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Katugampola fractional differential equation with Erdelyi-Kober integral boundary conditions

Naas Adjimi[a], Maamar Benbachir[b]

[a] *Laboratoire de Mathématiques et Sciences appliquées, University of Ghardaia, Algeria.*
[b] *Faculty of Sciences, Saad Dahlab University, Blida, Algeria.*

Abstract

This paper investigates the following Katugampola fractional differential equation with Erdelyi-Kober fractional integral boundary conditions:

$$\begin{cases} D^{\rho,\alpha}u(t) + h(t, u(t)) = 0, & 0 < t < T, \\ u(0) = 0, & \\ u'(T) = \lambda I_\eta^{\gamma,\delta} u'(\xi), & 0 < \xi < T, \end{cases}$$

where $D^{\rho,\alpha}$ is the Katugampola derivative of order $1 < \alpha < 2$, $\rho > 0$ and $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_\eta^{\gamma,\delta}$ denotes Erdelyi-Kober fractional integral of order $\delta > 0$, $\eta > 0$, λ, $\gamma \in \mathbb{R}$. Some new existence and uniqueness results are obtained using nonlinear's contraction principal and Krasnoselskii's and Leray-Schauder's fixed point theorems. Four examples are given in the last section to illustrate the obtained results.

Keywords: Katugampola fractional derivative; Erdelyi-Kober fractional integral; Boundary value problem; fixed point theorem.
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1. Introduction

2. Introduction

Fractional calculus is one of the most emerging areas and has attracted the attention of many researchers over the last few decades [2, 3, 6, 12, 13, 15, 16, 20, 17, 14]. Many researchers have been interested both in theoretical aspects and in various applications: physics, chemistry, engineering, biology, etc.

Email addresses: `naasadjimi@gmail.com` (Naas Adjimi), `mbenbachir2001@gmail.com` (Maamar Benbachir)

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Classical fractional order boundary conditions involve Riemman-Liouville or Katugampola type integral boundary conditions, which use Erdelyi-Kober fractional integral operators. Introduced by Arther Erdelyi and Herman Kober in 1940 [19], they play an important role in solving some problems in signal processing, dual and triple integral equations, and special function in mathematical physics [1, 7, 9, 11, 5, 18, 21].

In [1], the authors investigated the existence of solution of Caputo type fractional differential equation with nonlocal Riemann-Liouville and Erdelyi-Kober type integral boundary conditions, of the form

$$\begin{cases} {}^{C}D^{q}u(t) = f(t,u(t)), & 0 < t < T, \\ u(0) = \alpha I_{\eta}^{\gamma,\delta}u(\xi), & 0 < \xi < T, \\ u(T) = \beta J_{\eta}^{\psi}u(\zeta), & 0 < \zeta < T, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo derivative of order $1 < \alpha \leq 2$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_{\eta}^{\gamma,\delta}$ denotes Erdelyi-Kober fractional integral of order $\delta > 0, \eta > 0, \gamma \in \mathbb{R}$.

In [10], the authors investigated the existence results for Katugampola fractional differential equations via a measure of noncompactness:

$$\begin{cases} D^{\rho,\omega}u(t) = h(t,u(t)), & 0 < t < T, \\ u(0) = u_0, u'(0) = u_1, \end{cases}$$

where $D^{\rho,\omega}$ is Katugampola fractional derivative of order $1 < \rho < 2$, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\omega > 0$.

In [13], the authors established the fractional order boundary value problems with Katugampola fractional integral conditions as follow:

$$\begin{cases} {}^{C}D^{q}u(t) + f(t,u(t)) = 0, & 0 < t < T, \\ u(T) = \beta I^{\rho,q}(\epsilon), & 0 < \epsilon < T, \\ u'(T) = \gamma I^{\rho,q}u'(\eta), & 0 < \eta < T, \\ u''(T) = \delta I^{\rho,q}u''(\zeta), & 0 < \zeta < T, \end{cases}$$

where ${}^{\rho}I^{q}$ is the Katugampola integral, $q > 0, \rho > 0$, and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [19], the authors established the existence of solution for following nonlinear Riemann-Liouville fractional differential equation subject to nonlocal Erdelyi-Kober fractional integral conditions:

$$\begin{cases} D^{q}u(t) + f(t,u(t)) = 0 & 0 < t < T, \\ u(0) = 0, \\ \alpha u(T) = \Sigma_{i=1}^{m}\beta_i I_{\eta_i}^{\gamma_i,\delta_i}u(\xi_i), & 0 < \xi < T, \end{cases}$$

where D^{q} is the standard Riemann-Liouville fractional derivative of order $1 < q \leq 2$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_{\eta_i}^{\gamma_i,\delta_i}$ denote Erdelyi-Kober fractional integral of order $\delta_i > 0, \eta_i > 0, \gamma_i \in \mathbb{R}$ and $\alpha, \beta_i \in \mathbb{R}, \xi \in (0,T)$, $i = 1, 2, ...m$ are given constants.

In [4] the authors studied the existence and uniqueness of solution for a class nonlinear implicit fractional differential equation via Katugampola fractional derivative:

$$\begin{cases} D_{0^+}^{\rho,\alpha}u(t) = f(t,u(t),D_{0^+}^{\rho,\alpha}u(t)), & 0 < t < T, \\ u(0) = 0, \end{cases}$$

where $0 < \alpha \leq 1, \rho > 0$ and $T \leq (pc)\frac{1}{pc}$ for any $1 < p \leq \infty, c > 0$ is finite positive constant, p is the order of Lebesgue integral defined on a suitable space, the symbol $D_{0^+}^{\rho,\alpha}$ presents the Katugampola fractional derivative operator and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Motivated by the above papers, we investigate the existence of solution for the following Katugampola fractional differential equipped with Erdelyi-Kober fractional integral boundary conditions:

$$\begin{cases} D^{\rho,\alpha}u(t) + h(t,u(t)) = 0, & 0 < t < T, \\ u(0) = 0, \\ u'(T) = \lambda I_{\eta}^{\gamma,\delta}u'(\xi), & 0 < \xi < T, \end{cases} \tag{1}$$

where $1 < \alpha < 2$, $\rho > 0$, $\delta > 0$, $\eta > 0$, λ, $\gamma \in \mathbb{R}$, and $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The paper is organized as the follows. In section 2, we present some preliminaries and lemmas. Sections 3 and 4 are devoted to the existence and uniqueness solution of (1), which are obtained via contraction mapping principle, Krasnoselskii's fixed point theorem, and Leray-Schauder nonlinear alternative. In section 5, four illustrative examples are detailed.

3. Preliminaries

We recall here some basic definitions of fractional integral calculus and some auxiliary lemmas which will be needed later.

Consider the space $X_c^p[0, T], (c \in \mathbb{R}, 1 \leq p \leq \infty)$ of complex-valued Lebesgue measurable functions h on $[0, T]$ for which $\|h\|_{X_c^p} < \infty$, where the norm is defined by

$$\|h\|_{X_c^p} = \left(\int_0^T |s^c h(s)|^p \frac{ds}{s}) \right)^{\frac{1}{P}} < \infty,$$

for $c \in \mathbb{R}, 1 \leq p \leq \infty$. For $p = \infty$, we have

$$\|h\|_{X_c^\infty} = ess \sup_{0 \leq t \leq T} |s^c h(s)| .$$

By $X = C[0, T]$ we denote the Banach space of all continuous functions $x : [0, T] \to \mathbb{R}$ with the norm

$$\|x\|_C = \sup_{t \in [0,T]} |x(t)|.$$

Remark 3.1. *[4] Let $p, c, T \in \mathbb{R}_+^\star$ be such that $p \geq 1$, $c > 0$ and $T \leq (pc)^{\frac{1}{pc}}$. One can easily see that $\forall h \in C[0, T]$*

$$\|h\|_{X_c^p} = \left(\int_0^T |s^c h(s)|^p \frac{ds}{s} \right)^{\frac{1}{P}} \leq \left(\|h\|_c^p \int_0^T s^{pc-1} ds \right)^{\frac{1}{P}} = \frac{T^C}{(pc)^{\frac{1}{P}}} \|h\|_C,$$

and if $p = \infty$

$$\|h\|_{X_c^p} = ess \sup_{0 \leq t \leq T} (t^c |h(t)|) \leq T^c \|h\|_C,$$

which implies that $C[0, T] \hookrightarrow X_c^p[0, T]$, and $\|h\|_{X_c^p} \leq \|h\|_C$ for all $T \leq (pc)\frac{1}{pc}$.

Definition 3.1. *[13] The Riemann-Liouville fractional integral of order $\alpha > 0$ of continuous function $h : (0, \infty) \to \mathbb{R}$ is defined by*

$$J^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds,$$

where $\Gamma(\cdot)$ is the gamma function, defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha - 1} e^{-t} dt.$$

Definition 3.2. *[1] The Riemann-Liouville fractional derivative of order $\alpha > 0, n - 1 < \alpha < n$, $n \in \mathbb{N}$ of a continuous function $h : (0, \infty) \to \mathbb{R}$ is defined by*

$$D_{0^+}^\alpha h(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t - s)^{n - \alpha - 1} h(s) ds,$$

where the function h has absolutely continuous derivative up to order $(n - 1)$.

Definition 3.3. *[4] The left-sided Hadamard fractional derivative of order $\alpha > 0$ of a continuous function $h : (0,\infty) \to \mathbb{R}$ is defined by*

$$^{H}D^{\alpha}_{0^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_0^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} h(s)\frac{ds}{s}, n = [\alpha]+1.$$

Definition 3.4. *[4] The left-sided Hadamard fractional integral of order $\alpha > 0$ of a continuous function $h : [0,T] \to \mathbb{R}$ is defined by*

$$^{H}J^{\alpha}_{0^+}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\log\frac{t}{s}\right)^{\alpha-1} h(s)\frac{ds}{s}.$$

Definition 3.5. *[13] The Katugampola fractional integral of order $\alpha > 0$ and $\rho > 0$ of a function $h(t)$ for all $0 < t < \infty$, is defined by*

$$J^{\rho,\alpha}_{0^+}h(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}h(s)ds, \quad t \in [0,T],$$

for $\rho > 0$. This integral is called left-sided integral.

Lemma 3.1. *[13] Let be the constants $\rho, q > 0$ and $p > 0$. Then the following formula holds:*

$$J^{\rho,q}t^p = \frac{\Gamma(\frac{p+\rho}{\rho})}{\Gamma(\frac{p+\rho q+\rho}{\rho})}\frac{t^{p+\rho q}}{\rho^q}.$$

Remark 3.2. *[13] The above definition (3.5) of Katugampola fractional integral corresponds to the Riemann-Liouville fractional integral of order $\alpha > 0$, when $\rho = 1$, while the famous Hadamard fractional integral follows for $\rho \to 0$; that is:*

$$\lim_{\rho\to 0} J^{\rho,\alpha}_{0^+}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds.$$

Definition 3.6. *[4] The generalized fractional derivative of order $\alpha > 0$ corresponding to the Katugampola fractional integral is defined for any $0 < t < \infty$ by:*

$$\begin{aligned} D^{\rho,\alpha}_{0^+}h(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \left(J^{\rho,n-\alpha}_{0^+}h\right)(t), \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}}h(s)ds, \quad t \in [0,T], \end{aligned}$$

where $n = [\alpha]+1$ and $\rho > 0$ (when the integral exists).

Remark 3.3. *[4] As a basic example, we quote for $\alpha, \rho > 0$ and $\mu > -\rho$*

$$D^{\rho,\alpha}_{0^+}t^\mu = \frac{\rho^{\alpha-1}\Gamma(1+\frac{\mu}{\rho})}{\Gamma(1-\alpha+\frac{\mu}{\rho})}t^{\mu-\alpha\rho}.$$

Definition 3.7. *[1] The Erdelyi-Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$, of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by:*

$$I^{\gamma,\delta}_{\eta}h(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^t \frac{s^{\eta\gamma+\eta-1}}{(t^\eta - s^\eta)^{1-\delta}}h(s)ds,$$

provided that the right side is pointwise defined on $\mathbb{R}^+$.

Remark 3.4. *[1] For $\eta = 1$ the above operator is reduced to the Kober operator*

$$I_1^{\gamma,\delta} h(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma}{(t-s)^{1-\delta}} h(s) ds.$$

That was introduced for the first time by Kober. For $\gamma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight

$$I_1^{0,\delta} h(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{h(s)}{(t-s)^{1-\delta}} ds, \delta > 0.$$

Lemma 3.2. *[1] Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (\frac{q}{\eta}) + 1)}{\Gamma(\gamma + (\frac{q}{\eta}) + \delta + 1)}.$$

Theorem 3.1. *The operator $J_{a^+}^{\rho,\alpha}$ is linear and bounded from $C([a,b])$ to $C([a,b])$, then*

$$\|J_{a^+}^{\rho,\alpha} x\|_C \leq K_{\alpha,\rho} \|x\|_C,$$

with $K_{\alpha,\rho} = \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} (b^\rho - a^\rho)^\alpha$.

Proof. For any $x \in C[0,T]$; one has

$$\begin{aligned} \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} x(s) ds \right| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \|x\|_C \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} ds, \\ &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} (b^\rho - a^\rho)^\alpha \|x\|_C. \end{aligned}$$

□

4. Main Results

Lemma 4.1. *[10] Let $\alpha, \rho > 0$, if $u \in C[0,T]$, then we have the following properties.*

(i) *The fractional differential equation $D_{0^+}^{\rho,\alpha} u(t) = 0$ admits a solution defined by:*

$$u(t) = c_0 + c_1 t^\rho + c_2 t^{2\rho} + ... + c_n t^{(n-1)\rho},$$

where $c_i \in \mathbb{R}$, with $i = 0, 1, 2, ..., n, n = [\alpha] + 1$.

(ii) *Let $\alpha > 0$, then*

$$J^{\rho,\alpha} D_{0^+}^{\rho,\alpha} u(t) = u(t) + c_0 + c_1 t^\rho + c_2 t^{2\rho} + ... + c_n t^{(n-1)\rho},$$

where $c_i \in \mathbb{R}$ and $n = [\alpha] + 1$.

Lemma 4.2. *Let $1 < \alpha < 2$ and $\lambda \in \mathbb{R}$. A function $u \in C([0,T], \mathbb{R})$ is a solution of nonlinear Katugampola fractional integral equation*

$$u(t) = \frac{t^\rho}{A} J^{\rho,\alpha-1} g(T) - \frac{t^\rho \lambda}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1} g(\xi) - J^{\rho,\alpha} g(t), \tag{2}$$

if and only if u is a solution of the katugampola fractional differential equation with Erdelyi-Kober fractional integral conditions

$$\begin{cases} D^{\rho,\alpha} u(t) + g(t) = 0, & 0 < t < T, \\ u(0) = u(0) = 0, \\ u'(T) = \lambda I_\eta^{\gamma,\delta} u'(\xi), & 0 < \xi < T. \end{cases} \tag{3}$$

Proof. Applying Lemma (4.1) to equation (3), we obtain

$$u(t) = -c_0 - c_1 t^{\rho} - J^{\rho,\alpha} g(t), \tag{4}$$

with $c_0, c_1 \in \mathbb{R}$. The condition $u(0) = 0$ implies that $c_0 = 0$.
Thus

$$u'(t) = -\rho c_1 t^{\rho-1} - J^{\rho,\alpha-1} g(t). \tag{3.4}$$

Combining the Erdelyi-Kober fractional integral with (3.4), we get

$$\lambda I_{\eta}^{\gamma,\delta} J^{\rho,\alpha-1} u'(\xi) = -\rho c_1 \lambda \xi^{\rho-1} \frac{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} - \lambda I_{\eta}^{\gamma,\delta} J^{\rho,\alpha-1} g(\xi).$$

$$\begin{aligned} u'(T) &= -\rho c_1 T^{\rho-1} - J^{\rho,\alpha-1} g(T), \\ &= -\rho c_1 \lambda \xi^{\rho-1} \frac{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} - \lambda I_{\eta}^{\gamma,\delta} J^{\rho,\alpha-1} g(\xi). \end{aligned}$$

Solving the above equation for c_1 and choosing

$$A = -\rho \lambda \xi^{\rho-1} \frac{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} - T^{\rho-1},$$

we obtain

$$c_1 = \frac{1}{A} J^{\rho,\alpha-1} g(T) - \frac{\lambda}{A} I_{\eta}^{\gamma,\delta} J^{\rho,\alpha-1} g(\xi).$$

Substituting the constant c_1 into (4), we find (2). □

Also, we consider the notations:

$$\phi = \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha} |\lambda| \xi^{\rho(\alpha-1)} \Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^{\rho} + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} T^{\rho\alpha}, \tag{3.5}$$

$$\Omega_1 = \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha} |\lambda| \xi^{\rho(\alpha-1)} \Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^{\rho}. \tag{3.6}$$

In the following section, we investigate existence and uniqueness results for the boundary value problem (1).

5. Existence and uniqueness results

We defined the operator $H : X \to X$ associated to the problem (1) as

$$(Hu)(x) = -\frac{t^{\rho}}{A} J^{\rho,\alpha-1} h(s, u(s))(T) + \frac{t^{\rho}\lambda}{A} I_{\eta}^{\gamma,\delta} J^{\rho,\alpha-1} h(s, u(s))(\xi) - J^{\rho,\alpha} h(s, u(s))(t). \tag{4.1}$$

We use the following expressions:

$$\begin{aligned} J^{\rho,\alpha} h(s, u(s))(z) &= \frac{\rho^{\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} h(s, u(s)) ds, \\ I_{\eta}^{\gamma,\delta} J^{\rho,\alpha} h(s, u(s))(\xi) &= \frac{\eta \xi^{-\eta(\gamma+\delta)} \rho^{1-\alpha}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^z \int_0^r \frac{r^{\eta\gamma+\eta+1} (r^{\rho} - s^{\rho})^{\alpha-2} s^{\rho-1}}{(\xi^{\rho} - r^{\rho})^{\alpha-1} s^{\rho-1}} h(s, u(s)) dr ds, \end{aligned}$$

where $\xi \in [0, T]$.

Theorem 5.1. *Let $h : [0, T] \to \mathbb{R}$ be a continuous function. Assume that:*

(H_1) *there exists a positive constant L such that*

$$|h(t, u) - h(t, v)| \leq L\|u - v\|,$$

for each $t \in [0, T]$ and $u, v \in \mathbb{R}$.

(H_2) *$L\phi < 1$, where ϕ is defined by (3.5).*

Then the boundary value problem (1) has a unique solution on $[0, T]$.

Proof. By using the operator H defined by the formula (4.1) and applying the Banach contraction mapping principle, we will show that the operator H has a unique fixed point.

For any $u, v \in X$ and for each $t \in [0, T]$, we have

$$\begin{aligned}
|Hu(t) - Hv(t)| &\leq \frac{T^\rho}{A} J^{\rho,\alpha-1}|h(s, u(s)) - h(s, v(s))|(T) \\
&+ \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}|h(s, u(s)) - h(s, v(s))|(\xi) \\
&+ J^{\rho,\alpha}|h(s, u(s)) - h(s, v(s))|(t), \\
&\leq \frac{L\|u - v\|T^\rho}{A} J^{\rho,\alpha-1}(1)(T) \\
&+ \frac{L\|u - v\|T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}(1)(\xi) \\
&+ L\|u - v\|J^{\rho,\alpha}(1)(T), \\
&\leq L\|u - v\| \left\{ \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha}|\lambda|\xi^{\rho(\alpha-1)}\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^\rho + \frac{\rho^{-\alpha}}{\Gamma(\alpha + 1)} T^{\rho\alpha} \right\}, \\
&= L\phi\|u - v\|.
\end{aligned}$$

This implies that $\|Hu - Hv\| \leq L\phi\|u - v\|$ because $L\phi < 1$.

The operator $H : X \to X$ is a contraction mapping, therefore, we deduce by Banach's contraction principle mapping, that the operator H has a fixed point which is the unique solution of problem (1) on $[0, T]$. □

Definition 5.1. *[1] Let X be a Banach space and let $H : X \to X$ be a mapping. H is called a nonlinear contraction if there exists a continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\varphi(r) < r$ for all $r > 0$, with the property*

$$\|Hu - Hv\| \leq \varphi(\|u - v\|), \forall u, v \in E.$$

Lemma 5.1. *[1] (Boy and Wong) Let X be a Banach space and let $H : X \to X$ be a nonlinear contraction. Then H has a fixed point in X.*

Theorem 5.2. *[1] Let $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following condition holds:*

(H_3) $|h(t, u) - h(t, v)| \leq k(t)\frac{\|u-v\|}{B+\|u-v\|}$, *for $t \in [0, T]$, where $k : [0, T] \to \mathbb{R}^+$ is a given function.*

Then the problem (1) has a unique solution on $[0, T]$.

Proof. Let us define the continuous and nondecreasing function, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\begin{cases} \varphi(r) = \frac{Br}{B+r}, & \forall r > 0, \\ \varphi(0) = 0, & \varphi(r) < r, \end{cases}$$

where $B := \frac{T^\rho}{A} J^{\rho,\alpha-1}k(T) + \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}k(\xi) + J^{\rho,\alpha}k(T)$.

For any $u, v \in X$ and for each $t \in [0,T]$, one has

$$\begin{aligned}
|Hu(t) - Hv(t)| &\leq \frac{T^\rho}{A} J^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(T) \\
&+ \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(\xi) \\
&+ J^{\rho,\alpha}|h(s,u(s)) - h(s,v(s))|(t), \\
&\leq \frac{T^\rho}{A} J^{\rho,\alpha-1}\left(k(s)\frac{\|u-v\|}{B+\|u-v\|}\right)(T) \\
&+ \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}\left(k(s)\frac{\|u-v\|}{B+\|u-v\|}\right)(\xi) \\
&+ J^{\rho,\alpha}\left(k(s)\frac{\|u-v\|}{B+\|u-v\|}\right)(T), \\
&\leq \frac{\varphi(\|u-v\|)}{B}\left\{\frac{T^\rho}{A} J^{\rho,\alpha-1}k(T) + \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}k(\xi) + J^{\rho,\alpha}k(T)\right\}, \\
&= \varphi(\|u-v\|).
\end{aligned}$$

This implies that $\|Tu - Tv\| \leq \varphi(\|u-v\|)$. Therefore T is a nonlinear contractions. Hence by Lemma(5.1) the operator T has a fixed point which is solution of the problem (1), which completes the proof. □

Theorem 5.3. *[13] (Krassnoselski) Let M be a closed bounded, convex and nonempty subset of a Banach space X. Let A, B be two operators such that,*

(a) *$Ax + By \in M$, whenever $x, y \in M$,*

(b) *A is compact and continuous,*

(c) *B is a contraction mapping.*
Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 5.4. *Let $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that the condition (H_1) holds and the function h satisfies the assumptions:*

(H_4) *There exists a nonnegative function $\Theta \in (C[0,T],\mathbb{R})$ such that $|h(t,u(t))| \leq \Theta(t)$ for any $(t,u) \in [0,T] \times \mathbb{R}$,*

(H_5) *$L\Omega_1 < 1$, where Ω_1 is defined by (3.6).*

Then the boundary value problem (1) has a least one solution in $[0,T]$.

Proof. We first define two new operators T_1 and T_2 as

$$(T_1u)(t) = -\frac{t^\rho}{A} J^{\rho,\alpha-1}h(s,u(s))(T), \tag{9}$$

$$(T_2u)(t) = -\frac{t^\rho\lambda}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}h(s,u(s))(\xi) - J^{\rho,\alpha}h(s,u(s))(t), \quad t \in [0,T]. \tag{10}$$

Then we consider a closed, bounded, convex and nonempty subset of the Banach space X as $B_d = \{u \in X, \|u\| \leq d\}$ with, $\|\Theta\|\phi \leq d$, where ϕ is defined by (3.5).

Now for any $u, v \in B_d$, we have

$$
\begin{aligned}
|T_1u(t) + T_2v(t)| \leq & \frac{T^\rho}{A} J^{\rho,\alpha-1}|h(s,u(s))|(T) \\
& + \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}|h(s,u(s))|(\xi) \\
& + J^{\rho,\alpha}|h(s,u(s))|(T), \\
\leq & \frac{T^\rho\|\Theta\|}{A} J^{\rho,\alpha-1}(1)(T) \\
& + \frac{T^\rho|\lambda|\|\Theta\|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}(1)(\xi) \\
& + \|\Theta\| J^{\rho,\alpha}(1)(T), \\
\leq & \|\Theta\| \left\{ \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha}|\lambda|\xi^{\rho(\alpha-1)}\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^\rho + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} T^{\rho\alpha} \right\}, \\
= & \|\Theta\|\phi \leq d.
\end{aligned}
$$

Therefore, it's clear that $\|T_1u(t) + T_2v(t)\| \leq d$. Hence $T_1u(t) + T_2v(t) \in B_d$.

The next step concerns the compactness and continuity of the operator T_1. Continuity of h implies that the operator T_1 is continuous and uniformly bounded on B_d as

$$
\|T_1\| \leq \|\Theta\| \frac{\rho^{-\alpha} T^{\rho\alpha}}{\Gamma(\alpha+1)}.
$$

Now we prove the compactness of the operator T_1. For $t_1, t_2 \in [0, T]$, $t_1 < t_2$, we have

$$
|T_1u(t_2) - T_1u(t_1)| \leq \|\Theta\| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} |t_2^\alpha - t_1^\alpha|,
$$

which is independent of u and tends to zero when $t_2 - t_1 \to 0$. Thus T_1 is equicontinuous. By Arzela-Ascoli theorem, T_1 is compact on B_d.

Now, we prove that T_2 is a contraction mapping. For $u, v \in X$ and for each $t \in [0, T]$, we have

$$
\begin{aligned}
|T_2u(t) - T_2v(t)| \leq & J^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(T) \\
& + \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(\xi), \\
\leq & L\|u - v\| J^{\rho,\alpha-1}(1)(T) \\
& + \frac{T^\rho|\lambda| \leq L\|u - v\|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}(1)(\xi), \\
\leq & \frac{\rho^{1-\alpha} L\|u - v\|}{A\Gamma(\alpha)} T^{\rho(\alpha-1)} + L\|u - v\| \frac{\rho^{1-\alpha}|\lambda|\xi^{\rho(\alpha-1)}\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^\rho, \\
= & L\|u - v\| \left\{ \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha}|\lambda|\xi^{\rho(\alpha-1)}\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^\rho \right\},
\end{aligned}
$$

which implies that $\|T_2u(t) - T_2v(t)\| \leq L\Omega_1\|u - v\|$. As $L\Omega_1\|u - v\| < 1$, the operator T_2 is a contraction. Thus all the assumption of Theorem (5.3) are satisfied. So this implies that the problem (1) has at least one solution on $[0, T]$. □

Theorem 5.5. *[13] (Leray-Schauder's nonlinear alternative). Let X be a Banach space, C a closed, convex subset of X and U an open subset of C such that $0 \in U$. Let's assume that $A : \overline{U} \to C$ is a continuous compact map. Then either*

(i) *A has a fixed point in* $\overline{U}$, *or*

(ii) *There exists* $u \in \partial\overline{U}$ *(the boundary of* U *in* C*) and* $\lambda \in (0,1)$*; witch satisfies* $u = \lambda A(u)$.

Theorem 5.6. *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be continuous function. Assume that*

(H_6) *There exist a nonnegative function* $z \in C([0,T],\mathbb{R})$ *and a continuous nondecreasing function* $\Theta : [0,\infty) \to [0,\infty)$ *such that* $|f(t,u)| \leq z(t)\Theta(\|u\|)$, *for all* $(t,u) \in [0,T] \times \mathbb{R}$,

(H_7) *There exists a constant* $N > 0$ *such that*

$$\frac{N}{\phi\|z\|\Theta(N)} > 1,$$

where ϕ *is defined as in* (3.5).

Then the problem (1) has a least one solution on $[0,T]$.

Proof. Let $B_R = \{u \in X / \|u\| \leq R\}$ be a closed bounded subset in $X = C([0,T],\mathbb{R})$. Let H be the operator defined by (4.1). As a first step, we show that the operator H maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. Then for $t \in [0,T]$, we have

$$\begin{aligned}
|Hu(t)| &\leq \frac{T^\rho}{A} J^{\rho,\alpha-1}|h(s,u(s))|(T) \\
&\quad + \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}|h(s,u(s))|(\xi) \\
&\quad + J^{\rho,\alpha}|h(s,u(s))|(T), \\
&\leq \frac{T^\rho\Theta(\|u\|)}{A} J^{\rho,\alpha-1} z(s)(T) \\
&\quad + \frac{T^\rho|\lambda|\Theta(\|u\|)}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1} z(s)(\xi) \\
&\quad + \Theta(\|u\|) J^{\rho,\alpha} z(s)(T), \\
&\leq \frac{T^\rho\Theta(\|u\|)}{A} J^{\rho,\alpha-1}\|z\|(T) \\
&\quad + \frac{T^\rho|\lambda|\Theta(\|u\|)}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}\|z\|(\xi) \\
&\quad + \Theta(\|u\|) J^{\rho,\alpha}\|z\|(T), \\
&\leq \Theta(\|u\|)\|z\| \left\{ \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha}|\lambda|\xi^{\rho(\alpha-1)}\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^\rho + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} T^{\rho\alpha} \right\}, \\
&= \phi\Theta(\|u\|)\|z\|.
\end{aligned}$$

Consequently, $\|Hu(t)\| \leq \phi\Theta(\|u\|)\|z\|$.

Next, we show that the map $H : X \to X$ is completely continuous. Therefore, we will prove that the operator H maps bounded sets into equicontinuous sets of $X = C([0,T],\mathbb{R})$. Indeed let $t_1, t_2 \in [0,T]$, with

$t_1 < t_2$ and $u \in B_R$, then we have

$$\begin{aligned}
|Hu(t_2) - Hu(t_1)| &\le \frac{|t_1^\rho - t_2^\rho|}{A} J^{\rho,\alpha-1}|h(s,u(s))|(T) + |J^{\rho,\alpha}h(s,u(s))(t_2) - J^{\rho,\alpha}h(s,u(s))(t_1)| \\
&+ \frac{|\lambda||t_1^\rho - t_2^\rho|}{A} I_\eta^{\gamma,\delta} J^{\rho,\alpha-1}|h(s,u(s))|(\xi), \\
&\le \frac{|t_1^\rho - t_2^\rho|}{A}\Theta(\|u\|)z(s)J^{\rho,\alpha-1}(1)(T) + \Theta(\|u\|)z(s)|J^{\rho,\alpha}(1)(t_2) - J^{\rho,\alpha}(1)(t_1)| \\
&+ \frac{|\lambda||t_1^\rho - t_2^\rho|}{A}\Theta(\|u\|)z(s)I_\eta^{\gamma,\delta}J^{\rho,\alpha-1}(1)(\xi), \\
&\le \frac{|t_1^\rho - t_2^\rho|}{A}\Theta(\|u\|)\|z\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}T^{\rho(\alpha-1)} + \Theta(\|u\|)\|z\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}|(t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)})| \\
&+ \frac{|\lambda||t_1^\rho - t_2^\rho|}{A}\Theta(\|u\|)\|z\|\frac{\rho^{1-\alpha}|\lambda|\xi^{\rho(\alpha-1)}\Gamma(\gamma+(\frac{\rho-1}{\eta})+1)}{\Gamma(\alpha)\Gamma(\gamma+(\frac{\rho-1}{\eta})+\delta+1)}, \\
&\le \frac{|t_1^\rho - t_2^\rho|}{A}\Theta(R)\|z\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}T^{\rho(\alpha-1)} + \Theta(R)\|z\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}|(t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)})| \\
&+ \frac{|\lambda||t_1^\rho - t_2^\rho|}{A}\Theta(R)\|z\|\frac{\rho^{1-\alpha}|\lambda|\xi^{\rho(\alpha-1)}\Gamma(\gamma+(\frac{\rho-1}{\eta})+1)}{\Gamma(\alpha)\Gamma(\gamma+(\frac{\rho-1}{\eta})+\delta+1)}.
\end{aligned}$$

It is clear that the right-hand side of above inequality tends to zero independently of $u \in B_R$ as $t_2 - t_1 \to 0$. Therefore by the Ascoli-Arzela theorem, the operator $H : X \to X$ is completely continuous.

In the last step we show that the operator H has a fixed point. Let u be a solution of $H(u) = u$, then for each $t \in [0,T]$,

$$\|Hu\| = \|u\| \le \phi\|z\|\Theta(\|u\|),$$

which implies that

$$\frac{\|u\|}{\phi\|z\|\Theta(\|u\|)} \le 1.$$

From (H_7), there exists $N > 0$ such that $\|u\| \ne N$. Let us set $G = \{u \in X : \|u| \ < N\}$.

Then the operator $H : \overline{G} \to X$ is continuous and completely continuous. Consequently, there doesn't exist any $u \in \partial G$ such that $u = \mu Hu$ for some $\mu \in (0,1)$. Assume that there exists $u \in \partial G$ such that $u = \mu Hu$ for some $\mu \in (0,1)$. Then

$$\|u\| = \|\mu Hu\| \le \|Hu\| \le \phi\|z\|\Theta(\|u\|),$$

$$\frac{\|u\|}{\phi\|z\|\Theta(\|u\|)} \le 1.$$

This contradicts $\frac{\|u\|}{\phi\|z\|\Theta(\|u\|)} > 1$. Consequently, by nonlinear alternative Leray-Schauder principal, we conclude, that H has a fixed point $u \in \overline{G}$, which is a solution of problem (1), this completes the proof. □

6. Examples

Example 6.1. *Consider the following nonlinear Katugampola fractional differential equation with Erdelyi-Kober fractional integral conditions:*

$$\begin{cases} D^{1,\frac{3}{2}}u(t) = \left(\frac{|u|}{|u|+1}\right)\frac{e^{-\sin t}}{\pi+2} + \frac{1}{2} \ , \ t \in [0,1], \\ u(0) = 0 \ , \\ u'(1) = \frac{3}{5}I_{\frac{1}{5}}^{\frac{3}{4},\frac{\sqrt{2}}{2}}u'(\frac{1}{2}). \end{cases} \tag{5}$$

Here, $\alpha = \frac{3}{2}$, $\rho = 1$, $\gamma = \frac{3}{4}$, $\eta = \frac{1}{5}$, $\delta = \frac{\sqrt{2}}{2}$, $\xi = \frac{1}{2}$, $\lambda = \frac{3}{5}$.

$$f(t,u) = \left(\frac{|u|}{|u|+1}\right)\frac{e^{-\sin t}}{\pi+2} + \frac{1}{2}.$$

Hence, we have

$$|f(t,u) - f(t,v)| \leq \frac{1}{\pi+2}\|u-v\|.$$

Assumption (H_1) *is satisfied with* $L = \frac{1}{\pi+2}$. *Using the given value, we get* $\phi = 1,4701$. *Therefore* $L\phi = 0,2859 < 1$, *which implies that assumption* (H_2) *holds. Using theorem (5.1), we deduce that the boundary value problem (5) has a unique solution on* $[0,1]$.

Example 6.2. *Consider the following nonlinear Katugampola fractional differential equation with Erdelyi-Kober fractional integral conditions:*

$$\begin{cases} D^{1,\frac{7}{4}}u(t) = \frac{t^2}{\pi\sqrt{t^2+9}}\left(\frac{|u|}{|u|+5}\right) + \frac{e^t+t}{2} \ , \ t \in [0,\frac{1}{2}], \\ u(0) = 0 \ , \\ u'(\frac{1}{2}) = \frac{8}{3} I_{\frac{12}{7}}^{\frac{\sqrt{5}}{3},\frac{1}{\sqrt{6}}} u'(\frac{3}{11}) \ , \end{cases} \tag{6}$$

Here, $\alpha = \frac{7}{4}$, $\rho = 1$, $\gamma = \frac{\sqrt{5}}{3}$, $\eta = \frac{12}{7}$, $\delta = \frac{1}{\sqrt{6}}$, $\xi = \frac{3}{11}$, $\lambda = \frac{8}{3}$, $B = 0,0751$ *and*

$$f(t,u) = \frac{t^2}{\pi\sqrt{t^2+9}}\left(\frac{|u|}{|u|+5}\right) + \frac{e^t+t}{2}.$$

Choosing $k(t) = \frac{t^2}{3\pi}$, *we get*

$$|f(t,u) - f(t,v)| \leq \frac{t^2}{3\pi}\left(\frac{|u-v|}{0,0751+|u-v|}\right) + \frac{e^t+t}{2}.$$

Clearly, all the assumptions of Theorem (5.2) are satisfied, which implies that the problem (6) has at least one solution on $[0,\frac{1}{2}]$.

Example 6.3. *Consider the following nonlinear Katugampola fractional differential equation with Erdelyi-Kober fractional integral conditions:*

$$\begin{cases} D^{2,\frac{9}{5}}u(t) = \sin\left(\frac{|u|}{|u|+1}\right)\frac{e^{-2t}}{3(\pi+7)} + \frac{t^2+1}{2}, \ t \in [0,2] \\ u(0) = 0, \\ u'(2) = \frac{3}{8} I_{\frac{7}{3}}^{\frac{\sqrt{2}}{2},\frac{1}{4}} u'(\frac{3}{2}). \end{cases} \tag{7}$$

Here, $\alpha = \frac{9}{5}$, $\rho = 2$, $\gamma = \frac{\sqrt{2}}{2}$, $\eta = \frac{7}{3}$, $\delta = \frac{1}{4}$, $\xi = \frac{3}{2}$, $\lambda = \frac{3}{8}$, *and*

$$|f(t,u) - f(t,v)| \leq \frac{1}{3(\pi+7)}|u-v|,$$

$$|f(t,u)| \leq \frac{e^{-2t}}{3(\pi+7)} + \frac{t^2+1}{2},$$

with $L = \frac{1}{3(\pi+7)}$, $\phi = 4,6905$, $L\phi = 0,1542$, $\Omega_1 = 2,6134$, $L\Omega_1 = 0,0859 < 1$.

Again, the hypothesis of Theorem (5.4) are satisfied and, as a consequence, the problem (7) has at least one solution on $[0,2]$.

Example 6.4. *Consider the following nonlinear Katugampola fractional differential equation with Erdelyi-Kober fractional integral conditions :*

$$\begin{cases} D^{3,\frac{11}{6}}u(t) = \left(\frac{u^2(t)}{|u|+1}+1\right)\left(\frac{\sqrt{t}+1}{8}\right),\ t\in[0,\frac{9}{16}], \\ u(0)=0, \\ u'(\frac{9}{16}) = \frac{5}{11}I_{\frac{1}{9}}^{\frac{4}{13},\frac{1}{5}}u'(\frac{3}{7}). \end{cases} \tag{8}$$

Here $\alpha=\frac{11}{6}$, $\rho=3$, $\gamma=\frac{4}{13}$, $\eta=\frac{1}{9}$, $\delta=\frac{1}{5}$, $\xi=\frac{3}{8}$,$\lambda=\frac{5}{11}$.

Moreover

$$|f(t,u)| = \left|\left(\frac{u^2(t)}{|u|+1}+1\right)\left(\frac{\sqrt{t}+1}{8}\right)\right| \le \frac{\sqrt{t}+1}{8}(|u|+1).$$

We choose $z(t)=\frac{\sqrt{t}+1}{8}$ *and* $\Theta(\|u\|)=\|u\|+1$. *We have* $\|z\|=\frac{7}{32}$ *and* $\phi=0,0442$. *Now, we need to show that there exists* $N>0$ *such that*

$$\frac{N}{\Theta(N)\|z\|\phi} > 1,$$

and such $N>0$ *exists if* $1-\|z\|\phi>0$. *A straightforward calculus give* $\|z\|\phi=0,0097<1$, *assumption* H_7 *is satisfied. Hence using Theorem (5.6), the boundary value problem (8) has at least one solution on* $[0,\frac{9}{16}]$.

7. Conclusion

In this paper, with the help of standard fixed point theorems type, we obtained conditions for existence of at least one solution of a Katugampola fractional differential equation with Erdelyi-Kober fractional integral boundary conditions. In the future it seems interesting to obtain sufficient conditions to ensure Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

References

[1] B. Ahmad, S.K. Ntouyas, J. Tariboon, A. Alsaedi: Caputo type fractional differential equations with nonlocal Riemann-Liouville and Erdelyi-Kober integral boundary conditions, Filomat (2017), 4515-4529.

[2] B. Ahmad, S.K. Ntouyas, A. Alsaedi: New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, Adv. Difference Equ., 2011, Art. ID 107384, 11 pp.

[3] B. Ahmad, S.K. Ntouyas: A four-point nonlocal integral boundary value problem for fractional differential equations of arbitrary order, Electron. J. Qual. Theory Differ. Equ. 22(2011), pp. 1-15.

[4] Y. Arioua, B. Basti and N. Benhamidouche: Initial value problem for nonlinear implicit fractional differential equation with Katugampola derivative. Appl. Math. E-Notes, 19(2019), 397-412.

[5] B. Basti, Y. Arioua, N. Benhamidouche: Existence and Uniqueness of Solutions for Nonlinear Katugampola Fractional Differential Equations, Journal of Mathematics and Applications, No 42, pp 35-61 (2019).

[6] M. Benchohra, S. Hammani and S.K. Ntouyas: Boundary value problems for differential equation with fractional order and nonlocal conditions, Nonlinear Anal. TMA 71 (2009), 2391-2396.

[7] A. Boutiara, M. Benbachir, K. Guerbati: Boundary value problems for hilfer fractional differential equations with katugampola fractional integral and anti-periodic conditions, to appear (http://math.ubbcluj.ro/ mathjour/accepted.html).

[8] D.W. Boyd and J.S.W. Wong: On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458-464.

[9] M. Janaki, E. M. Elsayed, K. Kanagarajan: Katugampola-type fractional differential equations with delay and impulses, Open Access Journal of Mathematical and Theoretical Physics, Volume 1 Issue 3 - 2018.

[10] M. Janaki, K. kanagarajan, D. Viek: Existence results for Katugampola fractional differential equations via measure of noncompactness, Journal nonlinear Analysis and Applications 2018, No.2(2018)184-191.

[11] S.L. Kalla, V.S. Kiryakova: An H-function generalized fractional calculus based upon compositions of Erdelyi-Kober operators in LP; Math. Japonica 35 (1990), 1-21.

[12] K. Logeswari, C. Ravichandran: A new exploration on existence of fractional neutral integro- differential equations in the concept of Atangana-Baleanu derivative, Physica A: Statistical Mechanics and its Applications, Volume 544, 15 April 2020, 123454.

[13] N.I. Mahmudov, S. Emin: Fractional-order boundary value problems with Katugampola fractional integral conditions. Adv Differ Equ 2018, 81 (2018).

[14] K. Rajendra Prasad , L. D and M. Khuddush , "Existence and Uniqueness of Positive Solutions for System of (p,q,r)-Laplacian Fractional Order Boundary Value Problems", Advances in the Theory of Nonlinear Analysis and its Application, vol. 5, no. 1, pp. 138-157, Mar. 2021, doi:10.31197/atnaa.703304

[15] C. Ravichandran, N. Valliammal, Juan J. Nieto: New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces, Journal of the Franklin Institute, 356(3), 2019, 1535-1565.

[16] A. Saadi and M. Benbachir: Positive solutions for three-point nonlinear fractional boundary value problems, Electron. J. Qual. Theory Differ. Equ. 2011, No. 2, 1-19.

[17] F. Si Bachir , A. Said , M. Benbachir and M. Benchohra , "Hilfer-Hadamard Fractional Differential Equations; Existence and Attractivity", Advances in the Theory of Nonlinear Analysis and its Application, vol. 5, no. 1, pp. 49-57, Mar. 2021, doi:10.31197/atnaa.848928

[18] I.N. Sneddon, The use in mathematical analysis of Erdelyi-Kober operators and some of their applications, pp. 37-79 in Fractional calculus and its applications (West Haven, Connecticut, 1974), Lecture Notes in Math.

[19] N. Thongsalee, S.K. Ntouyas, J. Tariboon: Nonlinear Riemann-Liouville fractional differential equations with nonlocal Erdelyi-Kober fractional integral conditions, Frac. Calc. Appl. Anal. Volume 19, Issue 2 (2016).

[20] N. Valliammal, C. Ravichandran, Ju H. Park: On the controllability of fractional neutral integrodifferential delay equations with nonlocal conditions, Math. Methods Appl. Sci. 40 (14) (2017), 5044-5055.

[21] S. Zeng, D. Baleanu, Y. Bai, G. Wu: Fractional differential equations of Caputo-Katugampola type and numerical solutions, Applied Mathematics and computation 315(2017) 549-554.