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# Some Integrals Connected with a New Quadruple Hypergeometric Series

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**Abstract**

Hypergeometric function of four variables was introduced by Bin-Saad and Younis. In the present paper a new integral representations of of Euler-type and Laplace-type involving double and triple hypergeometric series for these functions are derived.

## 1. Introduction

In mathematics, there are various special functions that are used in numerous applications [8, 13, 17, 20]. In addition, some special functions have also been shown to have applications in diverse areas as statistical physics, quantum physics, quantum mechanics, fluid dynamics, acoustics, electrical current, heat conduction, astronomy, economics [1, 7, 15, 21]. Hypergeometric functions have a large variety of applications in many areas of mathematics such as in algebraic geometry, Lie algebras, difference equations, group theory, representation theory, partition theory and Hodge theory [1–6, 9–12, 16, 22]. Moreover, multiple hypergeometric functions can be used to solve physical and chemical problems in many areas of applied mathematics [1, 14, 19]. In the present study we aim to obtain certain integral representations of Euler-type and Laplace-type involving new quadruple hypergeometric series namely by $X_i^{(4)}$( $i = 11, 12, 13, 14, 15$).
Recall the Gaussian hypergeometric function defined by [19]

$$ {}_2F_1\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\quad (|x|<1),$$

where $(a)_n$ is the well-known Pochhammer symbol given by

$$(a)_n:=\begin{cases}1, & (n=0)\\ a(a+1)...(a+n-1), & (n\in\mathbb{N}:=\{1,2,...\})\end{cases}=\frac{\Gamma(a+n)}{\Gamma(a)},$$

$\Gamma(a)$ is the well-known Gamma function defined by

$$\Gamma(a)=\int_0^{\infty}e^{-t}t^{a-1}dt,\quad (Re(a)>0). \tag{1.1}$$

The Appell series $F_1, F_2$ and the Horn's series $H_3$ of two variables are defined as follows [19]:

$$F_1\left(a,b,c;d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\qquad (\max\ \{|x|,|y|\}<1),$$

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$$F_2\left(a,b,c;d,e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!},\qquad (|x|+|y|<1)$$

and

$$H_3(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\qquad \left(|x|<r,|y|<s,r+\left(s-\frac{1}{2}\right)^2=\frac{1}{4}\right).$$

The Exton's triple hypergeometric functions $X_5,X_6,X_7$ and $X_{14}$ are given by [10]

$$X_5\left(a,b,c;d;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\qquad \left(r<\frac{1}{4}\wedge\max\{s,t\}<\frac{1}{2}+\frac{1}{2}\sqrt{(1-4r)},|x|\le r,|y|\le s,|z|\le t\right),$$

$$X_6\left(a,b,c;d,e;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\left(t+2\sqrt{r}<1\wedge s<\frac{1}{2}(1-t\ )+\frac{1}{2}\sqrt{(1-t)^2-4r},|x|\le r,|y|\le s,|z|\le t\right),$$

$$X_7\left(a,b,c;d,e;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_m(e)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\left(s<1\wedge t<1\wedge r<\frac{1}{4}\min\{(1-s)^2,(1-t)^2\},|x|\le r,|y|\le s,|z|\le t\right)$$

and

$$X_{14}\left(a,b,c;d,e;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\left(r<\frac{1}{4}\wedge t<1\wedge s<(1-t)\left[\frac{1}{2}+\frac{1}{2}\sqrt{(1-4r)}\right],|x|\le r,|y|\le s,|z|\le t\right).$$

The following Srivastava's function of three variables $H_A$ is defined in [19] as

$$H_A\left(a,b,c;d,e;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n}(b)_{n+p}(c)_{p+m}}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\qquad (r<1\wedge s<1\wedge t<(1-r)(1-s),|x|\le r,|y|\le s,|z|\le t).$$

Lauricella hypergeometric function of four variables $F_C^{(4)}$ [19] which is defined by

$$F_C^{(4)}\left(a,b;c_1,c_2,c_3,c_4;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\qquad (\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}+\sqrt{|u|}<1).$$

More recently, Bin-Saad et al. [2–5] introduced new hypergeometric series of four variables namely by $X_1^{(4)},X_2^{(4)},...,X_{10}^{(4)}$ and investigated their certain properties including integral representations, symbolic representations, generating functions, etc. Motivated largely by the aforementioned works of Bin-Saad et al. [4] and [5], we defined further quadruple hypergeometric functions as follows:

$$X_{11}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\tag{1.2}$$

$$X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\tag{1.3}$$

$$X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\tag{1.4}$$

$$X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n+p}(c_2)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\tag{1.5}$$

$$X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c)_{m+n+p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.\tag{1.6}$$

The structure of this article is as follows. In Section 2, we give several Euler-type integrals involving the new quadruple series $X_i^{(4)},(\ i=11,12,13,14,15)$. Certain integral representations of Laplace-type for our series rae given in section 3.

## 2. Integral representations of Euler-Type

This section gives various integral representations of Euler-Type for the series $X_{11}^{(4)}, X_{12}^{(4)}, ..., X_{15}^{(4)}$ in terms of the classical Gauss hypergeometric function ${}_2F_1$, the Appell's double hypergeometric functions $F_1$ and $F_2$, Horn's function $H_3$ of two variables, the Srivastava's triple series $H_A$, the Exton's hypergeometric series of three variables $X_5, X_6, X_7$ and $X_{14}$, and the quadruple series $X_{11}^{(4)}, X_{12}^{(4)}$ and $F_C^{(4)}$ as follows:

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{2\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)} \int_0^{\frac{\pi}{2}} (sin^2\alpha)^{a_1-\frac{1}{2}} (cos^2\alpha)^{c_2-a_1-\frac{1}{2}} \left(1-z\, sin^2\alpha\right)^{-a_3} \\ &\times X_{14}\left(1+a_1-c_2,a_2,a_3;c_1,c_3;x\, tan^4\alpha,-y\, tan^2\alpha,\frac{u}{(1-z\, sin^2\alpha)}\right) d\alpha \\ &(Re(a_1)>0, Re(c_2-a_1)>0), \end{aligned} \tag{2.1}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_3)(S-T)^{a_2}(R-T)^{c_3-a_2}}{\Gamma(a_2)\Gamma(c_3-a_2)(S-R)^{c_3-a_3-1}} \int_R^S (\alpha-R)^{a_2-1}(S-\alpha)^{c_3-a_2-1} \\ &\times (\alpha-T)^{a_3-c_3}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)\right]^{-a_3} \\ &\times X_6(a_1,1+a_2-c_3,a_3;c_1,c_2;x,\lambda_1 y,\lambda_2 z)\, d\alpha \\ &\left(\lambda_1=-\frac{(S-T)(\alpha-R)}{(R-T)(S-\alpha)}, \lambda_2=\frac{(S-R)(\alpha-T)}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)]}\right), \\ &(Re(a_2)>0, Re(c_3-a_2)>0, T<R<S), \end{aligned} \tag{2.2}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_1)(1+M)^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)} \int_0^1 \alpha^{a_1-1}(1+M\alpha)^{1+a_1+a_2-2c_1} \\ &\times \left[(1-\alpha)(1+M\alpha)+(1+M)^2\alpha^2 x\right]^{c_1-a_1-1}\left[(1+M\alpha)-(1+M)\alpha y\right]^{-a_2} \\ &\times F_2(a_3,1+a_1-c_1,a_2;c_2,c_3;\lambda_1 z,\lambda_2 u)\, d\alpha \\ &\left(\lambda_1=-\frac{(1+M)\alpha(1+M\alpha)}{[(1-\alpha)(1+M\alpha)+(1+M)^2\alpha^2 x]},\ \lambda_2=\frac{(1+M\alpha)}{[(1+M\alpha)-(1+M)\alpha y]}\right), \\ &(\Re(a_1)>0, \Re(c_1-a_1)>0, M>-1), \end{aligned} \tag{2.3}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_2-a_1)\Gamma(c_3-a_2)} \int_0^1\int_0^1 \alpha^{a_1-1}(1-\alpha)^{c_2-a_1-1}\beta^{a_2-1} \\ &\times (1-\beta)^{c_3-a_2-1}(1-\alpha z-\beta u)^{-a_3} H_3\Big(1+a_1-c_2,1+a_2-c_3;c_1; \\ &\frac{\alpha^2 x}{(1-\alpha)^2},\frac{\alpha\beta y}{(1-\alpha)(1-\beta)}\Big)\, d\alpha d\beta \\ &(Re(a_1)>0, Re(a_2)>0, Re(c_2-a_1)>0, Re(c_3-a_2)>0), \end{aligned} \tag{2.4}$$

$$\begin{aligned} X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) &= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1-a_1)\Gamma(c_2-a_3)} \int_0^\infty\int_0^\infty \alpha^{a_1-1}(1+\alpha)^{1+a_1+a_2-2c_1}\beta^{a_3-1} \\ &\times (1+\beta)^{1+a_1-c_1-c_2}\left[(1+\alpha)(1+\beta)+\alpha^2(1+\beta)x+\alpha(1+\alpha)\beta z\right]^{c_1-a_1-1} \\ &\times [(1+\alpha)-\alpha y]^{-a_2}\, {}_2F_1\left(a_2,1+a_3-c_2;c_3;-\frac{(1+\alpha)\beta u}{[(1+\alpha)-\alpha y]}\right) d\alpha d\beta \\ &(Re(a_1)>0, Re(a_3)>0, Re(c_1-a_1)>0, Re(c_2-a_3)>0), \end{aligned} \tag{2.5}$$

$$\begin{aligned}X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u) &= \frac{\Gamma(a_1+a_2+a_3)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a)\Gamma(c_1-a)}\int_0^1\int_0^1\int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}\\ &\times(1-\beta)^{a_3-1}\gamma^{a-1}(1-\gamma)^{c_1-a-1}F_C^{(4)}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};c_2,\right.\\ &a,c_1-a,c_3;\lambda_1 x,\lambda_2 y,\lambda_3 z,\lambda_4 u)\,d\alpha d\beta d\gamma\\ &\Big(\lambda_1=4\alpha^2\beta^2,\lambda_2=4\alpha\beta^2\gamma(1-\alpha),\lambda_3=4\alpha\beta(1-\beta)(1-\gamma),\\ &\lambda_4=4(1-\alpha)\beta(1-\beta)\Big),\\ &(Re(a_i)>0,i=(1,2,3),Re(a)>0,Re(c_1-a)>0),\end{aligned} \tag{2.6}$$

$$\begin{aligned}X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u) &= \frac{2\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a_1-\frac{1}{2}}(cos^2\alpha)^{c_2-a_1-\frac{1}{2}}\\ &\times\left(1+x\,sin^2\alpha\,tan^2\alpha\right)^{c_2-a_1-1}H_A(a_3,a_2,1+a_1-c_2;c_3,c_1;u,\lambda y,\lambda z)\,d\alpha\\ &\left(\lambda=-\frac{tan^2\alpha}{(1+x\,sin^2\alpha\,tan^2\alpha)}\right),\\ &(Re(a_1)>0,Re(c_2-a_1)>0),\end{aligned} \tag{2.7}$$

$$\begin{aligned}X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u) &= \frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^{\infty}(e^{-\alpha})^{a_1}\left[(1-e^{-\alpha})+x\,e^{-2\alpha}\right]^{c_2-a_1-1}\\ &\times H_A(a_3,a_2,1+a_1-c_2;c_3,c_1;u,\lambda y,\lambda z)\,d\alpha\\ &\left(\lambda=-\frac{e^{-\alpha}}{\left[(1-e^{-\alpha})+x\,e^{-2\alpha}\right]}\right),\\ &(Re(a_1)>0,Re(c_2-a_1)>0),\end{aligned} \tag{2.8}$$

$$\begin{aligned}X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u) &= \frac{\Gamma(c_3)(S-T)^{a_2}(R-T)^{c_3-a_2}}{\Gamma(a_2)\Gamma(c_3-a_3)(S-R)^{c_3-a_3-1}}\int_R^S(\alpha-R)^{a_2-1}(S-\alpha)^{c_3-a_2-1}\\ &\times(\alpha-T)^{a_3-c_3}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)\right]^{-a_3}\\ &\times X_7(a_1,1+a_2-c_3,a_3;c_2,c_1;x,\lambda_1 y,\lambda_2 z)\,d\alpha\\ &\left(\lambda_1=-\frac{(S-T)(\alpha-R)}{(R-T)(S-\alpha)},\lambda_2=\frac{(S-R)(\alpha-T)}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)u)]}\right),\\ &(Re(a_2)>0,Re(c_3-a_2)>0,T<R<S),\end{aligned} \tag{2.9}$$

$$\begin{aligned}X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u) &= \frac{\Gamma(c_1}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^1\alpha^{a_1-1}(1-\alpha)^{c_1-a_1-1}(1-\alpha y)^{-a_2}\\ &\times(1-\alpha z)^{-a_3}\,{}_2F_1\left(\frac{1+a_1-c_1}{2},\frac{a_1-c_1}{2}+1;c_2;\frac{4\alpha^2 x}{(1-\alpha)^2}\right)\\ &\times{}_2F_1\left(a_2,a_3;c_3;\frac{u}{(1-\alpha y)(1-\alpha z)}\right)d\alpha\\ &(Re(a_1)>0,Re(c_1-a_1)>0),\end{aligned} \tag{2.10}$$

$$X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right)=\frac{\Gamma(a_1+a_2+a_3)\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a)\Gamma(b)\Gamma(c_1-a)\Gamma(c_2-b)}\int_0^1\int_0^1\int_0^1\int_0^1\alpha^{a_1-1}$$
$$\times(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}\gamma^{a-1}(1-\gamma)^{c_1-a-1}\zeta^{b-1}(\zeta-1)^{c_2-b-1}$$
$$\times F_C^{(4)}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};a,c_1-a,b,c_2-b;\lambda_1 x,\lambda_2 y,\right.$$
$$\left.\lambda_3 z,\lambda_4 u\right)d\alpha d\beta d\gamma d\zeta$$
$$\left(\lambda_1=4\alpha^2\beta^2\gamma,\lambda_2=4\alpha\beta^2(1-\alpha)(1-\gamma),\lambda_3=4\alpha\beta\zeta(1-\beta),\right.$$
$$\left.\lambda_4=4(1-\alpha)\beta(1-\beta)(1-\zeta)\right),$$
$$(Re(a_i)>0,i=(1,2,3),Re(a)>0,Re(b)>0,$$
$$Re(c_1-a)>0,Re(c_2-b)>0),\tag{2.11}$$

$$X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right)=\frac{2\Gamma(c_1)(1+M)^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}}$$
$$\times\left[\left(1+Msin^2\alpha\right)+(1+M)^2xsin^2\alpha\, tan^2\alpha\right]^{c_1-a_1-1}$$
$$\times\left(1+Msin^2\alpha\right)^{1+a_1+a_2-2c_1}\left[\left(1+Msin^2\alpha\right)-(1+M)ysin^2\alpha\right]^{-a_2}$$
$$\times F_1\left(a_3,1+a_1-c_1,a_2;c_2;\lambda_1 z,\lambda_2 u\right)d\alpha\tag{2.12}$$
$$\left(\lambda_1=-\frac{(1+M)\left(1+Msin^2\alpha\right)tan^2\alpha}{\left[\left(1+Msin^2\alpha\right)+(1+M)^2xsin^2\alpha\, tan^2\alpha\right]},\right.$$
$$\left.\lambda_2=\frac{\left(1+Msin^2\alpha\right)}{\left[\left(1+Msin^2\alpha\right)-(1+M)ysin^2\alpha\right]}\right),$$
$$(Re(a_1)>0,Re(c_1-a_1)>0,M>-1),$$

$$X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^{\infty}\alpha^{a_1-1}(1+\alpha)^{1+a_1+a_2-2c_1}\left[(1+\alpha)+\alpha^2x\right]^{c_1-a_1-1}$$
$$\times\left[(1+\alpha)-\alpha y\right]^{-a_2}F_1\left(a_3,1+a_1-c_1,a_2;c_2;-\frac{\alpha(1+\alpha)z}{\left[(1+\alpha)+\alpha^2x\right]},\right.$$
$$\left.\frac{(1+\alpha)u}{\left[(1+\alpha)-\alpha y\right]}\right)d\alpha$$
$$(Re(a_1)>0,Re(c_1-a_1)>0),\tag{2.13}$$

$$X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right)=\frac{\Gamma(c_2)\,(S-T)^{a_3}\,(R-T)^{c_2-a_3}}{\Gamma(a_3)\Gamma(c_2-a_3)\,(S-R)^{c_2-a_1-a_2-1}}\int_R^S(\alpha-R)^{a_3-1}(S-\alpha)^{c_2-a_3-1}$$
$$\times(\alpha-T)^{a_1+a_2-c_2}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)z\right]^{-a_1}$$
$$\times\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u\right]^{-a_2}H_3\left(a_1,a_2;c_1;\lambda_1 x,\lambda_2 y\right)d\alpha$$
$$\left(\lambda_1=\frac{(S-R)^2(\alpha-T)^2}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)z\right]^2},\right.$$
$$\lambda_2=\frac{(S-R)^2(\alpha-T)^2}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)z\right]}$$
$$\left.\times\frac{1}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u\right]}\right),$$
$$(Re(a_3)>0,Re(c_2-a_3)>0,T<R<S),\tag{2.14}$$

$$X_{13}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u)=\frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^1\alpha^{a_3-1}(1-\alpha)^{c_2-a_3-1}(1-\alpha z)^{-a_1}$$
$$\times(1-\alpha u)^{-a_2}H_3\left(a_1,a_2;c_1;\frac{x}{(1-\alpha z)^2},\frac{y}{(1-\alpha z)(1-\alpha u)}\right)d\alpha \tag{2.15}$$
$$(Re(a_3)>0,Re(c_2-a_3)>0),$$

$$X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_1)}{\Gamma(a)\Gamma(c_1-a)}\int_0^\infty\frac{\alpha^{a-1}}{(1+\alpha)^{c_1}}X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,$$
$$a_3;a,a,c_1-a,c_2;\frac{\alpha x}{(1+\alpha)},\frac{\alpha y}{(1+\alpha)},\frac{z}{(1+\alpha)},u\Big)d\alpha, \tag{2.16}$$
$$(Re(a)>0,Re(c_2-a)>0),$$

$$X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{2\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a_3-\frac{1}{2}}(cos^2\alpha)^{c_2-a_3-\frac{1}{2}}\left(1-u\,sin^2\alpha\right)^{-a_2}$$
$$\times X_5\left(a_1,a_2,1+a_3-c_2;c_1;x,\frac{y}{(1-u\,sin^2\alpha)},-z\,tan^2\alpha\right)d\alpha$$
$$(Re(a_3)>0,Re(c_2-a_3)>0), \tag{2.17}$$

$$X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^\infty(e^{-\alpha})^{a_3}(1-e^{-\alpha})^{c_2-a_3-1}\left(1-ue^{-\alpha}\right)^{-a_2}$$
$$\times X_5\left(a_1,a_2,1+a_3-c_2;c_1;x,\frac{y}{(1-ue^{-\alpha})},-\frac{ze^{-\alpha}}{(1-ue^{-\alpha})}\right)d\alpha \tag{2.18}$$
$$(Re(a_3)>0,Re(c_2-a_3)>0),$$

$$X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^1\alpha^{a_1-1}\left[(1-\alpha)+\alpha^2x\right]^{c_1-a_1-1}$$
$$\times(1-\alpha y)^{-a_2}(1-\alpha z)^{-a_3}\,{}_2F_1\left(a_2,a_3;c_2;\frac{u}{(1-\alpha y)(1-\alpha z)}\right)d\alpha \tag{2.19}$$
$$(Re(a_1)>0,Re(c_1-a_1)>0),$$

$$X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{2\Gamma(c_1)M^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{1+a_1+a_2+a_3-2c_1}$$
$$\times\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}}\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]^{c_1-a_1-1}$$
$$\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]^{-a_2}\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mzsin^2\alpha\right]^{-a_3}$$
$$\times{}_2F_1(a_2,a_3;c_2;\lambda u)\,d\alpha$$
$$\left(\lambda=\frac{\left(cos^2\alpha+Msin^2\alpha\right)^2}{\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mzsin^2\alpha\right]}\right)$$
$$(Re(a_1)>0,Re(c_1-a_1)>0,M>0), \tag{2.20}$$

$$X_{15}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}\int_0^\infty\int_0^\infty\frac{\alpha^{a-1}}{(1+\alpha)^c}\frac{\beta^{b-1}}{(1+\beta)^{c-a}}$$
$$\times X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;b,a,a,c-a-b;\lambda_1x,\lambda_2y,\lambda_2z,\lambda_3u)$$
$$\times d\alpha d\beta \tag{2.21}$$
$$\left(\lambda_1=\frac{\beta}{(1+\alpha)(1+\beta)},\lambda_2=\frac{\alpha}{(1+\alpha)},\lambda_3=\frac{1}{(1+\alpha)(1+\beta)}\right),$$
$$(Re(a)>0,Re(b)>0,Re(c-a)>0,Re(c-a-b)>0),$$

$$X_{15}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u)=\frac{\Gamma(c)(1+M)^{a_2}}{\Gamma(a_2)\Gamma(c-a_2)}\int_0^1 \alpha^{a_2-1}(1-\alpha)^{c-a_2-1}(1+M\alpha)^{a_1+a_3-c}$$
$$\times\left[(1+M\alpha)-(1+M)\alpha y\right]^{-a_1}\left[(1+M\alpha)-(1+M)\alpha u\right]^{-a_3}$$
$$\times H_3\left(a_1,a_3;c-a_2;\lambda_1 x,\lambda_2 z\right)d\alpha$$
$$\left(\lambda_1=\frac{(1-\alpha)(1+M\alpha)}{\left[(1+M\alpha)-(1+M)\alpha y\right]^2},\right. \tag{2.22}$$
$$\left.\lambda_2=\frac{(1-\alpha)(1+M\alpha)}{\left[(1+M\alpha)-(1+M)\alpha y\right]\left[(1+M\alpha)-(1+M)\alpha u\right]}\right),$$
$$\left(Re(a_2)>0,Re(c-a_2)>0,M>-1\right),$$

$$X_{15}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u)=\frac{2\Gamma(c)}{\Gamma(a_2)\Gamma(c-a_2)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a_2-\frac12}(cos^2\alpha)^{c-a_2-\frac12}\left(1-y\,sin^2\alpha\right)^{-a_1}$$
$$\times\left(1-u\,sin^2\alpha\right)^{-a_3}H_3\left(a_1,a_3;c-a_2;\frac{xcos^2\alpha}{\left(1-y\,sin^2\alpha\right)^2},\right. \tag{2.23}$$
$$\left.\frac{zcos^2\alpha}{\left(1-y\,sin^2\alpha\right)\left(1-u\,sin^2\alpha\right)}\right)d\alpha$$
$$\left(Re(a_2)>0,Re(c-a_2)>0\right),$$

$$X_{15}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u)=\frac{\Gamma(c)}{\Gamma(a_2)\Gamma(c-a_2)}\int_0^{\infty}(e^{-\alpha})^{a_1+a_2+a_3}(1-e^{-\alpha})^{c-a_2-1}(e^{\alpha}-y)^{-a_1}$$
$$\times(e^{\alpha}-u)^{-a_3}H_3\left(a_1,a_3;c-a_2;\frac{(1-e^{-\alpha})x}{(1-ye^{-\alpha})^2},\frac{(1-e^{-\alpha})z}{(1-ye^{-\alpha})(1-ue^{-\alpha})}\right)d\alpha$$
$$\left(Re(a_2)>0,Re(c-a_2)>0\right), \tag{2.24}$$

$$X_{15}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u)=\frac{\Gamma(c)}{\Gamma(a_3)\Gamma(c-a_3)}\int_0^{\infty}(e^{-\alpha})^{a_3}(1-e^{-\alpha})^{c-a_3-1}\left(1-ze^{-\alpha}\right)^{-a_1}$$
$$\times\left(1-ue^{-\alpha}\right)^{-a_2}H_3\left(a_1,a_2;c-a_3;\frac{e^{\alpha}(e^{\alpha}-1)x}{(e^{\alpha}-z)^2},\frac{e^{\alpha}(e^{\alpha}-1)y}{(e^{\alpha}-z)(e^{\alpha}-u)}\right)d\alpha \tag{2.25}$$
$$\left(Re(a_3)>0,Re(c-a_3)>0\right).$$

*Proof.* Once substituting the series definition of the special function in each integrand and then, changing the order of the integral and the summation, and finally taking into account the following integral representations of the Beta function and their various associated Eulerian integrals [8, 18, 20], we derive each of the integral representations from (2.1) to (2.25).

$$B(a,b)=\begin{cases}\displaystyle\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}\,dt & (Re(a)>0,Re(b)>0),\\ \displaystyle\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} & (a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{cases}$$

$$B(a,b)=\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}\,d\alpha=\int_0^{\infty}(e^{-\alpha})^a(1-e^{-\alpha})^{b-1}\,d\alpha,\qquad (Re(a)>0,Re(b)>0),$$

$$B(a,b)=2\int_0^{\frac{\pi}{2}}(sin\alpha)^{2a-1}(cos\alpha)^{2b-1}\,d\alpha=\int_0^{\infty}\frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}\,d\alpha,\qquad (Re(a)>0,Re(b)>0),$$

$$B(a,b)=\frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S\frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,d\alpha=(M+1)^a\int_0^1\frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)}\,d\alpha,$$
$$(T<R<S,M>-1,Re(a)>0,Re(b)>0).$$

□

## 3. Integral representations of Laplace-Type

In this section, we introduce Laplace integral representations of the hypergeometric series of four variables (1.2) to (1.6) by

$$\begin{aligned} & X_{11}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u\right) \\ & =\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^{\infty}\int_0^{\infty} e^{-(s+t)}s^{a_1-1}\,t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty\right)\Psi_2\left(a_3;c_2,c_3;sz,tu\right)dsdt, \\ & \left(Re(a_1)>0,Re(a_2)>0\right), \end{aligned} \tag{3.1}$$

$$\begin{aligned} & X_{12}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right) \\ & =\frac{1}{\Gamma(a_2)\Gamma(a_3)}\int_0^{\infty}\int_0^{\infty} e^{-(s+t)}s^{a_2-1}\,t^{a_3-1}\mathrm{H}_7\left(a_1;c_2,c_1;x,sy+tz\right){}_0F_1\left(-;c_3;stu\right)dsdt, \\ & \left(Re(a_2)>0,Re(a_3)>0\right), \end{aligned} \tag{3.2}$$

$$\begin{aligned} & X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right) \\ & =\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^{\infty}\int_0^{\infty} e^{-(s+t)}s^{a_1-1}\,t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty\right){}_1F_1\left(a_3;c_2;sz+tu\right)dsdt, \\ & \left(Re(a_1)>0,Re(a_2)>0\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} & X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right) \\ & =\frac{1}{\Gamma(a_2)\Gamma(a_3)}\int_0^{\infty}\int_0^{\infty} e^{-(s+t)}s^{a_2-1}\,t^{a_3-1}\mathrm{H}_6\left(a_1;c_1;x,sy+tz\right){}_0F_1\left(-;c_2;stu\right)dsdt, \\ & \left(Re(a_2)>0,Re(a_3)>0\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} & X_{15}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c,c,c,c;x,y,z,u\right) \\ & =\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(s+t+v)}s^{a_1-1}\,t^{a_2-1}\,v^{a_3-1}{}_0F_1\left(-;c;s^2x+sty+svz+tuv\right)dsdtdv, \\ & \left(Re(a_1)>0,Re(a_2)>0,Re(a_3)>0\right). \end{aligned} \tag{3.5}$$

where ${}_0F_1, {}_1F_1, \mathrm{H}_6, \mathrm{H}_7$ and $\Psi_2$ are the confluent hypergeometric functions defined by

$${}_0F_1\left(-;c;x\right)=\sum_{m=0}^{\infty}\frac{1}{(c)_m}\frac{x^m}{m!},\quad (|x|<\infty),$$

$${}_1F_1\left(a;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m}{(c)_m}\frac{x^m}{m!},\quad (|x|<\infty),$$

$$\mathrm{H}_6\left(a;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\left(|x|<\frac{1}{4},\quad |y|<\infty\right),$$

$$\mathrm{H}_7\left(a;b,c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\left(|x|<\frac{1}{4},\quad |y|<\infty\right)$$

and

$$\Psi_2\left(a;b,c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\left(|x|<\infty,\quad |y|<\infty\right).$$

*Proof.* It is noted that each of the integral representations (3.2) to (3.5) can be proved mainly by expressing the series definition of the involved the confluent hypergeometric functions in each integrand and changing the order of the integral sign and the summation, and finally using the formula (1.1). □

## 4. Conclusion

Concerning the hypergeometric functions of four variables, we mainly introduced five of them. On the basis of the definitions of the four variables hypergeometric functions, we succeed in establishing 25 integral formulas of Euler-type and five integral formulas of Laplace-type involving hypergeometric functions of two and three variables and confluent hypergeometric functions of two variables.

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