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Existence of almost fixed points for random operators with application in game theory

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Abstract

We introduce the notion of the random ε-fixed point for random operators. Then we prove some existence theorems in order to apply this results in game theory for example we will prove existence of ε-random Nash equilibrium in some class of random games. Such games can be applied in the environment in which Bayesian equilibria are considered, i.e. in games in which payoffs of players are known up to dependence on assignment of the players to specific types.

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1. Introduction - Multivalued mappings

We will assume that all considered topological spaces are metric spaces. We recall some well known definitions and facts about multivalued mappings. We begin introduction with definitions of multivalued mapping.

Let $(\mathbb{X}, d_1)$, $(\mathbb{Y}, d_2)$ be two metric spaces.

Definition 1. *We say that the mapping $\varphi : \mathbb{X} \to 2^{\mathbb{Y}}$ is a multivalued mapping if for every $x \in \mathbb{X}$, the set $\varphi(x)$ is a closed and nonempty subset of $\mathbb{Y}$. We denote such a mapping by $\varphi : \mathbb{X} \multimap \mathbb{Y}$ instead of $\varphi : \mathbb{X} \to 2^{\mathbb{Y}}$.*

In few following definitions by the φ symbol we will denote a mapping $\varphi : \mathbb{X} \multimap \mathbb{Y}$.

Definition 2. *The set $\{(x, y) \in \mathbb{X} \times \mathbb{Y},\ y \in \varphi(x)\}$ is called a graph of a mapping φ and we denote it by Γ_φ.*

Definition 3. *The map φ is called an upper semi-continuous mapping if for every open $U \subset \mathbb{Y}$ the set $\varphi^{-1}(U) = \{x \in \mathbb{X},\ \varphi(x) \subset U\}$ is an open subset of $\mathbb{X}$. For a shorter notation we will denote any upper semi-continuous mapping by usc.*

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Definition 4. *The map $\varphi : \mathbb{X} \multimap \mathbb{Y}$ is called a compact mapping if the set $\varphi(X) \subset \mathbb{Y}$ is a relatively compact subset of the space $\mathbb{Y}$, i.e, the closure of $\varphi(\mathbb{X})$ in $\mathbb{Y}$ is a compact set.*

Theorem 5. *A compact mapping φ is an usc mapping if and only if $\Gamma_\varphi \subset \mathbb{X} \times \mathbb{Y}$ is a closed subset of $\mathbb{X} \times \mathbb{Y}$.*

Definition 6. *Let $A \subset \mathbb{X}$ be a closed subset of $\mathbb{X}$ and $\varphi : A \multimap \mathbb{X}$ be a multivalued mapping. We shall say that a point $x_0 \in \mathbb{X}$ is a fixed point of φ provided $x_0 \in \varphi(x_0)$.*

By fix(φ) we denote the set of all fixed points of the mapping φ.

The set of all fixed points is a closed subset of A.

Let us add that, if φ is a compact mapping, then the set fix(φ) must be compact as well.

The notion of fixed points have been developed in many ways. The most important one, in context of this paper are almost fixed points or so called ε-fixed points. Thus we will recall the definition of ε-fixed points.

Definition 7. *Let $\varepsilon > 0$. We say that the point $x_0 \in \mathbb{X}$ is an ε-fixed point of a mapping φ, if $dist(x_0, \varphi(x_0)) < \varepsilon$, where $dist(x, A) = \inf_{y \in A} d(x, y)$.*

By fix$^\varepsilon(\varphi)$ we denote the set of all fixed points of a mapping φ.

We present a simple example of even a single-valued mapping that has no fixed point however for every $\varepsilon > 0$ the set of ε-fixed points is a non-empty and open subset of $\mathbb{R}$. Thus this example proves that the set fix$^\varepsilon$ might not be closed.

Example 8. *Consider the following mapping:*

$$\varphi(x) = \frac{x^2 + 1}{x}, \; x \in (0, +\infty)$$

Obviously such a mapping has no fixed points.

Let $\varepsilon > 0$. Then fix$^\varepsilon(f) = (\frac{1}{\varepsilon}, \infty)$, which is not a closed subset of $\mathbb{R}$. This example is clearly showing the difference between ε-fixed points and fixed points.

2. Random multivalued mappings

The very beginning of random fixed points for multivalued mappings is joint with two papers, first written by Papageorgiou [8] and second written by Benavides *et al.* [3].

We will consider the fixed points existence problem for random multivalued mappings.

Let us start this section with reminding the definition of measurable mapping.

Assume that $(\mathbb{X}, d)$ is a separable metric space and the space (Ω, μ) is a measurable space. Let us recall the definition of measurable mapping:

Definition 9. *The mapping $\varphi : \Omega \times \mathbb{X} \multimap \mathbb{Y}$ is called measurable, if φ is a product measurable mapping, i.e. for any open subset $U \subset \mathbb{Y}$, the set*

$$\varphi^{-1}(U) = \{(\omega, x) \in (\Omega \times \mathbb{X});\; F(\omega, x) \subset U\},$$

is a measurable subset of $\Omega \times \mathbb{X}$ (in $\mathbb{X}$ we consider σ-field of Borel's sets).

In further considerations we will need the notion of random usc mapping, which is both usc and measurable mapping of appropriate spaces.

Definition 10. *Let $A \subset \mathbb{X}$ be a closed subset. The measurable mapping $\varphi : \Omega \times A \multimap \mathbb{X}$ is called a random usc operator, if for almost every $\omega \in \Omega$, $\varphi(\omega, \cdot) : A \multimap \mathbb{X}$ is usc (with closed values).*

In some cases we might skip usc and simply call such a mapping random operator.

In order to introduce the definition of random ε-fixed point we must remind definition of the random fixed point at first.

Definition 11. *The measurable mapping $\eta : \Omega \to A$ is called a random fixed point of φ, if for almost every $\omega \in \Omega$, $\eta(\omega) \in \varphi(\omega, \eta(\omega))$.*

Finally, we are able to formulate the notion of random ε-fixed points for random operators.

Definition 12. *The measurable mapping $\eta : \Omega \to A$ is called a random ε-fixed point of φ, if for almost every $\omega \in \Omega$, $dist(\varphi(\omega, \eta(\omega)), \eta(\omega)) < \varepsilon$ and $\varepsilon > 0$.*

Lastly in this section we recall a theorem that guarantee existence of fixed point. We will apply this theorem to game theory in section 4.

Theorem 13. *Let $\mathbb{X}$ be a compact absolute approximative retract, then every usc mapping $\varphi : \mathbb{X} \multimap \mathbb{X}$ with R_δ values has fixed point.*

This theorem was proved in [6] (Theorem 4.8). For the notion of absolute approximative retract see for example [5].

3. Almost random fixed points theorems

We will generalize theorems proposed in [4] and [11], however we will make slightly stronger assumptions on considered multifunctions.

Let us begin this section with a theorem, which is crucial in order to prove those generalizations. We will use well-known Aumann selection theorem which can be find for example in [2].

Theorem 14. *Let $\varepsilon > 0$ and $\varphi : \Omega \times \mathbb{X} \multimap \mathbb{X}$ will be a random operator. Assume that for almost every $\omega \in \Omega$, $\varphi(\omega, \cdot) : A \multimap \mathbb{X}$ has an ε-fixed point, then there exists η a random ε-fixed point of mapping φ.*

Proof. Let $\varepsilon > 0$. We put $F : \Omega \multimap \mathbb{X}$:

$$F(\omega) = Fix^{\varepsilon}(\varphi(\omega, \cdot)).$$

The mapping F is well defined because we assume existence ε-fixed points of the mapping φ. To deduce that the graph Γ_F of F is measurable put $f : \Omega \times A \to [0, +\infty)$ as follows: $f(\omega, x) = \mathrm{dist}(x, \varphi(\omega, x))$. Then we have $\Gamma_F = f^{-1}((-\infty, \varepsilon))$. So by Aumann selection theorem there exists a measurable selector $\eta : \Omega \to A$ of F i.e. $\eta(\omega) \in F(\omega)$ for almost every $\omega \in \Omega$ so η is a random ε-fixed point of φ. □

Obviously, Theorem 14 remains true also for fixed points (compare [1]).

In further proposition we will formulate connection between fixed points and ε-fixed points.

Proposition 15. *Let $\varphi : \Omega \times \mathbb{X} \multimap \mathbb{X}$ be a random compact operator. Then there exists a random fixed point of φ if and only if there is a random ε-fixed point of φ for every $\varepsilon > 0$.*

Proof. Its obviously true that existence of random fixed point implies existence of ε-fixed point for every $\varepsilon > 0$.

Put $\varepsilon_n = \frac{1}{n}$. By assumptions there exist: $\eta_n : \Omega \to \mathbb{X}$ a random ε_n-fixed point (of φ). Let us fix ω. Almost surely

$$\tilde{d}(\eta_n(\omega), \varphi(\omega, \eta_n(\omega))) < \frac{1}{n},$$

what is equivalent to existence of $y_n \in \varphi_n(\omega, \eta(\omega))$ such that $d(y_n, \varphi_n(\omega, \eta_n(\omega))) < \frac{1}{n}$.

Since φ is a compact mapping then without losing generality we can assume that there exists $\lim_{n\to\infty} y_n(\omega) = y(\omega)$. The following inequality holds:

$$d(\varphi_n(\omega, y_n(\omega))) < \frac{1}{n}.$$

Put $\eta(\omega) = \lim_{n\to\infty} y_n(\omega)$. Then $\eta(\omega)$ is a fixed point of $\varphi(\omega, \cdot)$. This reasoning remains true for almost every ω so by Theorem 14 the proof is complete.

□

The following theorem is random case of existence theorem proved in [4]:

Theorem 16. *Let $\mathbb{E}$ be a reflexive and separable real Banach space and let A be a bounded and convex subset of $\mathbb{E}$ (with non-empty interior). Assume that $\varphi : \Omega \times A \multimap \mathbb{E}$ is a random operator, then for every $\varepsilon > 0$ there exists an ε-random fixed point of φ.*

Proof. For almost every $\omega \in \Omega$, the mapping $\varphi(\omega, \cdot) : A \multimap \mathbb{E}$ is fulfilling the assumption of Theorem 2.2 in [4]. Then this theorem is providing fixed point existence for almost every $\omega \in \Omega$. So by Theorem 14 the proof is finished. □

Let us remark that results proved in [11] can be generalized as follows:

Theorem 17. *Let A be a nonempty closed subset of a separable Banach space $\mathbb{E}$ and $\varphi : \Omega \times A \multimap \mathbb{E}$. Moreover assume that there is a finite subset $D = \{x_1, \ldots, x_n\}$ of A such that $conv(D) \subset A$ and $\varphi(\Omega \times A) \subset \bigcup_{i=1}^{+\infty} V_i$, where V_i is an open neighborhood of x_i in E. Then φ has a random ε-fixed point for every $\varepsilon > 0$.*

Theorem 18. *Let A be a nonempty convex subset of a separable Banach space and $\varphi : \Omega \times A \multimap \overline{A}$ be a random operator with convex values such that $\varphi(\Omega \times A)$ is totally bounded. The set $\overline{A}$ is the smallest closed subset containing A (smallest in respect to inclusion relation). Then φ has a random ε-fixed point for every $\varepsilon > 0$.*

Theorem 19. *Let $\mathbb{E}$ be a separable Banach space and $B(r) = \{x \in \mathbb{E};\ \|x\| < r\}$ for some $r > 0$. Moreover assume that $\varphi : \Omega \times B \multimap \overline{B}$ is a random operator with R_δ values. If $\varphi(\Omega \times B(\frac{1}{n}))$ is contained in a compact subset $C_n \subset \overline{B}$, for every n, then there exists ε-fixed point of φ for every $\varepsilon > 0$.*

Proof. Proofs of theorems 16-18 are similar to the proof of the Theorem 16. It is enough to apply Theorem 14 and Theorem 3.1, Theorem 3.2, or Theorem 3.3 in [11], respectively. □

4. Application to games – random approximate Nash equilibrium

In this section we will show how to apply ε-random fixed point theorems to game theory. We begin with definition of n-person game with random parameter. Then we will propose the definition of ε-random Nash equilibrium for such a game.

Lastly we will proof some existence results of ε-random Nash equilibrium.

Definition 20. *The strategic n-person game with a random parameter is a tuple $\Gamma = (\mathbb{X}, \Omega, u)$, where $\mathbb{X}$ is the set of all strategies profiles ($\mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_n$), Ω is a measurable space which determines randomness of the game, the mapping $u : \Omega \times \mathbb{X} \to \mathbb{R}^n$ is a payoff function and $u_i : \Omega \times \mathbb{X} \to \mathbb{R}$ is the i-th player payoff function.*

For this section let us make an agreement and if $x \in \mathbb{X}$, $y \in \mathbb{X}_i$, then we write (x_{-i}, y) instead of $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$.

Definition 21. *Let $\varepsilon > 0$, we say that $x^* \in \mathbb{X}$ is an ε-Nash equilibrium for a game Γ if for almost every $\omega \in \Omega$ for every $i \in \{1, \ldots, n\}$ $u_i(\omega, x^*) + \varepsilon \geq \sup_{x \in \mathbb{X}_i}(\omega, (x^*_{-i}, x))$.*

We denote the set of all ε-random Nash equilibrium in game Γ by $NE^\varepsilon(\Gamma)$.

Equivalently to ε-random Nash equilibrium we can formulate the ε-best response multifunction.
The ε-best response multifunction for the i-th player we define as follows:

$$B_i^{\varepsilon} : \Omega \times \mathbb{X} \multimap \mathbb{X}_i, \text{ where}$$

$$B_i^{\varepsilon}(\omega, x) = \{t \in \mathbb{X}_i \mid u_i(\omega, (x_{-i}, t)) \geq \sup_{y \in \mathbb{X}_i} u_i(\omega, (x_{-i}, y)) - \varepsilon\}.$$

Then we impose ε-best response multifunction.

$$B^{\varepsilon}(\omega, x) = B_1^{\varepsilon}(\omega, x) \times B_2^{\varepsilon}(\omega, x) \times \ldots \times B_n^{\varepsilon}(\omega, x).$$

Observe that if for almost every $\omega \in \Omega$ and $x^* \in B^\varepsilon(\omega, x^*)$, then $x^* \in \mathrm{NE}^\varepsilon(\Gamma)$. We will prove sufficient condition for ε-Nash equilibrium existence.

Theorem 22. *Let* Γ *be any game that is satisfying following conditions:*

1. *the payoff function* $u(\omega, \cdot) : \mathbb{X} \to \mathbb{R}^n$ *is uniformly continuous for almost every* $\omega \in \Omega$*, i.e.* $\forall_{\varepsilon>0}\, \exists_{\delta>0}$ *such that, if* $d(x, y) < \delta$*, then* $\forall_{i\in\{1,\ldots,n\}} |u_i(\omega, x) - u_i(\omega, y)| < \varepsilon$*;*

2. *for all* $\varepsilon > 0,\ \delta > 0$ *the aggregate multifunction* B^ε *possesses at least one* δ*-random fixed point;*

Then there exists at least one ε*-random Nash equilibrium.*

In metric space $\mathbb{X}$ we consider the max metric.

Proof. Let $\varepsilon > 0$. We will prove that for almost every $\omega \in \Omega$, there exists $y \in \mathbb{X}$, such that

$$u_i(\omega, y) \geq \sup_{t\in\mathbb{X}_i} u_i(\omega, (y_{-i}, t)) \text{ for every } i \in \{1, \ldots, n\}. \tag{4.1}$$

Because $u_i(\omega, \cdot) : \mathbb{X} \multimap \mathbb{R}$ is a composition of uniformly continous function and a projection function for i-th coordinate (what is obviously uniformly continuous function), then u_i is uniformly continuous. So there exists $\eta > 0$ such that, if only $x, y \in \mathbb{X}$ and $d(x, y) < \eta$, then $|u_i(\omega, x) - u_i(\omega, y)| < \frac{\varepsilon}{2}$ for each $i \in \{1, \ldots, n\}$.

By (2) there exists $x^* \in \mathrm{Fix}^\eta(B^{\frac{\varepsilon}{2}})$. Then, there exists $\overline{x} \in B^{\frac{\varepsilon}{2}}(x*)$ satisfying $d(x^*, \overline{x}) < \eta$, by the definition of $B^{\frac{\varepsilon}{2}}$ mapping we obtain:

$$d(x^*, (x^*_{-i}, \overline{x}_i)) < \eta. \tag{4.2}$$

Using uniform continuity of mapping u we obtain:

$$|u_i(\omega, x^*) - u_i(\omega, (x^*_{-1}, \overline{x}_i))| < \frac{\varepsilon}{2}. \tag{4.3}$$

Since $\overline{x} \in B^{\frac{\varepsilon}{2}}(x^*)$, then

$$u_i(\omega, x^*_{-i}, \overline{x}_i) + \frac{\varepsilon}{2} \geq \sup_{t\in\mathbb{X}_i} u_i(\omega, (x^*_{-i}, t_i)). \tag{4.4}$$

Because of (4.3) and (4.4), following inequalities holds:

$$\sup_{t\in\mathbb{X}_i} u_i(\omega, (x^*_{-i}, t_i)) - \varepsilon \leq u_i(\omega, (x^*_{-i}, \overline{x}_i)) - \frac{\varepsilon}{2} < u_i(\omega, x^*),$$

what implies that $x^* \in \mathrm{NE}^\varepsilon(\Gamma)$. □

Next theorem give us a sufficient condition of ε-random Nash equilibrium existence. For details, see e.g. [9], Chapter 3, Section 6.

Theorem 23. *Let* $\Gamma = (N, \Omega, \mathbb{X}, u)$ *be a n-person game that satisfies following conditions: the set of players* $N = \{1, 2, \ldots, n\}$*,* $\mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_n$ *and* $\mathbb{X}_i$ *is an absolute approximative retract for every* $i \in N$*. Additionally we assume that for every* $\varepsilon > 0$ *the* ε*-best response multifunction* $B^\varepsilon : \Omega \times \mathbb{X} \multimap \mathbb{X}$ *is a random operator such that for almost every* $\omega \in \Omega$ *the mapping* $B^\varepsilon(\omega, \cdot) : \mathbb{X} \multimap \mathbb{X}$ *has* R_δ *values then the game* Γ *has an* ε*-random Nash equilibrium for every* $\varepsilon > 0$.

Proof. Proof is similar to the proof of Theorem 16. It is enough to apply Theorem 14 and Theorem 13 to obtain thesis. □

5. Final remarks and comments

Random games can be applied, among other thing, in the following ways:

– in the environment in which Bayesian equilibria are considered(see [10]), i.e. in which payoffs of players are known up to dependence on assignment of the players to specific types; Ω stands for the type space;

– directly to correlated equilibria (see [2]) — Ω stands for the correlating device;

– belief distorted Nash equlibria with commonly observed histories on which the beliefs are based; Ω stands for set of histories [12].

So, the results of this paper, may turn out to be useful to prove the existence of concepts of ε-correlated equilibria and ε-belief distorted Nash equilibria. Definition of such new concepts is, however, out of the scope of this paper, so we leave this for further research.

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