Statistical cluster point and statistical limit point
sets of subsequences of a given sequence

Harry I. Miller, Leila Miller-Van Wieren*

Faculty of Engineering and Natural Sciences, International University of Sarajevo, Sarajevo, 71000,
Bosnia-Herzegovina

Abstract

cluster points and statistical limit points of a given sequence $x$. Here we show that almost
all subsequences of $x$ have the same statistical cluster point set as $x$. Also, we show an
analogous result for the statistical limit points of $x$.

Mathematics Subject Classification (2010). 40D25, 40G99, 28A12

Keywords. sequences, subsequences, statistical cluster points, statistical limit points

1. Introduction

Fridy [1] has proven that $\Gamma_x$, the set of statistical cluster points of $x = (x_n)$, is always
a closed set and $\Gamma_x$ is non-empty if $x$ is bounded. However $\Lambda_x$, the set of statistical limit
points of $x$, need not be closed. In [2] H.I. Miller studied statistical convergence and
relations between statistical convergence of a sequence $x$ and statistical convergence of
the subsequences of $x$. In particular, in [2], it is shown that if $L$ is the statistical limit of
$x$, then almost all subsequences of $x$ have $L$ as their statistical limit. Here we combine
two notions, statistical cluster points and subsequences, showing that $\Gamma_x$ is equal to the
statistical cluster point set of almost all subsequences of $x$. This is a continuation of the
results in [3] that also combine statistical cluster points and subsequences. Namely, in [3]
it is shown that if $\Gamma_x \neq \emptyset$ and $F$ is a non-empty closed subset of $\Gamma_x$, then there exists
a subsequence $y$ of $x$ such that $\Gamma_y = F$. Additionally we show that $\Lambda_x$ is equal to the
statistical limit point set of almost all subsequences of $x$. This is a continuation of the
results in [4] that also combine statistical limit points and subsequences.

2. Preliminaries

If $t \in (0,1]$, then $t$ has a unique binary expansion $t = \sum_{n=1}^{\infty} \frac{e_n}{2^n}$, $e_n \in \{0,1\}$, with infinitely
many ones. Next if $x = (x_n)$ is a sequence of reals, for each $t \in (0,1]$, let $x(t)$ denote the
subsequence of $x$ obtained by the following rule: $x_n$ is in the subsequence if and only if
$e_n = 1$. Clearly the mapping $t \to x(t)$ is a one-to-one onto mapping between $(0,1]$ and the
collection of all subsequences of $x$.

*Corresponding Author.

Email addresses: himiller@hotmail.com (H.I. Miller), lmiller@ius.edu.ba (L.M. Wieren)

Received: 10.07.2016; Accepted: 06.10.2016
If $K$ is a subset of the positive integers $N$, then following Fridy [1], $K_n$ denotes the set \{$k \in K : k \leq n$\} and $|K_n|$ denotes the number of elements in $K_n$. The natural density of $K$ (see [5]) is given by $\delta(K) = \lim_{n \to \infty} n^{-1}|K_n|$, provided this limit exists. In the case that $\delta(K) = 0$ we say that $K$ is thin, and otherwise we say that $K$ is non-thin.

Statistical convergence of a sequence is defined as follows.

We say that $L$ is the statistical limit of the sequence $x$, if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$ 

Statistical convergence and its connection to subsequences is studied in [2].

Statistical limit points and statistical cluster points of a sequence $x$ are defined as follows.

We say that a number $\lambda$ is a statistical limit point of a sequence of reals $x = (x_n)$ if $\lim_{k \to \infty} x_{n_k} = \lambda$ for some non-thin subsequence of $(x_n)$.

We say that a number $\gamma$ is a statistical cluster point of a sequence of reals $(x_n)$ if for every $\epsilon > 0$ the set $\{k \in N : |x_k - \gamma| < \epsilon\}$ is non-thin.

In [1], given a sequence $x$, three sets are considered. $L_x$, the set of limit points of $x$; $\Lambda_x$, the set of statistical limit points of $x$, and $\Gamma_x$, the set of statistical cluster points of $x$. Also, if $x$ is bounded, then $\Gamma_x$ is closed and non-empty.

In this paper we want to examine, $\Gamma_x$ and its relation to $\Gamma_x(t)$. Additionally we also consider $\Lambda_x$ and its relation to $\Lambda_x(t)$.

3. Results

Our main result is the following.

**Theorem 3.1.** If $x = (x_n)$ is a bounded sequence, then $\Gamma_x = \Gamma_x(t)$ for almost all $t \in (0, 1]$ (in the sense of Lebesgue measure).

**Proof.** Since $\Gamma_x$ is closed, it is either finite or separable, i.e. there is a countable subset of $\Gamma_x$, $\{l_n : n \in N\}$ such that its closure is $\Gamma_x$. We consider only the second case, the proof in the first case is much simpler.

First we show that $\Gamma_x \subseteq \Gamma_x(t)$ for almost all $t$. It is sufficient to show that $m(B_n) = 1$ for $n = 1, 2, \ldots$ where $B_n = \{t \in (0, 1) : l_n \in \Gamma_x(t)\}$. This is true since in that case $m(B) = 1$ for $B = \bigcap_{n=1}^\infty B_n$ and then $\{l_n : n \in N\} \subseteq \Gamma_x(t)$ for all $t \in B$ and consequently $\Gamma_x \subseteq \Gamma_x(t)$ for all $t \in B$.

Since $l_n \in \Gamma_x$, then for every $\epsilon > 0$, $\{k \in N : |x_k - l_n| < \epsilon\}$ is non-thin . If $\epsilon = \frac{1}{2}$ we can denote the above set by $\{k_1^i, k_2^i, k_3^i, \ldots\}$. Then, since it is non-thin there exists $\delta_j > 0$ such that

$$\frac{1}{p} |\{i : k_j^i \leq p\}| > \delta_j$$

for infinitely many $p$. We can assume that $p = k_j^i$ for infinitely many sufficiently large $M$. Now for each $j$, by the Law of Large Numbers, the limiting frequency of $x_{k_j^i}$ $i = 1, 2, \ldots$ among the sequence $x(t)$ is $\frac{1}{2}$ for almost all $t \in (0, 1]$, i.e. if $t = \sum_{m=1}^{\infty} \frac{\epsilon_m}{2^m}$, then $\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} t_{k_j^i}^i = \frac{1}{2}$ for almost all $t \in (0, 1]$. That is, $m(D_j) = 1$, where

$$D_j = \{t \in (0, 1] : \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} t_{k_j^i}^i = \frac{1}{2}\}$$

(3.1)

for all $j$. Hence if $D = \bigcap_{j=1}^{\infty} D_j$, $m(D) = 1$. Now we will check that $l_n$ is a statistical cluster point for each $t$ in $D$.

To see this we will show that $\{i \in N : |x(t) - l_n| < \frac{1}{j}\}$ is non-thin for every $j \in N$ and every $t \in D_j$. 

Consider the earlier mentioned $p = k^j_M$ for $M$ large enough. Then the number of such $i \leq p$, with $|x_i - l_n| < \frac{1}{j}$ is greater than $p\delta_j$. Now take $t \in D_j$. By (3.1),
$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} t^j_{k_i} = \frac{1}{2}.$$ So for large $M$, $p = k^j_M$, we have
$$\frac{1}{p} \{ i : p : |x(t)_i - l_n| < \frac{1}{j} \} > \frac{\delta_j}{4},$$ i.e. this holds for infinitely many $p$, i.e. \{ $i \in N : |x(t)_i - l_n| < \frac{1}{j}$ \} is non-thin for every $j \in N$ and every $t \in D_j$. Hence $l_n$ is a statistical cluster point for every $t \in D$. This completes the proof that $\Gamma_x \subseteq \Gamma_x(t)$ for almost all $t$.

Next we show that $\Gamma_x(t) \subseteq \Gamma_x$ for almost all $t$. We will show that this inclusion holds for all normal $t \in (0, 1]$, i.e. for all $t = \sum_{n=1}^{\infty} \frac{e_n}{M}$ for which $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} e_i = \frac{1}{2}$. It is well known that almost all $t \in (0, 1]$ are normal (see [5]).

Suppose that $l$ is a statistical cluster point of $x(t)$ for some normal $t$. Then for any $\epsilon > 0$, \{ $i : |(x(t))_i - l| < \epsilon$ \} is non-thin, i.e. there exists $\delta_\epsilon > 0$ such that
$$\frac{1}{n} |i \leq n : |(x(t))_i - l| < \epsilon| > 2\delta_\epsilon$$
for infinitely many $n$. This implies that
$$\frac{1}{n} |i \leq n : |x_i - l| < \epsilon| > \frac{1}{2} \delta_\epsilon$$
for infinitely many $n$, and hence $l$ is a statistical cluster point of $x$. Therefore $\Gamma_x(t) \subseteq \Gamma_x$ for all normal $t$, and consequently for almost all $t \in (0, 1]$. Therefore we conclude that $\Gamma_x(t) = \Gamma_x$ for almost all $t \in (0, 1]$.

Next, we will prove an analogous result for the set of statistical limit points of $x$ and its subsequences. The set $\Lambda_x$ is not necessarily closed (see [4]). However the following useful theorem was proved by Kostyrko, Mačaj, Šalat and Strauch [4].

**Theorem 3.2.** For every bounded sequence $x$, the set $\Lambda_x$ is an $F_\sigma$-set in $R$.

In the proof of the above theorem, the authors show that
$$\Lambda_x = \bigcup_{j=1}^{\infty} \Lambda(x, \frac{1}{j})$$
where $\Lambda(x, \frac{1}{j}) = \{ l, \exists k_i, i = 1, 2, \ldots, \lim_{i \to \infty} x_{k_i} = l, \bar{\delta}(\{ k_i \}) \geq \frac{1}{j} \}$ where $\bar{\delta}$ denotes the upper statistical density (i.e. $\bar{\delta}(\{ k_i \}) = \limsup_{i \to \infty} \frac{1}{k_i}$) and $\Lambda(x, \frac{1}{j})$ is closed for all $j$.

Here is our second result.

**Theorem 3.3.** If $x = (x_n)$ is a bounded sequence, then $\Lambda_x = \Lambda_x(t)$ for almost all $t \in (0, 1]$ (in the sense of Lebesgue measure).

**Proof.** We proceed in a similar manner as in the proof of Theorem 3.1.

First we show that $\Lambda_x \subseteq \Lambda_x(t)$ for almost all $t$.

As mentioned earlier, $\Lambda_x = \bigcup_{j=1}^{\infty} T_j$, where
$$T_j = \Lambda(x, \frac{1}{j}) = \{ l, \exists k_i, i = 1, 2, \ldots, \lim_{i \to \infty} x_{k_i} = l, \bar{\delta}(\{ k_i \}) \geq \frac{1}{j} \}.$$ Supposed $j \in N$ is fixed. Using the above notation (from [4]), $T_j$ is closed and separable so there exists a set \{ $l_{ij} : i \in N$ \} such that its closure is $T_j$. Let $i \in N$. If $l = l_{ij}$, then by the Law of Large Numbers, $l \in \Lambda(x(t), \frac{1}{j})$, for all $t \in B_{ij}$, where $m(B_{ij}) = 1$. Let $B_j = \bigcap_{i=1}^{\infty} B_{ij}$. Then $m(B_j) = 1$. Hence \{ $l_{ij} : i \in N$ \} $\subseteq \Lambda(x(t), \frac{1}{j})$ for every $t \in B_j$. Now since $T_j$ and $\Lambda(x(t), \frac{1}{j})$ are both closed we get that $T_j \subseteq \Lambda(x(t), \frac{1}{j})$ for every $t \in B_j$.
Therefore \( \Lambda_x = \bigcup_{j=1}^{\infty} T_j \subseteq \bigcup_{j=1}^{\infty} \Lambda(x(t), \frac{1}{T_j}) = \Lambda_{x(t)} \) for all \( t \in \bigcap_{j=1}^{\infty} B_j \). Since \( m(\bigcap_{j=1}^{\infty} B_j) = 1 \), we have shown that \( \Lambda_x \subseteq \Lambda_{x(t)} \) for almost all \( t \).

Next we show that \( \Lambda_{x(t)} \subseteq \Lambda_x \) for almost all \( t \). Again we show that this inclusion holds for all normal \( t \in (0, 1) \). Suppose that \( l \) is a statistical limit point of \( x(t) \) for some normal \( t \). Then \( x(t) \) has a non-thin subsequence that converges to \( l \) (in the normal sense). It is easy to see that this subsequence \( x(t)_i = x_{k_i} \) is then also a non-thin subsequence of \( x \) and therefore \( l \) is also a statistical limit point of \( x \). This completes the proof. \( \square \)

4. Concluding remarks

We mentioned that \( m(\nu) = 1 \), where \( \nu \) is the set of normal numbers in \((0, 1]\). However \( \nu \) is a set of first Baire category. In light of this we suspect that a category analogue of our Theorem 3.1 is not true.

Also, one could examine possible analogues of our results using permutations rather than subsequences.

References