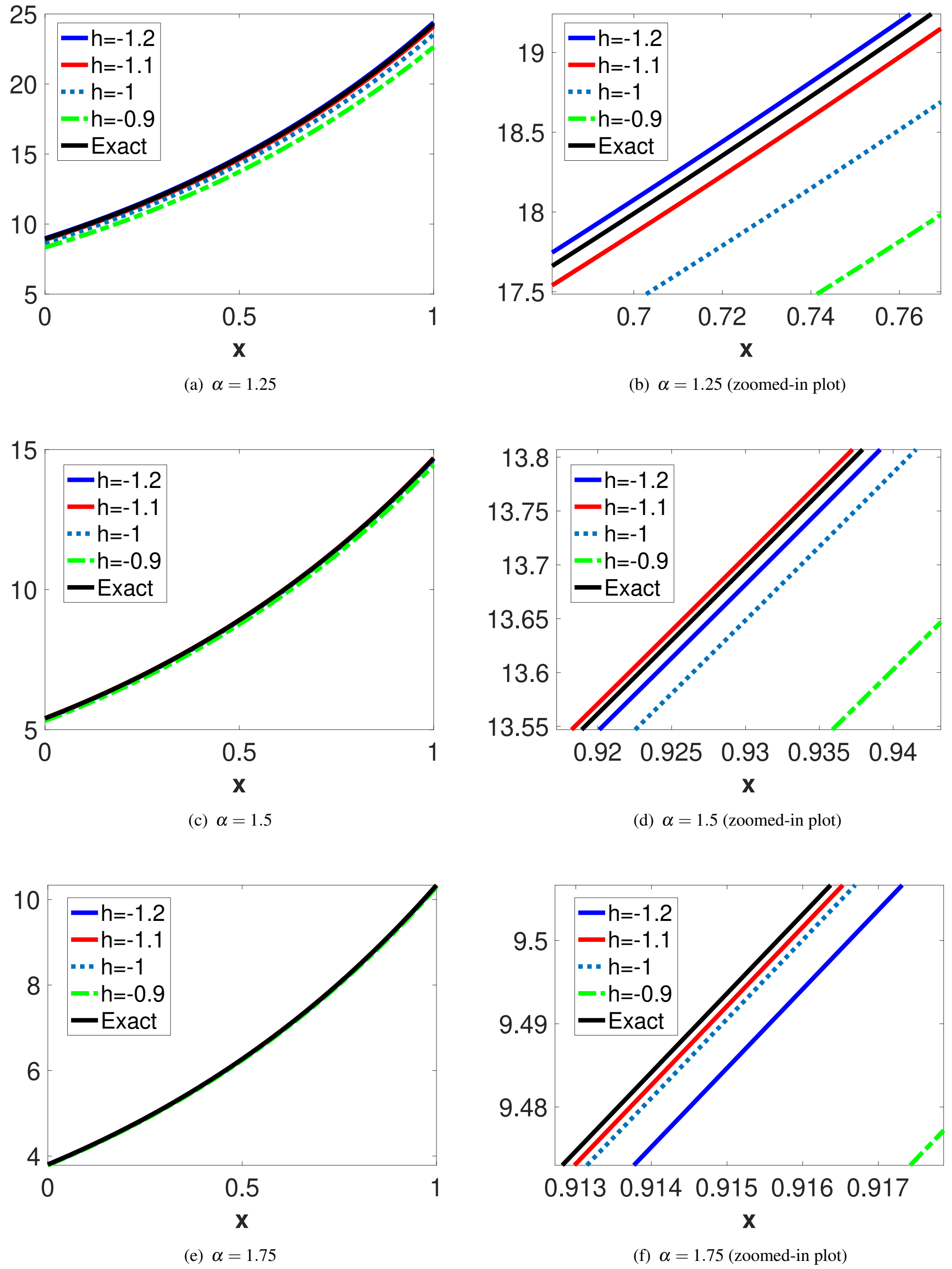

Figure 3.2: Comparison of the 4-term approximate and exact solutions at the final time $T = 1$ in example 3. Here $\alpha = 1.25, 1.5$ and 1.75. In (a)-(f), the blue, red, dark blue, green and black lines represent the approximate solutions with $h = -1.2, -1.1, -1, -0.9$ and the exact solution, respectively. (a): solutions when $\alpha = 1.25$. (b): zoomed-in plot of solutions when $\alpha = 1.25$. (c): solutions when $\alpha = 1.5$. (d): zoomed-in plot of solutions when $\alpha = 1.5$. (e): solutions when $\alpha = 1.75$. (f): zoomed-in plot of solutions when $\alpha = 1.75$.

From the first equation in (3.25), we get $C_{m,1} = C_{3,1} = -3$. The third equation in (3.25) leads to $C_{m,m} = (-3)^{m-2}C_{2,2} = (-3)^m$. For $j = 2$ in the second equation of (3.25), we have $C_{m,2} = C_{m-1,2} - 3C_{m-1,1} = C_{m-1,2} + 9$. So we can derive $C_{m,2} = C_{2,2} + 9(m-2) = 9(m-1)$. Similarly, we can also show that $C_{m,3} = -\frac{27}{2}(m-1)(m-2)$ and $C_{m,4} = \frac{27}{2}(m^3 - 3m^2 + 2m - 18)$. The analytical formula of $C_{m,j}$ with $5 \le j \le m-1$ can be calculated in the similar manner, but we omit the detail here. The exact solution to the original problem is given by $\sum_{m=0}^{\infty} u_m(x,t)$.

If we take $h = -1$, then according to equation (3.23), $u_m(x,t) = C_{m,m}h^m \frac{t^{m\alpha}e^x}{\Gamma(1+m\alpha)} = \frac{3^m t^{m\alpha}e^x}{\Gamma(1+m\alpha)}$. Therefore, the solution becomes $u(x,t) = e^x E_\alpha(3t^\alpha)$, where E_α is the Mittag-Leffler function defined in (2.4). For this example, we compute the 4-term approximate solutions with various values of $\alpha \in (1,2)$ and $h < 0$, and plot the solutions in Figure 3.2. The top row in Figure 3.2 shows the 4-term approximate solutions for $h = -1.2, -1.1, -1$ and -0.9 when $\alpha = 1.25$. We observe that the approximation solution with $h = -0.9$ is the least accurate solution, followed by the solution with $h = -1$. The 4-term approximation solution with $h = -1.2$ is slightly more accurate than that with $h = -1.1$. We can see that with appropriate choice of h, a 4-term approximation solution can be very accurate. When $\alpha = 1.5$, similar results can be observed except that in this case the approximation solution with $h = -1.1$ is more accurate than that with $h = -1.2$. The bottom row in Figure 3.2 shows that when $\alpha = 1.75$, $h = -1.1$ leads to the most accurate approximate solution. Based on the discussion above, we see that the parameter h provides the flexibility of obtaining accurate solutions.

4. Conclusion

In this paper, we derive the analytical and the approximate analytical solutions of the one and two-dimensional time-fractional Boussinesq equation using the homotopy analysis method. The homotopy analysis method is a semi-analytical technique to solve the differential equations by representing the solution in the form of series with an auxiliary parameter h. We have demonstrated that the iterative method in [16] is a special case of the homotopy analysis method with $h = -1$. Numerical results show that the truncated N-term approximate solution with an appropriate value of N can be used as an accurate approximation of the analytical solution. In particular, we find that $N = 10$ with a suitable choice of h leads to accurate approximate solutions when the final time is less than or equal to 1, and we can obtain the approximate solutions with single-precision accuracy in the sense of L^∞ norm. Moreover, the accuracy can be further improved by selecting the optimal value of h, which is the advantage of using this method over using the iterative method with $h = -1$.

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