Bilge International Journal of Science and Technology Research

ISSN: 2651-401X
e-ISSN: 2651-4028
2020, Volume: 4, Issue: 2, 55-62
Received: 29.03.2020; Accepted: 21.09.2020

Exponentiated Weibull-Logistic Distribution

Uğur Murat [1*], Gamze Özel Kadılar [1]

Abstract: In this paper, exponentiated Weibull-logistic distribution is introduced. The main functions of proposed distribution are derived and plotted for different parameter values. Besides, skewness and kurtosis measures of proposed distribution are presented. Then, by finding moment generating function, expected value and variance are derived. A simulation study is given for showing performance of exponentiated Weibull-logistic distribution by the maximum likelihood estimation approach. Finally, applications based on real datasets are presented and proved that, exponentiated Weibull-logistic distribution is better than existing distributions in literature.

Keywords: Weibull-logistic distribution, Weibull-G family, Maximum likelihood estimation, Hazard function.

[1]**Address:** Hacettepe University, Ankara, Turkey
***Corresponding author**: umurat@hacettepe.edu.tr
Citation: Murat, U., Özel Kadılar, G. (2020). Exponentiated Weibull-Logistic Distribution. Bilge International Journal of Science and Technology Research, 4 (2): 55-62.

1. INTRODUCTION

Many distributions are used to model life times and Weibull and Logistic distributions are some of them. The logistic function was proposed by Verhulst (Johnson et al. 1995) to define the growth curve in the 19th century and then has been applied to many areas over time. If the random variable X has Type-I Logistic (Lg) distribution its cumulative distribution function (cdf) is

$$G(x) = \frac{1}{1 + e^{-\lambda x}}, \qquad \lambda > 0.$$

Correspondingly, the probability density function (pdf) of Type-I Logistic distribution is given by

$$g(x) = \frac{\lambda e^{\lambda x}}{(1 + e^{\lambda x})^2}, \qquad \lambda > 0.$$

The Weibull distribution is used extensively to model lifetime. It was first introduced by Weibull (1951) and has been used in many fields, especially for lifetime data analysis and statistical models in engineering. Depending on the values of its shape parameter, it also models Rayleigh and exponential distributions in some cases, is also widely used in datasets regarding failure rates.

If X has two parameters Weibull (W) distribution, its cdf is given by

$$F(x) = 1 - e^{-\alpha x^{\beta}}, \qquad \alpha, \beta > 0.$$

Then, density function for Weibull distribution (for two parameters) is given by

$$f(x) = \alpha\beta\, x^{\beta-1}\, e^{-\alpha x^{\beta}}, \qquad \alpha, \beta > 0.$$

Although the classical distributions are widely used, these distributions are insufficient in some application areas and extended forms of these distributions are needed. Hence, various studies (Gurvich et al. 1997; Cordeiro et al. 2015) have been carried out to expand the Weibull distribution and Weibull-type families were introduced in some of these studies. Examples of these distribution families are the Weibull-G family (Bourguignon et al. 2014; Tahir et al. 2016a; Alizadeh et al. 2018), Weibull-H family (Cordeiro et al. 2017) and Weibull-X family (Ahmad et al. 2018).

In the Weibull-G family, G(x) / [1 – G(x)] conversion has been performed by (Hassan & Elgarhy, 2016; Tahir et al.

2016b; Nassar et al. 2017; Korkmaz, 2019). The cdf for X with the Weibull-G family distribution is obtained as

$$F(x) = 1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}, \qquad \alpha,\beta > 0.$$

When the Logistic distribution is used as the G(x) function, the cdf of the Weibull-Logistic (W-Lg) distribution becomes

$$F(x) = 1 - e^{-\alpha\left(\frac{\frac{1}{1+e^{-\lambda x}}}{1-\frac{1}{1+e^{-\lambda x}}}\right)^{\beta}},$$

$$\alpha,\beta,\lambda > 0.$$

Its simplification is as follows:

$$F(x) = 1 - e^{-\alpha\, e^{\lambda\beta x}}, \qquad \alpha,\beta,\lambda > 0.$$

Density function of Weibull-Logistic (W-Lg) distribution is

$$f(x) = \lambda\alpha\beta\, e^{\lambda\beta x-\alpha e^{\lambda\beta x}}, \qquad \alpha,\beta,\lambda > 0.$$

Some studies have been made to generalize the Weibull family distributions by adding a new shape parameter ($\theta > 0$) (Elgarhy et al. 2017; Hassan & Elgarhy, 2018; Korkmaz, 2018). The exponentiated Weibull-logistic (ExpW-Lg) distribution proposed in this study is the generalized version of the Weibull-Logistic (W-Lg) distribution with parameter θ. A new cdf is obtained by taking θ exponent of the relevant cdf, i.e. $F(x) = G(x)^{\theta}$. The new parameter added defines skewness, kurtosis and tail values. Here, G(x) is base-line distribution and F(x) is cumulative density for exponentiated-G distribution (Bursa & Özel, 2017).

The cdf for exponentiated distribution is

$$F(x) = G(x)^{\theta}, \qquad \theta > 0.$$

The pdf of the exponentiated distribution is

$$f(x) = \theta\, G(x)^{\theta-1} g(x), \qquad \theta > 0.$$

Here, g(x) is the pdf of G(x) function.

In Section 2, main functions for proposed distribution are given and graphs for the density and hazard functions are presented comparatively. The quantile function for proposed distribution, skewness and kurtosis values are obtained for different parameter values. In Section 3, moment generating function for novel distribution is derived. Then, mean and variance are calculated in this context. A simulation study is done using maximum likelihood estimation is shown in Section 4. Applications on real datasets are done in Section 5. Conclusions of the study are discussed in Section 6.

2. EXPONENTIATED WEIBULL-LOGISTIC DISTRIBUTION

Cumulative density of exponentiated Weibull-Logistic (ExpW-Lg) distribution is obtained as

$$F(x) = \left(1 - e^{-\alpha\, e^{\lambda\beta x}}\right)^{\theta}, \qquad \alpha,\beta,\lambda,\theta > 0.$$

Density function for ExpW-Lg distribution is derived by

$$f(x) = \theta\left(1 - e^{-\alpha\, e^{\lambda\beta x}}\right)^{\theta-1}\left(\lambda\alpha\beta\, e^{\lambda\beta x-\alpha e^{\lambda\beta x}}\right),$$

$$\alpha,\beta,\lambda,\theta > 0.$$

The graphs for ExpW-Lg distribution for 0.5, 2, 4 and 10 values of θ are presented in Figure 1.

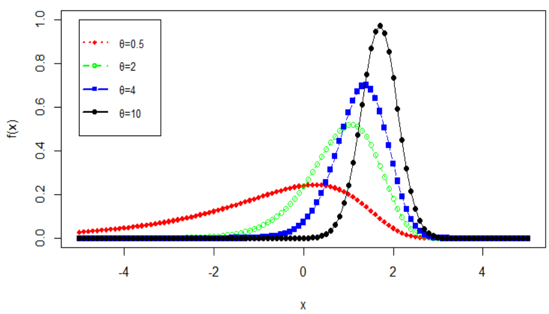

Figure 1. The graphs for density function of the ExpW-Lg distribution with parameters (α=0.5; β=0.5; λ=2; θ=0.5, 2, 4, 10).

As seen in Figure 1, with increasing θ value, the density function gets positive values and becomes narrow and variability decreases.

Density graphs for ExpW-Lg distribution for 0.5, 2, 4 and 10 values of α are shown in Figure 2.

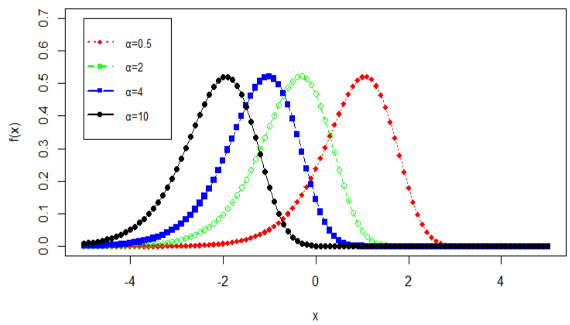

Figure 2. Density graphs for ExpW-Lg distribution with parameters (α=0.5, 2, 4, 10; β=0.5; λ=2; θ=2).

Figure 2 shows that although the shape of the plots of the pdf remains the same with the increase of α, they shift towards the x axis in the negative direction.

Graphs of ExpW-Lg distribution for 0.5, 2, 4 and 10 values of β are given in Figure 3.

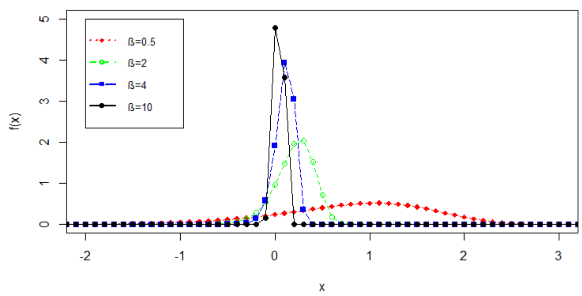

Figure 3. Graphs of ExpW-Lg distribution with parameters (α=0.5; β=0.5, 2, 4, 10; λ=2; θ=2).

Figure 3 presents that as β value increases, plots of the pdf become narrow and the variability decreases. Density graphs for ExpW-Lg distribution for 0.5, 2, 4 and 10 values of λ are given in Figure 4.

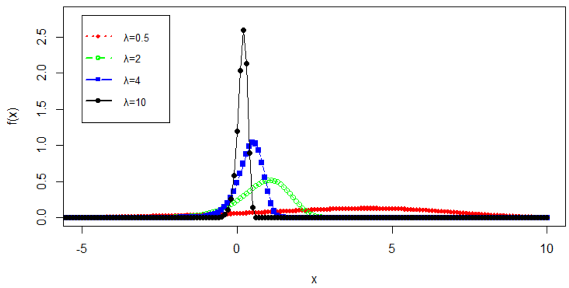

Figure 4. Density graphs for ExpW-Lg distribution for selected values of the model parameters (α=0.5; β=0.5; λ=0.5, 2, 4, 10; θ=2).

Figure 4 shows that if λ increases, plots of the pdf become narrow and the variability decreases. Now, the pdf plots of the ExpW-Lg distribution are given in Figures 5-6.

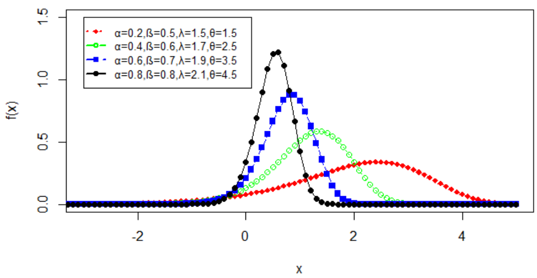

Figure 5. Density graphs for ExpW-Lg distribution for selected values of the model parameters.

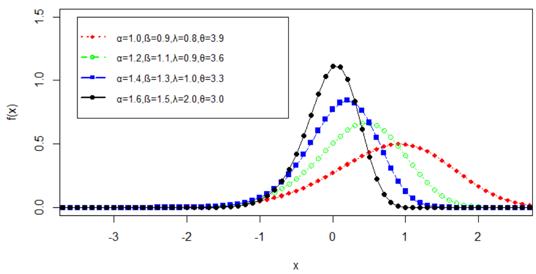

Figure 6. Density graphs of ExpW-Lg distribution.

2.1 Survival and Hazard Functions

Survival function (srf) of the ExpW-Lg distribution is

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \left(1 - e^{-\alpha\, e^{\lambda\beta x}}\right)^{\theta},$$

$$\alpha, \beta, \lambda, \theta > 0.$$

Then, the hazard function (hrf) for ExpW-Lg distribution is obtained as

$$h(x) = \frac{\theta\lambda\alpha\beta\left(1 - e^{-\alpha\, e^{\lambda\beta x}}\right)^{\theta-1}\left(e^{\lambda\beta x - \alpha e^{\lambda\beta x}}\right)}{1 - \left(1 - e^{-\alpha\, e^{\lambda\beta x}}\right)^{\theta}}$$

$$\alpha, \beta, \lambda, \theta > 0.$$

Hazard function graphs for ExpW-Lg distribution for 0.5, 2, 4 and 10 values of θ are given in Figure 7.

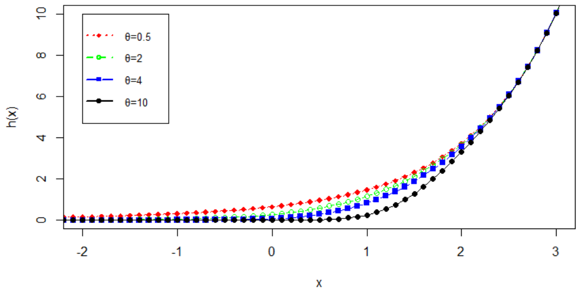

Figure 7. Hazard function graphs for ExpW-Lg distribution with parameters (α=0.5; β=0.5; λ=2; θ=0.5, 2, 4, 10).

Hazard function graphs for ExpW-Lg distribution for 0.5, 2, 4 and 10 values of α are depicted in Figure 8.

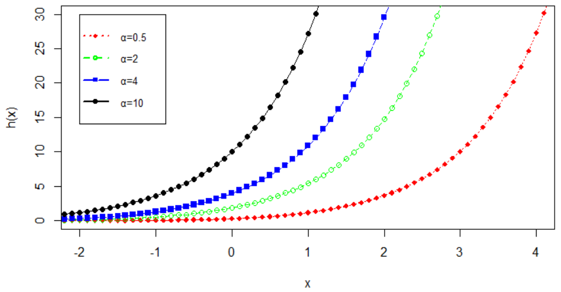

Figure 8. Hazard function graphs for ExpW-Lg distribution for selected values of the model parameters (α=0.5, 2, 4, 10; β=0.5; λ=2; θ=2).

Figure 9 shows hazard function graphs for ExpW-Lg distribution when the values of β are 0.5, 2, 4.

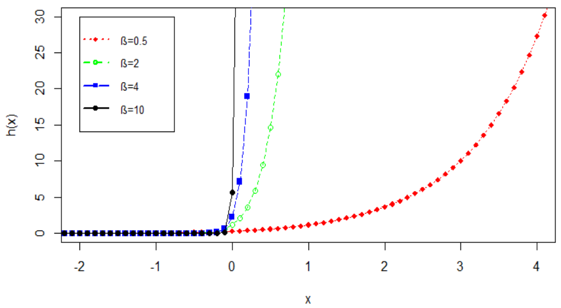

Figure 9. Hazard function graphs for ExpW-Lg distribution with parameters (α=0.5; β=0.5, 2, 4, 10; λ=2; θ=2).

Hazard function graphs for ExpW-Lg distribution for 0.5, 2, 4 and 10 values of λ are 0.5, 2, 4, 10 in Figure 10.

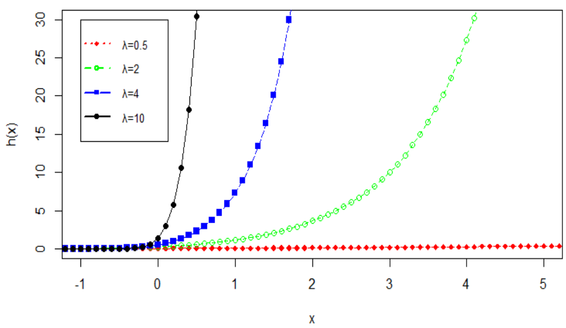

Figure 10. Hazard function graphs for ExpW-Lg distribution with parameters (α=0.5; β=0.5; λ=0.5, 2, 4, 10; θ=2).

Hazard function graphs for ExpW-Lg distribution are presented in Figures 11-12.

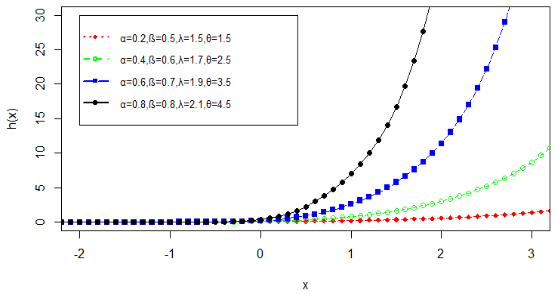

Figure 11. Hazard function graphs for ExpW-Lg distribution.

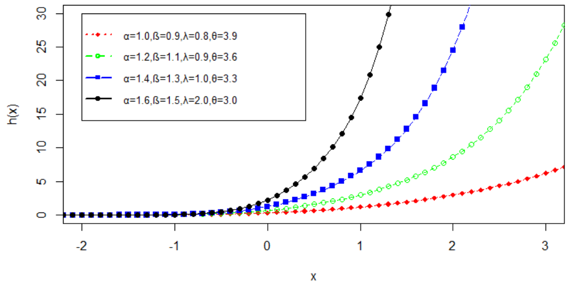

Figure 12. Hazard function graphs for ExpW-Lg distribution.

2.2 Quantile Function

p[th] quantile for ExpW-Lg distribution is found by inverting the F(p) function and is given by

$$Q(p) = F^{-1}(p) = x = \frac{\ln\left(-\frac{\ln\left(1-p^{\frac{1}{\theta}}\right)}{\alpha}\right)}{\lambda\beta},$$

$$0 \le p \le 1$$

The median of the ExpW-Lg distribution is obtained as

$$Medyan(X) = Q\left(\frac{1}{2}\right) = \frac{ln(-\frac{ln(1-p^{\frac{1}{\theta}})}{\alpha})}{\lambda\beta}$$

Skewness and kurtosis were computed for some parameters to present performance of parameters with ExpW-Lg distribution on skewness and kurtosis. For this aim, Bowley's formula for skewness (S) and Moor's formula for kurtosis (K) were used.

When S = 0, the distribution is symmetrical, when S > 0 is right-skewed and when S < 0 is left-skewed. The calculated skewness and kurtosis for different parameter for ExpW-Lg distribution are given in Table 1.

Table 1. Skewness and kurtosis of the ExpW-Lg distribution

θ	α	β	λ = 2		λ = 4	
			Skewness	Kurtosis	Skewness	Kurtosis
0.5	0.5	0.5	-0.1721597	-0.4664396	-0.1721597	-0.4664396
		2	-0.1721597	-0.4664396	-0.1721597	-0.4664396
	2	0.5	-0.1721597	-0.4664396	-0.1721597	-0.4664396
		2	-0.1721597	-0.4664396	-0.1721597	-0.4664396
2	0.5	0.5	-0.07421204	-0.2017165	-0.07421204	-0.2017165
		2	-0.07421204	-0.2017165	-0.07421204	-0.2017165
	2	0.5	-0.07421204	-0.2017165	-0.07421204	-0.2017165
		2	-0.07421204	-0.2017165	-0.07421204	-0.2017165
4	0.5	0.5	-0.03985193	-0.1086256	-0.03985193	-0.1086256
		2	-0.03985193	-0.1086256	-0.03985193	-0.1086256
	2	0.5	-0.03985193	-0.1086256	-0.03985193	-0.1086256
		2	-0.03985193	-0.1086256	-0.03985193	-0.1086256
10	0.5	0.5	-0.00619869	-0.01819422	-0.00619869	-0.01819422
		2	-0.00619869	-0.01819422	-0.00619869	-0.01819422
	2	0.5	-0.00619869	-0.01819422	-0.00619869	-0.01819422
		2	-0.00619869	-0.01819422	-0.00619869	-0.01819422

Table 1 proves that ExpW-Lg distribution is skewed to left because all skewness values are less than zero (S < 0). However, although

parameters α, β and λ change results, it is observed that they don't affect the skewness and kurtosis values computed from the Bowley and Moor formulas. The skewness and kurtosis values for ExpW-Lg distribution change only for θ. It is observed that the value of the skewness and kurtosis decrease while θ increases.

3. MOMENT GENERATING FUNCTION

Moment generating function for ExpW-Lg distribution is obtained as follows:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \theta\lambda\alpha\beta\,(1 - e^{-\alpha\, e^{\lambda\beta}})^{\theta-1}\left(e^{t+\lambda\beta-\alpha e^{\lambda\beta}}\right).$$

The variance and mean of X random variable are calculated. After the first and second derivatives are found, the mean of X random variable is found by the value t = 0 of the first derivative of the moment generating function as follows:

The value of $E(X^2)$ used in the variance calculation is found with the t = 0 value of the second derivative for moment generating function.

$$E(X^2) = \theta\lambda\alpha\beta\,(1 - e^{-\alpha\, e^{\lambda\beta}})^{\theta-1}\,(e^{\lambda\beta-\alpha e^{\lambda\beta}})$$

The mean, variance and standard deviation are, respectively, obtained for α = 0.5, β = 0.5, λ = 2, θ = 2 as follows:

$E(x) = \mu = 0.5189$

$Var(X) = \sigma^2 = 0.5189 - 0.2693 = 0.2496$

$\sigma = \sqrt{0.2496} = 0.4996$

4. SIMULATION STUDY

In the study, the parameters (α, β, λ, θ) were estimated by maximum likelihood estimation (MLE) method after these random numbers reaching the determined sample size and this procedure was repeated 1000 times. The mean and mean squared error (MSE) values of the estimated parameters were calculated at the end of the simulation. Within the scope of the study, parameter values were taken as α=0.1, β=1, λ=1, θ=2, and calculations were made for n

= 10, 50 and 200 sample sizes. Simulation results are given in Table 2.

.

Table 1. Simulation results

n	Mean				Mean Squared Error (MSE)			
	α (0.10)	β (1.00)	λ (1.00)	θ (2.00)	α	β	λ	θ
10	0.122	1.113	0.978	2.192	0.0100	0.0696	0.0817	0.3814
50	0.127	1.077	0.960	2.190	0.0067	0.0708	0.0756	0.3651
200	0.129	1.075	0.947	2.198	0.0065	0.0713	0.0704	0.3346

Table 2 shows that the sample size had little effect on variability in parameter estimates.

5. APPLICATION

ExpW-Lg distribution is applied on two datasets, the parameter estimations are done by MLE method, and the descriptive statistics are obtained. Parameter estimations were done for the W-Lg and ExpW-Lg distributions by MLE method. AIC, CAIC, BIC, Anderson Darling (A), Cramér-von Misses (CvM) statistics obtained.

As the first dataset, an experimental data of the strength values of 1.5 cm of glass fibre was used in the study of Smith and Naylor (1987) were obtained in the National Physical Laboratory in England. The dataset consisting of 63 observations is given below:

Dataset-1: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

The main statistics of the first dataset are given in Table 3.

Table 2. The main statistics of the first dataset

Mean	Median	Variance	Min.	Max.
1.507	1.590	0.105	0.55	2.24

Parameter estimations and goodness of fit statistics for first dataset are presented in Table 4.

.

Table 3. Estimated parameters and goodness of fit statistics for first dataset

Model	MLE				AIC	CAIC	BIC	A	CvM
	α	β	λ	θ					
W-Lg	0.015	1.284	2.118	-	47.335	47.742	53.764	0.859	0.151
ExpW-Lg	0.066	0.775	2.726	3.670	38.570	39.260	47.143	1.170	0.212

The histogram and empirical curves of the W-Lg and ExpW-Lg distributions for the first dataset are given in Figure 13. Table 4 and Figure 13 show that the ExpW-Lg distribution is better than the W-Lg distribution for first dataset.

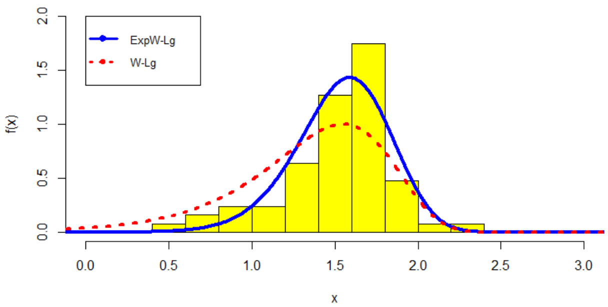

Figure 13. Histogram of the dataset and compliance of the distributions to the first data.

Then, graphs for ExpW-Lg distribution for the first dataset are given in Figure 14.

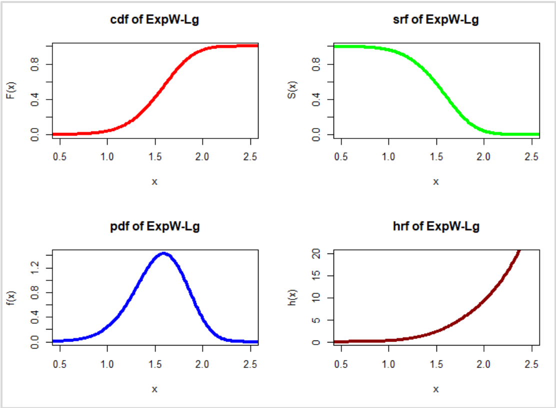

Figure 14. Graphs for ExpW-Lg distribution of first dataset.

As second application, the fracture toughness data of Alumina (Al_2O_3) material is used in the study of Nadarajah and Kotz (2008). The dataset consisting of 119 observations is given below:

Dataset-2: 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

The main statistics of the second dataset are presented in Table 5.

Table 4. The main statistics of the second dataset

n	Mean	Median	Variance	Min.	Max.
119	4.325	4.380	1.037	1.68	6.81

Parameter estimations and the goodness of fit statistics of the second dataset are given in Table 6.

Table 5. Estimated parameters for second dataset

Model	MLE				AIC	CAIC	BIC	A	CvM
	α	β	λ	θ					
W-Lg	0.008	1.273	0.788	-	351.940	352.149	360.278	0.610	0.100
ExpW-Lg	0.134	1.196	0.494	3.703	345.689	346.040	356.806	0.430	0.067

The histogram and empirical curves of the W-Lg and ExpW-Lg distributions for the second dataset are given in Figure 15. Table 6 and Figure 15 show that ExpW-Lg distribution fits more to the second dataset.

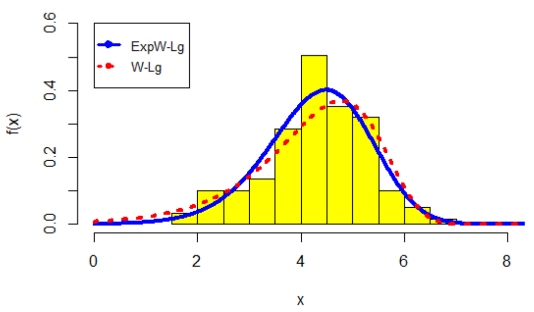

Figure 15. Histogram of the dataset and compliance of the distributions to the second data.

Finally, cdf, srf, pdf and hrf plots of the ExpW-Lg distribution of the second dataset are given in Figure 16.

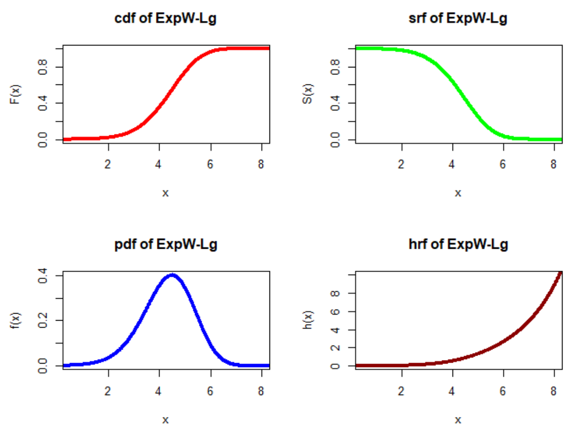

Figure 16. Graphs for cdf, srf, density function and hrf for ExpW-Lg distribution for second dataset.

4. DISCUSSION AND CONCLUSIONS

In the study, Exponentiated Weibull-Logistic (ExpW-Lg) distribution is introduced and the statistical properties related to the distribution are given by obtaining the cumulative distribution, probability density, survival and hazard functions of the proposed distribution. After deriving quantile function, skewness and kurtosis, the results show that the distribution is skewed to the left, based on the negative values of skewness. Then, the parameters of the ExpW-Lg distribution are estimated by maximum likelihood estimation method and the related statistics are given in a simulation study. Two applications of the ExpW-Lg to real datasets shows that the new distribution fits well the datasets more than the W-Lg distribution. It is considered that the proposed distribution can be used in various applications and in different datasets.

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