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Lower Bounds for the Blow up Time to a Coupled Nonlinear Hyperbolic Type Equations

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Abstract
The initial and Dirichlet boundary value problem of nonlinear hyperbolic type equations in a bounded domain is studied. We established a lower bounds for the blow up time.

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1. Introduction

In this paper, we deal with the lower bounds of the blow up time of solutions of the following hyperbolic type equations

$$\begin{cases} u_{tt} - div\left(\rho\left(|\nabla u|^2\right)\nabla u\right) - \Delta u_{tt} + |u_t|^{m-1} u_t = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} - div\left(\rho\left(|\nabla v|^2\right)\nabla v\right) - \Delta v_{tt} + |v_t|^{r-1} v_t = f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n\,(n = 1,2,3)$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$; $m,r \geq 1$ are constants, and $f_i(u,v) : R^2 \to R\ (i = 1,2)$ are functions which will be specified later. Also,

$$\rho(s) = b_1 + b_2 s^q, \quad q, b_1, b_2 \geq 0.$$

In the absence of the dispersion terms (Δu_{tt} and Δv_{tt}), eq. (1.1) reduces to the following system

$$\begin{cases} u_{tt} - div\left(\rho\left(|\nabla u|^2\right)\nabla u\right) - \Delta u_{tt} + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - div\left(\rho\left(|\nabla v|^2\right)\nabla v\right) - \Delta v_{tt} + |v_t|^{r-1} v_t = f_2(u,v). \end{cases} \tag{1.2}$$

In [1], Wu et al. considered the global existence and the blow up of the solution of the problem (1.2). Later, Fei and Hongjun [2] improved the blow up result in [1]. Finally, in [3], Pişkin and Polat studied the existence, the decay and the blow up of the solutions for the problem (1.2).

The aim of this paper note is to derive a lower bound for the blow up time occurs. Before stating our main theorem, we give some notations, lemmas and theorems.

2. Preliminaries

In this paper, we denote $\|.\| = \|.\|_{L^2(\Omega)}$ and $\|.\|_p = \|.\|_{L^p(\Omega)}$. Moreover, c_i $(i = 1, 2, \ldots)$ are arbitrary constants.

Let

$$f_1(u,v) = (p+1)\left[a|u+v|^{p-1}(u+v) + b|uv|^{\frac{p-1}{2}} v\right],$$

and

$$f_2(u,v) = (p+1)\left[a|u+v|^{p-1}(u+v) + b|uv|^{\frac{p-1}{2}} u\right],$$

where $a, b > 0$ are constant and p satisfies

$$\begin{cases} 1 < p & \text{if } n \leq 2, \\ 1 < p \leq \frac{n}{n-2} & \text{if } n > 2. \end{cases} \tag{2.1}$$

By a simple calculation, we have

$$uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v), \quad (u,v) \in R^2, \tag{2.2}$$

where

$$F(u,v) = \left[a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}\right]. \tag{2.3}$$

We define

$$J(t) = \frac{1}{2}\left[b_1\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\right] + \frac{1}{2q+2}\left[b_2\left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2}\right)\right] - \int_\Omega F(u,v)\,dx, \tag{2.4}$$

and

$$I(t) = \left[b_1\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\right] + \left[b_2\left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2}\right)\right] - (p+1)\int_\Omega F(u,v)\,dx. \tag{2.5}$$

We also define the energy functional as follows

$$\begin{aligned} E(t) \;=\; & \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\left[b_1\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\right] \\ & + \frac{1}{2q+2}\left[b_2\left(\|\nabla u\|_{2q+2}^{2q+2} + \|\nabla v\|_{2q+2}^{2q+2}\right)\right] + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \int_\Omega F(u,v)\,dx. \end{aligned} \tag{2.6}$$

We also define

$$W_- = \left\{(u,v) : (u,v) \in W_0^{1,2q+2}(\Omega) \times W_0^{1,2q+2}(\Omega),\ I(u,v) < 0\right\}. \tag{2.7}$$

The next lemma shows that our energy functional (2.6) is a nonincreasing function along the solution of (1.1).

Lemma 2.1. *Energy functional is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = -\left(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1}\right) \leq 0. \tag{2.8}$$

Proof. Multiplying the first equation in (1.1) by u_t and the second equation by v_t, integrating over Ω. Then integrating by parts, we get

$$E(t) - E(0) = -\int_0^t \left(\|u_\tau\|_{m+1}^{m+1} + \|v_\tau\|_{r+1}^{r+1}\right) d\tau \text{ for } t \geq 0 \tag{2.9}$$

Lemma 2.2. *(Sobolev-Poincare inequality) [4]. Let*

$$\begin{cases} 2 \le p < \infty; & n = 1,2, \\ 2 \le p \le \frac{2n}{n-2}; & n \ge 3 \end{cases}$$

then there is a constant $C_* = C_*(\Omega, p)$ *such that*

$$\|u\|_p \le C_* \|\nabla u\|, \ \ \forall u \in H_0^1(\Omega).$$

□

Lemma 2.3. *[5, 6] There exist two positive constants* c_1 *and* c_2 *such that*

$$\int_\Omega |f_1(u,v)|^2 dx \le c_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^p$$

and

$$\int_\Omega |f_2(u,v)|^2 dx \le c_2 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^p$$

are satisfied.

The local existence theorem which can be established combining the arguments of [3].

Theorem 2.4. *(Existence-uniqueness). Assume that (2.1) holds. Then further that* $u_0, v_0 \in W_0^{1,2q+2}(\Omega) \cap L^{p+1}(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$*.Then the system (1.1) has a unique local solution*

$$u, v \in C\left([0,T); W_0^{1,2q+2}(\Omega) \cap L^{p+1}(\Omega)\right).$$

Theorem 2.5. *[7]. Suppose that* $r > \max\{p,q\}$ *and* $E(0) < 0$ *hold. Then the solution u of the system blows up in finite time* T^*.

3. Lower bound for blow up time

In this section, our aim is to determine a lower bound for blow up time of the system (1.1).

Theorem 3.1. *Let* $u_0, v_0 \in W_0^{1,2q+2}(\Omega) \cap L^{p+1}(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, $(u_0, v_0) \in W_-$, *and* $1 < p, q < r$. *Assume that (2.1) holds. Then the solutions u of the problem (1.1) become unbounded at finite time* T^*. *Also, the lower bounds for the blow up time is given by*

$$\int_{\psi(0)}^{\infty} \frac{d\psi(z)}{\psi(\tau) + E(0) + \frac{c_1+c_2}{2}(p+1)^p \psi^p(\tau)} \le T^*.$$

Proof. We define

$$\psi(t) = \int_\Omega F(u,v)\, dx \tag{3.1}$$

By taking a derivative of (3.1), we get

$$\psi'(t) = \int_\Omega (u_t F_u + v_t F_v)\, dx \tag{3.2}$$

Thanks to Young's inequality, we have

$$\psi'(t) \le \frac{1}{2} \int_\Omega \left(u_t^2 + v_t^2\right) dx + \frac{1}{2} \int_\Omega \left(F_u^2 + F_v^2\right) dx.$$

By the Lemma 3, we get

$$\psi'(t) \le \frac{1}{2} \int_\Omega \left(u_t^2 + v_t^2\right) dx + \frac{c_1 + c_2}{2} \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^p \tag{3.3}$$

Since $I(t) < 0$, we have

$$b_1\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+b_2\left(\|\nabla u\|_{2q+2}^{2q+2}+\|\nabla v\|_{2q+2}^{2q+2}\right)\leq (p+1)\int_\Omega F(u,v)\,dx. \tag{3.4}$$

Inserting (3.4) into (3.3), we have

$$\begin{aligned}\psi'(t) &\leq \frac{1}{2}\int_\Omega \left(u_t^2+v_t^2\right)dx+\frac{c_1+c_2}{2}\left((p+1)\int_\Omega F(u,v)\,dx\right)^p \\ &= \frac{1}{2}\int_\Omega \left(u_t^2+v_t^2\right)dx+\frac{c_1+c_2}{2}(p+1)^p\left(\int_\Omega F(u,v)\,dx\right)^p \\ &= \frac{1}{2}\int_\Omega \left(u_t^2+v_t^2\right)dx+\frac{c_1+c_2}{2}(p+1)^p\,\psi^p(t)\end{aligned} \tag{3.5}$$

By the definition $E(t)$, we get

$$\begin{aligned}&(q+1)\left(\|u_t\|^2+\|v_t\|^2\right)+(q+1)\,b_1\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &+b_2\left(\|\nabla u\|_{2q+2}^{2q+2}+\|\nabla v\|_{2q+2}^{2q+2}\right)\ +(q+1)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right) \\ =\ &(2q+2)\,E(t)+(2q+2)\int_\Omega F(u,v)\,dx \\ \leq\ &(2q+2)\,E(0)+(2q+2)\,\psi(t)\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\psi'(t)\leq\psi(t)+E(0)+\frac{c_1+c_2}{2}(p+1)^p\,\psi^p(t)\,. \tag{3.7}$$

Applying Theorem 5, we have

$$\lim_{t\to T^*}\int_\Omega F(u,v)\,dx=\infty \tag{3.8}$$

According to (3.7), (3.8), we have

$$\int_{\psi(0)}^{\infty}\frac{d\psi(z)}{\psi(\tau)+E(0)+\frac{c_1+c_2}{2}(p+1)^p\,\psi^p(\tau)}\leq T^*.$$

This completes the proof. □

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