SOME NOTES ON LIFTS OF THE $F (v + 1), \lambda^2 (v - 1)$-STRUCTURE ON COTANGENT AND TANGENT BUNDLE

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Abstract. The $F^{v+1} - \lambda^2 F^{v-1} = 0$ structure $(v \geq 3)$ have been studied by Kim J. B. [14]. Later, Srivastava S.K studied on the complete lifts of $(1, 1)$ tensor field $F$ satisfying structure $F^{v+1} - \lambda^2 F^{v-1} = 0$ and extended in $M^n$ to cotangent bundle. This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete and horizontal lifts of $F^{v+1} - \lambda^2 F^{v-1} = 0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the complete and horizontal lifts of $F (v + 1), \lambda^2 (v - 1)$-structure and the conditions of almost holomorphic vector fields in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of $F^{v+1} - \lambda^2 F^{v-1} = 0$-structure. In the second part, all results obtained in the first section were investigated according to the complete and horizontal lifts of the $F^{v+1} - \lambda^2 F^{v-1} = 0$ structure in tangent bundle $T(M^n)$.

1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [25]. Later, a lot of authors studied on the topics of the bundle, Riemannian manifolds and $F$ structure too [1, 2, 3, 4, 5, 10, 12, 13, 16, 23]. There are a lot of structures in tangent and cotangent bundle. One of them is the $F (v + 1), \lambda^2 (v - 1)$-structure $(v \geq 3)$ have been studied by Kim J. B. [14].

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Later, Srivastava S.K studied on the complete lifts of \((1, 1)\) tensor field \(F\) satisfying structure \(F^{v+1} - \lambda^2 F^{v-1} = 0\) and extended in \(M^n\) to cotangent bundle \([21]\). In this context, a differentiable structure \(F^{2v+4} + F^2 = 0, (F \neq 0, v \neq 0)\) studied by K.K. Dube \([14]\) and Upadhyay and Gupta have obtained some integrability conditions of \(F(K, -(K - 2))\)-structure, satisfying \(F^K + F^{K-2} = 0,(F\) is a tensor field of type \((1, 1))\) \([24]\).

This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete and horizontal lifts of \(F ((v + 1), \lambda^2 (v - 1))\)-structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the complete and horizontal lifts of \(F^{v+1} - \lambda^2 F^{v-1} = 0\) structure and the conditions of almost holomorphic vector fields in cotangent bundle \(T^*(M^n)\). Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of \(F^{v+1} - \lambda^2 F^{v-1} = 0\) structure. In the second part, all results obtained in the first section were investigated according to the complete and horizontal lifts of the \(F ((v + 1), \lambda^2 (v - 1))\)-structure in tangent bundle \(T(M^n)\).

Let \(M^n\) be a differentiable manifold of class \(C^\infty\) and of dimension \(n\) and let \(T^*(M^n)\) denote the cotangent bundle of \(M\). Then \(T^*(M^n)\) is also a differentiable manifold of class \(C^\infty\) and dimension \(2n\).

The following are notations and conventions that will be used in this paper.

1. \(\mathcal{T}^r_s(M^n)\) denotes the set of the tensor fields \(C^\infty\) and of type \((r, s)\) on \(M^n\).

Similarly, \(\mathcal{T}^r_s(T^*(M^n))\) denotes the set of such tensor fields in \(T^*(M^n)\).

2. The map \(\pi\) is the projection of \(T^*(M^n)\) onto \(M^n\).

3. Vector fields in \(M^n\) are denoted by \(X, Y, Z\) and Lie differentiation by \(L_X\). The Lie product of vector fields \(X\) and \(Y\) is denoted by \([X, Y]\).

4. Suffixes \(a, b, c, \ldots, h, i, j\) take the values \(1\) to \(n\) and \(i = i + n\). Suffixes \(A, B, C, \ldots\) take the values \(1\) to \(2n\).

If \(A\) is point in \(M^n\), then \((\pi^*)^{-1}(A) : M^n \rightarrow T^*(M^n)\) is fiber over \(A\). Any point \(p \in (\pi^*)^{-1}(A)\) is denoted by the ordered pair \((A, p_A)\), where \(p = 1\)-form in \(M^n\) and \(p_A\) is the value of \(p\) at \(A\). Let \(U\) be a coordinate neighborhood in \(M^n\) such that \(A \in U\). Then \(U\) induces a coordinate neighborhood \((\pi^*)^{-1}(U)\) in \(T^*(M^n)\) and \(p \in \pi^{-1}(A)\).

1.1. **The complete lift of \(F^{v+1} - \lambda^2 F^{v-1} = 0\) on cotangent bundle.** Let \(M^n\) be an \(n\)-dimensional connected differentiable manifold of class \(C^\infty\). Let there be given in \(M^n\), a \((1, 1)\) tensor field \(F\) of class \(C^\infty\) satisfying \([14][21]\)

\[
F^{v+1} - \lambda^2 F^{v-1} = 0,
\]

where \(\lambda\) is non zero complex number. Also

\[
\text{rank}(F) = \frac{1}{2} \left( \text{rank} F^{v+1} + \dim M^n \right)
\]

\[
= (a \text{ constant every where on } M^n)
\]
Let the operators $l^*$ and $m^*$ be defined as
\[ l^* \text{ def } (F/\lambda)^{u-1}, \quad m^* = I - (F/\lambda)^{u-1}, \] (4)
where $I$ denotes the identity operator on $M^n$. Then the operators $I^*$ and $m^*$ applied to the tangent space at a point of the manifold be complementary projection operators.

Let $F^h_i$ be the component of $F$ at $A$ in the coordinate neighbourhood $U$ of $M^n$. Then the complete lift $F^C$ of $F$ is also a tensor field of type $(1, 1)$ in $T^*(M^n)$ whose components $\tilde{F}_B^A$ in $(\pi^*)^{-1}(U) : M^n \rightarrow T^*(M^n)$ are given by [17]
\[ \tilde{F}_i^h = F^h_i, \quad \tilde{F}_i^h = 0, \quad \tilde{F}_i^h = p_a[\partial F^a_i / \partial x^i - \partial F^a_i / \partial x^h] \] (7)
and
\[ \tilde{F}_i^h = F^i_h, \] (8)
where $(x^1, x^2, x^3, \ldots, x^n)$ are coordinates of $A$ in $U$ and $p_A$ has components $(p_1, p_2, p_3, \ldots p_n)$. Thus we can write
\[ F^C = (\tilde{F}_B^A) = \begin{bmatrix} F^h_i & 0 \\ p_a(\partial_i F^a_h - \partial_h F^a_i) & F^h_i \end{bmatrix}, \] (9)
where $\partial_i = \partial / \partial x^i$.

If we put
\[ \partial_i F^a_h - \partial_h F^a_i = 2\partial[i F^a_h], \] (10)
then the equation [9] can be written as
\[ F^C = (\tilde{F}_B^A) = \begin{bmatrix} F^h_i & 0 \\ 2p_a \partial[F^a_h] & F^h_i \end{bmatrix} \] (11)
\[ (F^C)^2 = (F^C)(F^C) = \begin{bmatrix} F^h_i F_j^i 0 \\ L_{hj} F_j^i F^i_h \end{bmatrix}. \] (12)

Squaring [12] again we get [17]
\[ (F^C)^4 = \begin{bmatrix} F^h_i F_j^i 0 \\ L_{hj} F_j^i F^i_h \end{bmatrix} \begin{bmatrix} F^h_i F_j^i 0 \\ L_{hj} F_j^i F^i_h \end{bmatrix}, \]
\[ = \begin{bmatrix} F^h_i F_j^i F_k^j L_{hj} + F^i_j F^k_l L_{ij} + F^i_j F^k_l F^h_i F^i_h \end{bmatrix}. \]
\[ (F^C)^6 = (F^C)^4(F^C)^2 = \begin{bmatrix} F^h_i F_j^i F_k^j F^k_l F^m_n Q_{hnm} 0 \\ F_m^n F_i^m F_k^j F_j^i F^i_h \end{bmatrix} \] (13)
\[ (F^C)^7 = (F^C)^6(F^C) = \begin{bmatrix} -\lambda^2 F_p^n 0 \\ -\lambda^2 p_a \partial[p F^a_h] -\lambda^2 F_p^h \end{bmatrix}. \] (14)
Thus it follows that

\[(F^{v+1})^C - \lambda^2 (F^{v-1})^C = 0\]  

(15)

Thus, we get the complete lift of \(F^{v+1} - \lambda^2 F^{v-1} = 0\)-structure on the cotangent bundle.

1.2. Horizontal lift of the structure \(F^{v+1} - \lambda^2 F^{v-1} = 0\) on cotangent bundle. Let \(F, G\) be two tensor field of type \((1,1)\) on the manifold \(M^n\). If \(F^H\) denotes the horizontal lift of \(F\), we have \([17,18,25]\)

\[F^H G^H + G^H F^H = (FG + GF)^H.\]  

(16)

Taking \(F\) and \(G\) identical, we get

\[(F^H)^2 = (F^2)^H.\]  

(17)

Thus, multiplying both sides by \(F^H\) and making use of the same (16), we get

\[(F^H)^3 = (F^3)^H.\]  

(18)

Thus it follows that

\[(F^H)^4 = (F^4)^H, \quad (F^H)^5 = (F^5)^H.\]  

(19)

Thus,

\[(F^{v+1})^H - \lambda^2 (F^{v-1})^H = 0.\]  

(20)

In view of (17), we can write \((F^H)^{v+1} - \lambda^2 (F^H)^{v-1} = 0\).

Thus, we get the horizontal lift of \(F^{v+1} - \lambda^2 F^{v-1} = 0\)-structure on the cotangent bundle.

**Proposition 1.** Let \(M^n\) be a Riemannian manifold with metric \(g\), \(\nabla\) be the Levi-Civita connection and \(R\) be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle \(T^*(M^n)\) of \(M^n\) satisfies the following

i) \([\omega^v, \theta^v] = 0,\]  

\[\text{(21)}\]

ii) \([X^H, \omega^v] = (\nabla_X \omega)^v,\]

\[\text{iii) \([X^H, Y^H] = [X, Y]^H + \gamma R(X, Y) = [X, Y]^H + (pR(X, Y))^v \text{ for all } X, Y \in \mathfrak{X}_0(M^n) \text{ and } \omega, \theta \in \mathfrak{X}_0^1(M^n). (See [25] p. 238, p. 277 for more details).}\]

2. Main Results

2.1. The Nijenhuis tensors of \(F ((v + 1), \lambda^2 (v - 1))\)-structure on cotangent bundle.

**Definition 2.** Let \(F\) be a tensor field of type \((1,1)\) satisfying \(F^{v+1} - \lambda^2 F^{v-1} = 0\) in \(M^n\). The Nijenhuis tensor of a \((1,1)\) tensor field \(F\) of \(M^n\) is given by


(22)
for any $X, Y \in \mathfrak{X}_0\!(M^n)$. The condition of $N_F(X, Y) = N(X, Y) = 0$ is essential to integrability condition in these structures.

The Nijenhuis tensor $N_F$ is defined local coordinates by

$$N_{ij}^k \partial_k = (F_i^s \partial_s^k F_j^k - F_j^s \partial_s^k F_i^k - \partial_i F_j^k F_i^k + \partial_j F_i^k F_s^k) \partial_k,$$

where $X = \partial_i$, $Y = \partial_j$, $F \in \mathfrak{X}_1\!(M^n)$.

**Proposition 3.** If $X, Y \in \mathfrak{X}_0\!(M^n)$, $\omega, \theta \in \mathfrak{X}_0\!(M^n)$ and $F, G \in \mathfrak{X}_1\!(M^n)$, then

$$[\omega^v, \theta^w] = 0, \quad [\omega^v, \gamma F] = (\omega \circ F)^v, \quad [\gamma F, \gamma G] = \gamma [F, G],$$

$$[X^C, \omega^v] = (L_X \omega)^v, \quad [X^C, \gamma F] = \gamma (L_X F), \quad [X^C, Y^C] = [X, Y]^C,$$

where $\omega \circ F$ is a 1–form defined by $(\omega \circ F)(Z) = \omega(FZ)$ for any $Z \in \mathfrak{X}_0\!(M^n)$ and $L_X$ the Lie derivative in direction of $X$.

**Theorem 4.** The Nijenhuis tensor $N_{(F^{u+1})^C(F^{u+1})^C}(X^C, \omega^v)$ of the complete lift of $F^{u+1}$ vanishes if the Lie derivative of the tensor field $F^{u-1}$ with respect to $X$ is zero and $F$ is an almost $\pi$–structure on $M$ (see [19] p.46).

**Proof.** In consequence of Definition 2 the Nijenhuis tensor of $F^{u+1}$ is given by

$$N_{(F^{u+1})^C(F^{u+1})^C}(X^C, \omega^v)$$

$$= \lambda^4 \bigl\{ (F^{u-1} X)^C + \gamma L_X F^{u-1}, (\omega \circ F^{u-1})^v \bigr\}$$

$$- \lambda^4 \bigl\{ (F^{u-1} X)^C + \gamma L_X F^{u-1}, \omega^v \bigr\}$$

$$- (F^{u-1})^C \bigl[C^{L_X (\omega \circ F^{u-1})} \omega + (F^{2u-2})^C (L_X \omega)^v \bigr]$$

$$= \lambda^4 \bigl\{ (L_{(F^{u-1} X)} (\omega \circ F^{u-1}))^v - ((\omega \circ F^{u-1}) \circ (L_X F^{u-1}))^v - (L_{(F^{u-1} X)} (\omega \circ F^{u-1}))^v - ((\omega \circ (L_X F^{u-1}))^v \circ F^{u-1} v - ((L_X (\omega \circ F^{u-1})) \circ (F^{u-1}))^v + ((L_X (\omega \circ F^{u-1})) \circ (F^{u-1})^v + ((L_X (\omega \circ F^{u-1})) \circ (F^{u-1}))^v \bigr\}$$

If the lie derivatives of the tensor field $F^{u-1}$ with respect to $X$ is zero, then the equation takes the form

$$= \lambda^4 \omega \circ (L_{F^{u-1} X} F^{u-1})^v - ((\omega \circ L_X F^{u-1}) F^{u-1})^v.$$

Let $F$ be almost $\pi$–structure on $M$ then $F^2 = \lambda^2 I$, where $I$ is unit tensor field. So $F^{u-1} = \lambda^2 I$ and we get

$$N_{(F^{u+1})^C(F^{u+1})^C}(X^C, \omega^v) = 0$$

The theorem is proved. \qed
Theorem 5. The Nijenhuis tensor \( N_{(F^v)^1C(F^v+1)^1C} \left( \omega^V, \theta^V \right) \) of the complete lift of \( F^{v+1} \) vanishes.

Proof. Because \( [\omega^V, \theta^V] = 0 \) and \( \omega \circ F^{v-1} \in \mathcal{Z}_0(M^n) \) on \( T^v(M^n) \), the Nijenhuis tensor \( N(\omega^V, \theta^V) \) for the complete lift of \( F^{v+1} \) is vanishes. \( \Box \)

2.2. Tachibana operators applied to vector and covector fields according to lifts of \( F ((v + 1), \lambda^2 (v - 1)) \)-structure on cotangent bundle.

Definition 6. Let \( \varphi \in \mathcal{Z}_1(M^n) \), and \( \mathcal{Z}(M^n) = \sum_{r,s=0}^\infty \mathcal{Z}_{r,s}(M^n) \) be a tensor algebra over \( R \). A map \( \phi_\varphi \mid_{(r+s):} \mathcal{Z}(M^n) \rightarrow \mathcal{Z}(M^n) \) is called a Tachibana operator or \( \phi_\varphi \) operator on \( M^n \) if

\[
\begin{align*}
\phi_\varphi & \text{ is } R-\text{linear}, \\
\phi_\varphi(K \otimes L) & = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L \text{ for all } K, L \in \mathcal{Z}(M^n), \\
\phi_\varphi X & = -(\partial Y \varphi)X \text{ for all } X, Y \in \mathcal{Z}_0(M^n), \text{ where } L_Y \text{ is the Lie derivative in direction of } Y \text{ (see [7,9,15])}, \\
\phi_\varphi Y & = (\partial Y \varphi)X + \eta((L_Y \varphi)X)
\end{align*}
\]

for all \( \eta \in \mathcal{Z}_0(M^n) \) and \( X, Y \in \mathcal{Z}_0(M^n) \), where \( \partial Y \varphi = \eta(Y) = \eta \otimes Y, \mathcal{Z}_{r,s}(M^n) \) the module of all pure tensor fields of type \( (r, s) \) on \( M^n \) with respect to the affinor field, \( \otimes \) is a tensor product with a contraction \( C \) \( [6,8,19] \) (see \( 22 \) for applied to pure tensor field).

Remark 7. If \( r = s = 0 \), then from \( c), d \) and \( e \) of Definition 6 we have \( \phi_\varphi X(\partial Y \varphi) = \phi X(\partial Y \varphi) = \phi X(\partial Y \varphi) - \phi(\partial Y \varphi)X \) for \( \partial Y \varphi = \eta(Y) = \eta \otimes Y, \mathcal{Z}_{r,s}(M^n) \) the module of all pure tensor fields of type \( (r, s) \) on \( M^n \) with respect to the affinor field, \( \otimes \) is a tensor product with a contraction \( C \) \( [6,8,19] \) (see \( 22 \) for applied to pure tensor field).

Consider \( M^n = R^2 \) with standard coordinates \( x, y \). Let \( \varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Consider the function \( f = 1 \). This may be written in many different ways as \( \partial Y \varphi = \eta(Y) \). Indeed taking \( \eta = dx \), we may choose \( Y = \frac{\partial}{\partial x} \) or \( Y = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \). Now the right-hand side of \( \phi X(\partial Y \varphi) = \phi X(\partial Y \varphi) = \phi X(\partial Y \varphi) - \phi(\partial Y \varphi)X \) is \( (\phi X)1 - X1 = 0 \) in the first case, and \( (\phi X)1 - X1 = X1 \) in the second case. For \( X = \frac{\partial}{\partial y} \), the latter expression is \( -1 \neq 0 \). Therefore, we put \( r + s > 0 \) \( [19] \).

Remark 8. From \( d) \) of Definition 6 we have

\[
\phi X Y = [\varphi X, Y] - \varphi[Y, X]. \quad (24)
\]

By virtue of

\[
f X Y = f g [X, Y] + f(Xg) Y - g(Y f) X \quad (25)
\]

\[
\phi_\varphi X Y = [\varphi X, Y] - \varphi[Y, X].
\]

By virtue of

\[
f X Y = f g [X, Y] + f(Xg) Y - g(Y f) X \quad (25)
\]
Theorem 9. Let \((F^{v+1})^C\) be a tensor field of type \((1, 1)\) on \(T^*(M^n)\). If the Tachibana operator \(\phi_{(F^{v+1})^C}\) applied to vector and covector fields according to the structure \((F^{v+1})^C - \lambda^2 (F^{v-1})^C = 0\) defined by \([13]\) on \(T^*(M^n)\), then we get the following results.

i) \(\phi_{(F^{v+1})^C} X \cdot Y^C = -\lambda^2 \{(LYF^{v-1}) \cdot X\}^C + \gamma (LY (LXF^{v-1}))^C - \gamma (L[Y,X]F^{v-1})\}

ii) \(\phi_{(F^{v+1})^C} X \cdot \omega^v = -\lambda^2 \{-(LF_{v-1}X \omega)^v + (\omega \circ (LXF^{v-1})^v + ((LX\omega) \circ F^{v-1})\}

iii) \(\phi_{(F^{v+1})^C} \omega^v \cdot X^C = -\lambda^2 (\omega (LXF^{v-1}))^v\),

iv) \(\phi_{(F^{v+1})^C} \omega^v \cdot \theta^v = 0\),

where the complete lifts \(X^C, Y^C \in \mathfrak{X}_0^1(T^*(M^n))\) of \(X, Y \in \mathfrak{X}_0^1(M)\) and the vertical lift \(\omega^v, \theta^v \in \mathfrak{X}_0^1(T^*(M^n))\) of \(\omega, \theta \in \mathfrak{X}_0^1(M)\) are given, respectively.

Proof. i)

\[\phi_{(F^{v+1})^C} X \cdot Y^C = -(LYC (F^{v+1})^C)X^C\]

\[= -LYC (F^{v+1})^C X^C + (F^{v+1})^C LYCX^C\]

\[= -\lambda^2 \{YC, (F^{v-1}X)^C\} - \lambda^2 [YC, \gamma LXF^{v-1}]\]

\[+ \lambda^2 (F^{v-1}[Y, X])^C + \lambda^2 \gamma (L[Y,X]F^{v-1})\]

\[= -\lambda^2 \{(LYF^{v-1}) X\}^C + \gamma (LY (LXF^{v-1}))^C - \gamma (L[Y,X]F^{v-1})\]\n
ii)

\[\phi_{(F^{v+1})^C} X \cdot \omega^v = -(L\omega^v (F^{v+1})^C)X^C\]

\[= -L\omega^v (F^{v+1})^C X^C + (F^{v+1})^C L\omega^v X^C\]

\[= -\lambda^2 \{\omega^v, (F^{v-1}X)^C + \gamma (LXF^{v-1})\} - \lambda^2 (F^{v-1})^C (LX\omega)^v\]

\[= -\lambda^2 \{-(LF_{v-1}X \omega)^v + (\omega \circ (LXF^{v-1})^v + ((LX\omega) \circ F^{v-1})\}

iii)

\[\phi_{(F^{v+1})^C} \omega^v \cdot X^C = -(LXC (F^{v+1})^C)\omega^v\]

\[= -LXC (F^{v+1})^C \omega^v + (F^{v+1})^C LXC\omega^v\]

\[= -\lambda^2 (LX (\omega \circ F^{v-1}))^V + \lambda^2 ((LX\omega) \circ F^{v-1})^V\]

\[= -\lambda^2 (\omega (LXF^{v-1}))^V\]

iv)

\[\phi_{(F^{v+1})^C} \omega^v \cdot \theta^v = -(L\theta^v (F^{v+1})^C)\omega^v\]

\[= -L\theta^v (F^{v+1})^C \omega^v + (F^{v+1})^C L\theta^v \omega^v\]
Proposition 10. The complete lift \( Y^C \) is an holomorphic vector field with respect to the structure \( (F^{v+1})^C - \lambda^2 (F^{v-1})^C = 0 \), if \( L_Y F^{v-1} = 0 \).

Proof. i) 
\[
(L_Y (F^{v+1})^C)X^C = L_Y (F^{v+1})^C X^C - (F^{v+1})^C L_Y C^X^C \\
= \lambda^2 ((L_Y F^{v-1}) X)^C + \lambda^2 (F^{v-1} (L_Y X))^C \\
+ \lambda^2 \gamma (L_Y (L_X F^{v-1})) - \lambda^2 (F^{v-1} (L_Y X))^C \\
- \lambda^2 \gamma (L_{[Y,X]} F^{v-1}) \\
= \lambda^2 \{((L_Y F^{v-1}) X)^C + \gamma (L_Y (L_X F^{v-1})) \\
- \gamma (L_{[Y,X]} F^{v-1}) \}
\]

ii) 
\[
(L_Y (F^{v+1})^C)\omega^v = L_Y (F^{v+1})^C \omega^v - (F^{v+1})^C L_Y \omega^v \\
= \lambda^2 L_Y (\omega \circ F^{v-1})^v - \lambda^2 ((L_Y \omega) \circ F^{v-1})^v \\
= \lambda^2 (\omega (L_Y F^{v-1}))^v
\]

where \( Y \in \mathfrak{X}_0^1 (M) \) and \( L_Y \) is the Lie derivative in direction of \( Y \). 

2.3. The purity conditions of Sasakian metric with respect to \((F^{v+1})^C\) on \( T^*(M^n) \). Let \( F \) be an affinor field on \( M^n \), i.e. \( F \in \mathfrak{X}_1(M^n) \). A tensor field \( t \) of \((r,s)\) is called pure tensor field with respect to \( F \) if
\[
t(FX_1, X_2, ..., X_s, \xi, \xi, ..., \xi) = t(X_1, FX_2, ..., X_s, \xi, \xi, ..., \xi) \\
... \\
... \\
= t(X_1, X_2, ..., FX_s, \xi, \xi, ..., \xi) \\
= t(X_1, X_2, ..., X_s, F\xi, \xi, ..., \xi) \\
= t(X_1, X_2, ..., X_s, \xi, F\xi, ..., \xi) \\
... \\
...
for any \(X_1, X_2, \ldots, X_s \in \mathcal{H}_0(M^n)\) and \(\xi, \zeta, \ldots, \zeta \in \mathcal{H}_0(M^n)\), where \(T\) is the adjoint operator of \(F\) defined by
\[
(Tx)(X) = \xi(FX) = (\xi \circ F)(X)
\]

**Definition 11.** A Sasakian metric \(Sg\) is defined on \(T^*(M^n)\) by the three equations [20]
\[
\begin{align*}
S g(\omega^v, \theta^v) &= (g^{-1}(\omega, \theta))^v = g^{-1}(\omega, \theta) \circ \pi, \quad (26) \\
S g(\omega^v, Y^H) &= 0, \quad (27) \\
S g(X^H, Y^H) &= (g(X, Y))^v = g(X, Y) \circ \pi. \quad (28)
\end{align*}
\]

For each \(x \in M^n\) the scalar product \(g^{-1} = (g^v)\) is defined on the cotangent space \(\pi^{-1}(x) = T^*_x(M^n)\) by
\[
g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j, \quad (29)
\]
where \(X, Y \in \mathcal{H}_0(M^n)\) and \(\omega, \theta \in \mathcal{H}_0(M^n)\). Since any tensor field of type \((0, 2)\) on \(T^*(M^n)\) is completely determined by its action on vector fields of type \(X^H\) and \(\omega^v\) (see [25], p.280), it follows that \(Sg\) is completely determined by equations \((26), (27)\) and \((28)\).

**Theorem 12.** Let \((T^*(M^n), Sg)\) be the cotangent bundle equipped with Sasakian metric \(Sg\) and a tensor field \((F^{v+1}C)\) of type \((1, 1)\) defined by [16] on \(T^*(M^n)\). Sasakian metric \(Sg\) is pure with respect to \((F^{v+1}C)\) if \(F^{v-1} = \lambda^2 I\) and \(\nabla F^{v-1} = 0\). (\(I = \text{identity tensor field of type } (1, 1)\))

**Proof.** We put
\[
S(\tilde{X}, \tilde{Y}) = S g((F^{v+1}C) \tilde{X}, \tilde{Y}) - g((F^{v+1}C) \tilde{X}, (F^{v+1}C) \tilde{Y}).
\]

If \(S(\tilde{X}, \tilde{Y}) = 0\) for all vector fields \(\tilde{X}\) and \(\tilde{Y}\) which are of the form \(\omega^v, \theta^v\) or \(X^H, Y^H\), then \(S = 0\). By virtue of \((F^{v+1}C) - \lambda^2 (F^{v-1}C) = 0\) and \((26), (27), (28)\), we get
\[
i)
S(\omega^v, \theta^v) = S g((F^{v+1}C) \omega^v, \theta^v) - g((F^{v+1}C) \omega^v, \theta^v),
\]
\[
= S g(\lambda^2 (F^{v-1}C) \omega^v, \theta^v) - g(\lambda^2 (F^{v-1}C) \omega^v, \theta^v),
\]
\[
= \lambda^2 \{ S g((\omega \circ F^{v-1})^v, \theta^v) - g(\omega, \theta \circ F^{v-1})^v) \}
\]
\[
= \lambda^2 \{ g^{-1} ((\omega \circ F^{v-1}), \theta) - g^{-1} (\omega, (\theta \circ F^{v-1}))^v) \}\]
\[
e)
S(X^H, \theta^v) = S g((F^{v+1}C) X^H, \theta^v) - g((F^{v+1}C) X^H, (F^{v+1}C) \theta^v),
\]
\[
= S g(\lambda^2 (F^{v-1}C) X^H, \theta^v) - g((F^{v-1}C) X^H, (F^{v-1}C) \theta^v),
\]
\[ \begin{align*}
&= \lambda^2 \left( S g((F^{v-1} X)^H, \theta^v) \right) + \lambda^2 \left( S g(p \left[ \nabla F^{v-1} \right]_X)^v, \theta^v) \right), \\
&= \lambda^2 \left( g(\left[ \nabla F^{v-1} \right]_X, \theta) \right),
\end{align*} \]

where \( \nabla_X F + F(\nabla_X) - \nabla F X = [\nabla F]_X \) (see [25] p. 279).

iii) \[ S (X^H, Y^H) = S g((F^{v+1})^C X^H, Y^H) - S g(X^H, (F^{v+1})^C Y^H) \]
\[ = \lambda^2 \{ S g((F^{v-1} X)^H, Y^H) + \gamma(\left[ \nabla F^{v-1} \right]_X, Y^H) \}
- S g(X^H, (F^{v-1} Y)^H) + \gamma(\left[ \nabla F^{v-1} \right]_Y, Y^H) \]
\[ = \lambda^2 \{ S g((F^{v-1} X)^H, Y^H) + S g((p(\left[ \nabla F^{v-1} \right]_X)^v, Y^H) \}
- S g(X^H, (F^{v-1} Y)^H) - S g(X^H, (p(\left[ \nabla F^{v-1} \right]_Y)^v) \}
= \lambda^2 \{ g((F^{v-1} X)^v, Y^v) - g(X, (F^{v-1} Y)^v) \}
\]

where \( F^C X^H = (FX)^H + \gamma([\nabla F]_X) \) for all \( X^H \in \mathfrak{h}(T^*(M^n)), F^C \in \mathfrak{z}(T^*(M^n)) \) and \( [\nabla F]_X \in \mathfrak{z}(M^n) \) (see [25], p. 279).

2.4. The structure \( (F^{v+1})^H - \lambda'(F^{v-1})^H = 0 \) on cotangent bundle. In this section, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of \( F^{v+1} - \lambda' F^{v-1} = 0 \) structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of the structure \( F^{v+1} - \lambda' F^{v-1} = 0 \) in cotangent bundle \( T^*(M^n) \). Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of \( F^{v+1} - \lambda' F^{v-1} = 0 \) structure.

**Theorem 13.** The Nijenhuis tensors of \( (F^{v+1})^H \) and \( F^{v-1} \) denote by \( \tilde{N} \) and \( N \), respectively. Thus, taking account of the definition of the Nijenhuis tensor, the formulas [2] stated in Proposition 1 and the structure \( (F^{v+1})^H - \lambda^2 (F^{v-1})^H = 0 \), we find the following results of computation.

i) \[ \tilde{N}_{(F^{v+1})^H(F^{v+1})^H} (X^H, Y^H) = \lambda^4 \{ (N_{F^{v+1} F^{v-1}} (X, Y))^H \\
+ \gamma(R(\left[ \nabla F^{v-1} \right]_X, F^{v-1} Y) - R(F^{v-1} X, Y) F^{v-1} \\
- R(X, F^{v-1} Y) F^{v-1} + R(X, Y) (F^{v-1})^2) \}, \]

ii) \[ \tilde{N}_{(F^{v+1})^H(F^{v+1})^H} (X^H, \omega^v) = \lambda^4 \{ (\omega(\left[ \nabla F^{v-1} \right] X F^{v-1}))^v \\
- (\omega(\left[ \nabla F^{v-1} \right] X F^{v-1}))^v \}, \]

iii) \[ \tilde{N}_{(F^{v+1})^H(F^{v+1})^H} (\omega^v, \theta^v) = 0. \]

**Proof.** The Nijenhuis tensor \( N(X^H, Y^H) \) for the horizontal lift of \( F^{v+1} \) is given by
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i)  
\[ \tilde{N}_{F^u+1} H_{F^u+1} H \left( X^H, Y^H \right) \]
\[ = [(F^u+1)^H X^H, (F^u+1)^H Y^H] - (F^u+1)^H [(F^u+1)^H X^H, Y^H] \]
\[ - (F^u+1)^H [X^H, (F^u+1)^H Y^H] + (F^u+1)^H (F^u+1)^H [X^H, Y^H] \]
\[ = \lambda^4 \left( (F^u-1) X + (F^u-1) Y \right)^H - \gamma R (F^u-1 X, (F^u-1) Y) \]
\[ - (F^u-1)^H \left( [F^u-1 X, Y]^H + \gamma R (F^u-1 X, Y) \right) \]
\[ - (F^u-1)^H \left( [X, (F^u-1) Y]^H + \gamma R (X, (F^u-1) Y) \right) \]
\[ + ((F^u-1)^2 [X, Y])^H + \gamma R (X, Y) (F^u-1)^2 \}
\[ = \lambda^4 \{(N_{F^u-1} + (X, Y))^H + \gamma R (F^u-1 X, F^u-1 Y) \}
- R (F^u-1 X, Y) F^u-1 - R (X, F^u-1 Y) F^u-1 + R (X, Y) (F^u-1)^2 \} \}

Let us suppose that the curvature tensor $R$ of $\nabla$ satisfies
\[ R (F^u-1 X, F^u-1 Y) - R (F^u-1 X, Y) F^u-1 - R (X, F^u-1 Y) F^u-1 + R (X, Y) (F^u-1)^2 = 0 \]
and the Nijenhuis tensor of the $F^u-1$ is zero. So, we get
\[ \tilde{N}_{F^u+1} H_{F^u+1} H \left( X^H, Y^H \right) = 0. \]

ii)  
\[ \tilde{N}_{F^u+1} H_{F^u+1} H \left( X^H, \omega^v \right) \]
\[ = [(F^u+1)^H X^H, (F^u+1)^H \omega^v] - (F^u+1)^H [(F^u+1)^H X^H, \omega^v] \]
\[ - (F^u+1)^H [X^H, (F^u+1)^H \omega^v] + (F^u+1)^H (F^u+1)^H [X^H, \omega^v] \]
\[ = \lambda^4 \{(F^u-1 X)^H + (\omega \circ F^u-1)^v \} - (F^u-1)^H [(F^u-1 X)^H, \omega^v] \]
\[ - (F^u-1)^H [X^H, (\omega \circ F^u-1)^v] + ((F^u-1)^2 [X, \omega])^H (\nabla_\omega^v) \}
\[ = \lambda^4 \{((\nabla_{F^u-1} X) (\omega \circ F^u-1))^v \} - ((\nabla_{F^u-1} X) (F^u-1)^v) \}
\[ - ((\omega \circ F^u-1)) (F^u-1)^v) + ((\omega \circ F^u-1) (F^u-1)^2 v) \}
\[ = \lambda^4 \{(\omega \circ F^u-1 X (F^u-1)^v) \} - ((\omega \circ F^u-1) (F^u-1)^v) \} \}

We now suppose $\nabla^v = 0$, then we see $\tilde{N}_{F^u+1} H_{F^u+1} H \left( X^H, \omega^v \right) = 0.$

iii)  
\[ \tilde{N}_{F^u+1} (F^u+1) \left( \omega^v, \theta^v \right) \]
\[ = [(F^u+1)^H \omega^v, (F^u+1)^H \theta^v] - (F^u+1)^H [(F^u+1)^H \omega^v, \theta^v] \]
\[ - (F^u+1)^H [\omega^v, (F^u+1)^H \theta^v] + (F^u+1)^H (F^u+1)^H [\omega^v, \theta^v] \]
\[ = \lambda^4 \{(\omega \circ F^u-1)^v, (\theta \circ F^u-1)^v \} - F^u-1 [(\omega \circ F^u-1)^v, \theta^v] \]
Proposition 14. Let \((F^{v+1})^H\) be a tensor field of type \((1,1)\) on \(T^*(M^n)\). If the Tachibana operator \(\phi_{(F^{v+1})^H}\) applied to vector and covector fields according to horizontal lifts of \(F^{v+1}\) defined by [20] on \(T^*(M^n)\), then we get the following results.

i) \(\phi_{(F^{v+1})^H} X^H \omega^v = \lambda^2 \{ -((L_Y F^{v-1})^H X)^H - (pR(Y, F^{v-1} X))^v + (pR(Y, X))^v F^{v-1} X^H \} \)

ii) \(\phi_{(F^{v+1})^H} X^H \omega^v = \lambda^2 \{ (\nabla_{F^{v-1}} X) \omega - (\nabla_X \omega) \circ F^{v-1} \} \)

iii) \(\phi_{(F^{v+1})^H} \omega^v X^H = -\lambda^2 (\omega \circ (\nabla_X F^{v-1}))^v \)

iv) \(\phi_{(F^{v+1})^H} \omega^v \theta^v = 0 \)

where horizontal lifts \(X^H, Y^H \in \mathfrak{X}_0^1(T^*(M^n))\) of \(X, Y \in \mathfrak{X}_0^1(M^n)\) and the vertical lift \(\omega^v, \theta^v \in \mathfrak{X}_0^1(T^*(M^n))\) of \(\omega, \theta \in \mathfrak{X}_0^1(M^n)\) are given, respectively.

Proof. i)

\[
\phi_{(F^{v+1})^H} X^H \omega^v = -(L_Y^H (F^{v+1})^H) X^H = -(L_Y (F^{v+1}) X^H + (F^{v+1})^H L_Y X^H = \lambda^2 \{ -((L_Y F^{v-1})^H X)^H - (pR(Y, F^{v-1} X))^v + (pR(Y, X))^v F^{v-1} X^H \}
\]

ii)

\[
\phi_{(F^{v+1})^H} X^H \omega^v = -(L_{\omega^v} (F^{v+1})^H) X^H = -(L_{\omega^v} (F^{v+1}) X^H + (F^{v+1})^H L_{\omega^v} X^H = -\lambda^2 L_{\omega^v} (F^{v-1})^H X^H + \lambda^2 (F^{v-1})^H (\nabla_X \omega)^v = \lambda^2 \{ (\nabla_{F^{v-1}} X) \omega - (\nabla_X \omega) \circ F^{v-1} \}
\]

iii)

\[
\phi_{(F^{v+1})^H} \omega^v X^H = -(L_X^H (F^{v+1})^H) \omega^v = -\lambda^2 (\nabla_X (\omega \circ F^{v-1}))^v + \lambda^2 ((\nabla_X \omega) \circ F^{v-1})^v = -\lambda^2 (\omega \circ (\nabla_X F^{v-1}))^v
\]

iv)

\[
\phi_{(F^{v+1})^H} \omega^v \theta^v = -(L_{\theta^v} (F^{v+1})^H) \omega^v
\]
Proof. (1) Let

\[ \theta \in \mathfrak{X}^\ast(Y^H), \quad a \in \mathfrak{X}^\ast(Y^H) \]

and a tensor field \( g \). Sasakian metric \( g \) is pure with respect to \( (F^\nu+1)^H \) if \( F^\nu-1 = \lambda^2 I \) (\( I = \text{Identity tensor field of type } (1,1) \)).

Proposition 15. The horizontal lift \( Y^H \) is an holomorphic vector field with respect to the structure \( (F^\nu+1)^H - \lambda^2 (F^\nu-1)^H = 0 \). If \( L_Y F^\nu - 1 = 0 \) and \( R(Y, F^\nu-1 X) = -R(Y, X) F^\nu-1 \).

Proof. i)

\[
(L_Y (F^\nu+1)^H) X^H = L_Y (F^\nu+1)^H X^H - (F^\nu+1)^H L_Y X^H
\]

\[
= \lambda^2 ([Y, F^\nu-1 X]^H + \gamma R(Y, F^\nu-1 X))
\]

\[
- \lambda^2 ((F^\nu-1 [Y, X])^H + \lambda R(Y, X) F^\nu-1)
\]

\[
= \lambda^2 \{ ([L_Y F^\nu-1] X)^H + \gamma R(Y, F^\nu-1 X)
\]

\[
- R(Y, X) F^\nu-1\}
\]

ii)

\[
(L_X (F^\nu+1)^H) \omega^v = L_X (F^\nu+1)^H \omega^v - (F^\nu+1)^H L_X \omega^v
\]

\[
= \lambda^2 (\nabla_X (\omega \circ F^\nu-1))^v - \lambda^2 ((\nabla_X \omega) F^\nu-1)^v
\]

\[
= \lambda^2 \{ ([L_X F^\nu-1] X)^v - ((\nabla_X \omega) F^\nu-1)^v \}
\]

Theorem 16. Let \( (T^\ast(M^n), S g) \) be the cotangent bundle equipped with Sasakian metric \( S g \) and a tensor field \( (F^\nu+1)^H \) of type \( (1,1) \) defined by [1]. Sasakian metric \( S g \) is pure with respect to \( (F^\nu+1)^H \) if \( F^\nu-1 = \lambda^2 I \). (\( I = \text{Identity tensor field of type } (1,1) \)).

Proof. We put

\[
S(\check{X}, \tilde{Y}) = S g((F^\nu+1)^H \check{X}, \tilde{Y}) - S g(\check{X}, (F^\nu+1)^H \tilde{Y}).
\]

If \( S(\check{X}, \tilde{Y}) = 0 \), for all vector fields \( \check{X} \) and \( \tilde{Y} \) which are of the form \( \omega^v, \theta^v \) or \( X^H, Y^H \), then \( S = 0 \). By virtue of \( (F^\nu+1)^H - \lambda^2 (F^\nu-1)^H = 0 \) and [26], [27], [28], we get

i)

\[
S(\omega^v, \theta^v) = S g((F^\nu+1)^H \omega^v, \theta^v) - S g(\omega^v, (F^\nu+1)^H \theta^v)
\]

\[
= \lambda^2 \{ S g((\omega \circ F^\nu-1)^v, \theta^v) - S g(\omega^v, (\theta \circ F^\nu-1)^v) \}
\]

\[
= \lambda^2 \{ (g^{-1}(\omega \circ F^\nu-1), \theta)^v - g^{-1}(\omega, (\theta \circ F^\nu-1))^v \}
\]

ii)

\[
S(X^H, \theta^v) = S g((F^\nu+1)^H X^H, \theta^v) - S g(X^H, (F^\nu+1)^H \theta^v)
\]
\[ S (X^H, Y^H) = S g((F^{v-1})^H X^H, Y^H) - S g(X^H, (F^{v-1})^H Y^H) \]
\[ = \lambda^2 (S g((F^{v-1})^H X^H, Y^H) - S g(X^H, (F^{v-1})^H Y^H)) \]
\[ = \lambda^2 (S g((F^{v-1})^H X^H, Y^H) - S g(X^H, (F^{v-1})^H Y^H)) \]

We now suppose \( F^{v-1} = \lambda^2 I \), then we get \( S g = 0 \). So, \( S g \) is pure with respect to \( (F^{v-1})^H \).

2.5. The structure \( (F^{v+1})^C - \lambda^2 (F^{v-1})^C = 0 \) on tangent bundle \( T (M^n) \). Let \( M^n \) be an \( n \)-dimensional connected differentiable manifold of class \( C^\infty \). Let there be given in \( M^n \), a \((1,1)\)-tensor field \( F \) of class \( C^\infty \) satisfying [14][21]

\[ F^{v+1} - \lambda^2 F^{v-1} = 0, \quad (30) \]

where \( \lambda \) is non zero complex number. Also rank \( (F) \)
\[ = \frac{1}{2} \left( \text{rank } F^{v+1} + \dim M^n \right) \]
\[ = r \left( \text{a constant every where on } M^n \right) \]

Let the operators \( l^* \) and \( m^* \) be defined as
\[ l^* \text{def} \ (F/\lambda)^{v-1}, \quad m^* = I - (F/\lambda)^{v-1}, \]
where \( I \) denotes the identity operator on \( M^n \). Then the operators \( l^* \) and \( m^* \) applied to the tangent space at a point of the manifold be complementary projection operators.

Let \( F^h_i \) be the component of \( F \) at \( A \) in the coordinate neighbourhood \( U \) of \( M^n \). Then the complete lift \( F^C \) of \( F \) is also a tensor field of type \((1,1)\) in \( T(M^n) \) whose components \( F^C_i \) in \( \pi^{-1}(U) : M^n \rightarrow T(M^n) \) are given by [25]

\[ F^C = \begin{pmatrix}
F^h_i & 0 \\
\partial F^h_i & F^h_i
\end{pmatrix}. \quad (31) \]

Let \( F, G \in \mathcal{A}_1 (M^n) \) then we have
\[ (FG)^C = F^C G^C. \quad (32) \]

Putting \( F = G \) we obtain
\[ (F^2)^C = (F^C)^2. \quad (33) \]

Putting \( G = F^2 \) in [32] and making use of [33] we get
\[ (F^3)^C = (F^C)^3. \quad (34) \]
Continuing the above process of replacing $G$ in equation (32) by some higher degree of $F$ we obtain
\[(F^{v+1})^C = (Fc)^{v+1}.\] (35)

Taking complete lift on both sides of equation (30) we get
\[(F^{v+1})^C - \lambda^2 (F^{v-1})^C = 0\] (36)
which in view of the equation (35) gives
\[(Fc)^{v+1} - \lambda^2 (Fc)^{v-1} = 0.\] (37)

The complete lift of a $F^{v+1} - \lambda^2 F^{v-1} = 0$ structure also has $F^{v+1} - \lambda^2 F^{v-1} = 0$ structure in tangent bundle.

**Lemma 17.** Let $X$ and $Y$ be any vector fields on a Riemannian manifold $(M^n, g)$, we have [25]
\[
\begin{align*}
[X^H, Y^H] &= [X, Y]^H - (R(X, Y) u)^v, \\
[X^H, Y^v] &= (\nabla_X Y)^v, \\
[X^v, Y^v] &= 0,
\end{align*}
\]
where $R$ is the Riemannian curvature tensor of $g$ defined by
\[R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.
\]

In particular, we have the vertical spray $u^v$ and the horizontal spray $u^H$ on $T(M^n)$ defined by
\[u^v = u^i (\partial_i)^v = u^i \delta_i, \quad u^H = u^i (\partial_i)^H = u^i \delta_i,
\]
where $\delta_i = \partial_i - u^j \Gamma_{ji} \partial_j$. $u^v$ is also called the canonical or Liouville vector field on $T(M^n)$.

**Theorem 18.** The Nijenhuis tensor $N_{(F^{v+1})^C(F^{v+1})^C} X^C, Y^C$ of the complete lift of $F^{v+1}$ vanishes if the Nijenhuis tensor of the $F^{v-1}$ is zero.

**Proof.** In consequence of Definition [2] the Nijenhuis tensor of $(F^{v+1})^C$ is given by
\[
\begin{align*}
N_{(F^{v+1})^C(F^{v+1})^C} X^C, Y^C &= \lambda^4 \{[F^{v-1}X]^C, (F^{v-1}Y)^C\} - (F^{v-1})^C ([F^{v-1}X]^C, Y^C) \\
&\quad - (F^{v+1})^C [X^C, (F^{v+1})^C Y^C] + (F^{v+1})^C (F^{v+1})^C [X^C, Y^C] \\
&= \lambda^4 \{[F^{v-1}X]^C, (F^{v-1}Y)^C\} - (F^{v-1})^C ([F^{v-1}X]^C, Y^C) \\
&\quad - (F^{v-1})^C [X^C, (F^{v-1}Y)^C] + (F^{v-1})^C (F^{v-1})^C [X^C, Y^C] \\
&= \lambda^4 \{[F^{v-1}X, F^{v-1}Y] - F^{v-1} [F^{v-1}X, Y] \\
&\quad - F^{v-1} [X, F^{v-1}Y] + F^{v-1} F^{v-1} [X, Y]^C \\
&= \lambda^4 N_{F^{v-1}F^{v-1}} (X, Y)^C
\end{align*}
\]
Theorem 19. The Nijenhuis tensor $N_{(F^{v+1})^C(F^{v+1})^C}(X^C, Y^v)$ of the complete lift of $F^{v+1}$ vanishes if the Nijenhuis tensor $F^{v-1}$ is zero.

Proof. 

\[
N_{(F^{v+1})^C(F^{v+1})^C}(X^C, Y^v) \\
= [(F^{v+1})^C X^C, (F^{v+1})^C Y^v] - (F^{v+1})^C [(F^{v+1})^C X^C, Y^v] \\
- (F^{v+1})^C [X^C, (F^{v+1})^C Y^v] + (F^{v+1})^C (F^{v+1})^C [X^C, Y^v] \\
= \lambda^4 ((F^{-1}X)^C, (F^{-1}Y)^v) - (F^{-1})^C [(F^{-1}X)^C, Y^v] \\
- (F^{-1})^C [X^C, (F^{-1}Y)^v] + (F^{-1})^C (F^{-1})^C [X, Y]^v \\
= \lambda^4 [(F^{-1}X, F^{-1}1)^v - (F^{-1})^C [F^{-1}X, Y]^v] \\
- (F^{-1})^C [X, (F^{-1}Y)^v] - (F^{-1}F^{-1})^C [X, Y]^v] \\
= \lambda^4 N_{F^{-1}F^{-1}}(X, Y)^v \]

\[\square\]

Theorem 20. The Nijenhuis tensor $N_{(F^{v+1})^C(F^{v+1})^C}(X^e, Y^v)$ of the complete lift of $F^{v+1}$ vanishes.

Proof. Because $[X^e, Y^v] = 0$ and $F^{-1}X \in \mathfrak{S}_0(M^n)$, easily we get the Nijenhuis tensor $N_{(F^{v+1})^C(F^{v+1})^C}(X^e, Y^v) = 0$. \[\square\]

2.6. The purity conditions of Sasakian metric with respect to $(F^{v+1})^C$ on $T(M^n)$.

Definition 21. The Sasaki metric $S$ is a (positive definite) Riemannian metric on the tangent bundle $T(M^n)$ which is derived from the given Riemannian metric on $M$ as follows:

\[
S_g(X^H, Y^H) = g(X, Y), \\
S_g(X^e, Y^e) = S_g(X^e, Y^H) = 0, \\
S_g(X^e, Y^v) = g(X, Y)
\]

for all $X, Y \in \mathfrak{S}_0(M^n)$ [20].

Theorem 22. The Sasaki metric $S$ is pure with respect to $(F^{v+1})^C$ if $\nabla F^{v-1} = 0$ and $F^{v-1} = \lambda^2 I$, where $I$ is the identity tensor field of type $(1, 1)$.

Proof. $S(\tilde{X}, \tilde{Y}) = S_g((F^{v+1})^C \tilde{X}, \tilde{Y}) - S_g(\tilde{X}, (F^{v+1})^C \tilde{Y})$ if $S(\tilde{X}, \tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form $X^e, Y^v$ or $X^H, Y^H$ then $S = 0$.

i) $S(X^e, Y^v) = S_g((F^{v+1})^C X^e, Y^v) - S_g(X^e, (F^{v+1})^C Y^v)$
Theorem 23. Let $(F^{v+1})^C$ be a tensor field of type $(1,1)$ on $T(M^n)$. If the Tachibana operator $\phi_{(F^{v+1})^C}$ applied to vector fields according to complete lifts of $F^{v+1}$ defined by (36) on $T(M^n)$, then we get the following results.

\begin{align*}
i & \phi_{(F^{v+1})^C} X^C Y^C = -\lambda^2 ((LY^{F^{v-1}})X)^C, \\
ii & \phi_{(F^{v+1})^C} X^C Y^v = -\lambda^2 ((LY^{F^{v-1}})X)^v, \\
iii & \phi_{(F^{v+1})^C} X^v Y^C = -\lambda^2 ((LY^{F^{v-1}})X)^v, \\
iv & \phi_{(F^{v+1})^C} X^v Y^v = 0,
\end{align*}

where $X, Y \in \mathfrak{X}(M)$, the complete lifts $X^C, Y^C \in \mathfrak{X}(T(M))$ and the vertical lift $X^v, Y^v \in \mathfrak{X}(T(M))$.

Proof. \(i\)

\begin{align*}
\phi_{(F^{v+1})^C} X^C Y^C &= -(LY^C (F^{v+1})^C)X^C \\
&= \lambda^2 \{-LY^C (F^{v-1})^C + (F^{v-1})^C LY^C X^C\} \\
&= -\lambda^2 ((LY^{F^{v-1}})X)^C
\end{align*}

\(ii\)

\begin{align*}
\phi_{(F^{v+1})^C} X^C Y^v &= -(LY^v (F^{v+1})^C)X^C
\end{align*}
\[
\begin{align*}
L_Y (F^u+1)^C X^C + (F^u+1)^C L_Y X^C \\
\lambda^2 \{-L_Y (F^u-1)^C + (F^u-1)^C L_Y X^C\} \\
-\lambda^2 ((L_Y F^u-1) X)^v \\
= -L_Y (F^u+1)^C X^v \\
-\lambda^2 \{-L_Y (F^u-1)^C + (F^u-1)^C L_Y X^v\} \\
= -\lambda^2 ((L_Y F^u-1) X)^v \\
= -2 (L_Y F^u-1) X^v \\
= 0
\end{align*}
\]

**Theorem 24.** The complete lift \( Y^C \) is an holomorphic vector field with respect to the structure \((F^u+1)^C - \lambda^2 (F^u-1)^C = 0\), if \( L_Y F^u-1 = 0 \).

**Proof.** i)
\[
(L_Y (F^u+1)^C) X^C = L_Y (F^u+1)^C X^C - (F^u+1)^C L_Y X^C \\
= \lambda^2 \{-L_Y (F^u-1)^C + (F^u-1)^C L_Y X^C\} \\
= \lambda^2 ((L_Y F^u-1) X)^C
\]

ii)
\[
(L_Y (F^u+1)^C) X^v = L_Y (F^u+1)^C X^v - (F^u+1)^C L_Y X^v \\
= \lambda^2 \{-L_Y (F^u-1)^C + (F^u-1)^C L_Y X^v\} \\
= \lambda^2 ((L_Y F^u-1) X)^v
\]

2.7. The structure \((F^u+1)^H - \lambda^2 (F^u-1)^H = 0\) on tangent bundle \( T(M^n) \).

Let \( F^b_i \) be the component of \( F \) at \( A \) in the coordinate neighbourhood \( U \) of \( M^n \). Then the horizontal lift \( F^H \) of \( F \) is also a tensor field of type \((1, 1)\) in \( T(M^n) \) whose components \( F^b_i \) in \( \pi^{-1}(U) : M^n \rightarrow T(M^n) \) are given by
\[
F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F^b_i \\ -\Gamma^b_i \end{pmatrix} = \begin{pmatrix} F^b_i - \Gamma^b_i \Gamma^b_i F^b_i \\ \Gamma^b_i \Gamma^b_i F^b_i \end{pmatrix}.
\]
Let $F$, $G$ be two tensor fields of type $(1,1)$ on the manifold $M$. If $F^H$ denotes the horizontal lift of $F$, we have

$$(FG)^H = F^HG^H$$

(40)

Taking $F$ and $G$ identical, we get

$$(F^H)^2 = (F^2)^H$$

(41)

Multiplying both sides by $F^H$ and making use of the same (41), we get

$$(F^H)^3 = (F^3)^H.$$  

Thus it follows that

$$(F^H)^{u+1} = (F^{u+1})^H$$

(42)

Taking horizontal lift on both sides of equation $F^{u+1} - \lambda^2 F^{-1} = 0$ we get

$$(F^{u+1})^H - \lambda^2 (F^{-1})^H = 0$$

(43)

In view of (42), we can write

$$(F^H)^{u+1} - \lambda^2 (F^H)^{-1} = 0.$$  

(44)

Thus the horizontal lift of $F^{u+1} - \lambda^2 F^{-1} = 0$ structure also has $F^{u+1} - \lambda^2 F^{-1} = 0$ structure in tangent bundle $T(M^n)$.

**Theorem 25.** The Nijenhuis tensor $N_{(F^{u+1})^H (F^{u+1})^H} (X^H, Y^H)$ of the horizontal lift of $F^{u+1}$ vanishes if the Nijenhuis tensor of the $F^{-1}$ is zero and

$$\{-(\hat{R}(F^{-1}X, F^{-1}Y)u) + (F^{-1}(\hat{R}(F^{-1}X, Y)u))
\}
+ (F^{-1}(\hat{R}(X, F^{-1}Y)u)) - ((F^{-1})^2(\hat{R}(X, Y)u))\} = 0$$

Proof.

$$N_{(F^{u+1})^H (F^{u+1})^H} (X^H, Y^H)$$

$$= ([F^{u+1}]^H X^H, [F^{u+1}]^H Y^H]
- ([F^{u+1}]^H X^H, Y^H]
- ([F^{u+1}]^H Y^H, [F^{u+1}]^H X^H]
+ ([F^{u+1}]^H Y^H, [F^{u+1}]^H X^H]
= \lambda^2
(\{[F^{u+1}X, F^{u+1}Y] - (F^{-1})^2 [F^{u+1}X, Y]
- (F^{-1}) [X, F^{u+1}Y] - (F^{-1}) [X, Y])^H
- (\hat{R}(F^{-1}X, F^{-1}Y)u) + (F^{-1}(\hat{R}(F^{-1}X, Y)u))v
+ (F^{-1}(\hat{R}(X, F^{-1}Y)u))v - ((F^{-1})^2(\hat{R}(X, Y)u))v\}
= \lambda^2
= \lambda^2((N_{F^{u+1}F^{-1}} (X, Y))^H - (\hat{R}(F^{u+1}X, F^{u+1}Y)u)^v$
\[
+ (F^{v-1}(\hat{R}(F^{v-1}X,Y)u))^v + (F^{v-1}(\hat{R}(X,F^{v-1}Y)u))^v
\]
\[\]
\[- ((F^{v-1})^2(\hat{R}(X,Y)u))^v}.\]

Proof. If \(N_{F^{v-1}F^{v-1}}(X,Y) = 0\) and \(\{ - \hat{R}(F^{v-1}X,F^{v-1}Y) u + (F^{v-1}(\hat{R}(F^{v-1}X,Y)u)) + (F^{v-1}(\hat{R}(X,F^{v-1}Y)u)) - ((F^{v-1})^2(\hat{R}(X,Y)u)) \}^v = 0\), then we get \(N_{(F^{v+1})^H(F^{v+1})^H} (X^H,Y^H) = 0\). The theorem is proved.

Where \(\hat{R}\) denotes the curvature tensor of the affine connection \(\nabla\) defined by \(\hat{\nabla}_X Y = \nabla_Y X + [X,Y]\) (see [25] p.88-89).

**Theorem 26.** The Nijenhuis tensor \(N_{(F^{v+1})^H(F^{v+1})^H} (X^H,Y^v)\) of the horizontal lift of \(F^{v+1}\) vanishes if the Nijenhuis tensor of the \(F^{v-1}\) is zero and \(\nabla F^{v-1} = 0\).

**Proof.**
\[
N_{(F^{v+1})^H(F^{v+1})^H} (X^H,Y^v) = \{(F^{v+1})^H X^H, (F^{v+1})^H Y^v\}
\]
\[\]
\[= \lambda^4 \{(F^{v-1}X,F^{v-1}Y)^v - ((F^{v-1})^2 [X,Y])^v
\]
\[+ (\nabla_{F^{v-1}Y} F^{v-1}X)^v - (F^{v-1} (\nabla_Y F^{v-1}X))^v
\]
\[\}

\[
= \lambda^4 \{(N_{F^{v-1}F^{v-1}} (X,Y))^v + (\nabla_{F^{v-1}Y} F^{v-1}X)^v
\]
\[\}

\]

**Theorem 27.** The Nijenhuis tensor \(N_{(F^{v+1})^H(F^{v+1})^H} (X^v,Y^v)\) of the horizontal lift of \(F^{v+1}\) vanishes.

**Proof.** Because \([X^v,Y^v] = 0\) for \(X,Y \in M\), we get \(N_{(F^{v+1})^H(F^{v+1})^H} (X^v,Y^v) = 0\).

**Theorem 28.** The Sasakian metric \(Sg\) is pure with respect to \((F^{v+1})^H\) if \(F^{v-1} = \lambda^2 I\), where \(I\) = identity tensor field of type \((1,1)\).

**Proof.** \(S(\tilde{X},\tilde{Y}) = S g((F^{v+1})^H \tilde{X},\tilde{Y}) = S g(\tilde{X},(F^{v+1})^H \tilde{Y})\) if \(S(\tilde{X},\tilde{Y}) = 0\) for all vector fields \(\tilde{X}\) and \(\tilde{Y}\) which are of the form \(X^v,Y^v\) or \(X^H,Y^H\) then \(S = 0\).

i) \(S(X^v,Y^v) = S g((F^{v+1})^H X^v,Y^v) = S g(X^v,(F^{v+1})^H Y^v)\)
Theorem 29. Let the Tachibana operator applied to vector fields according to horizontal lifts of \( F^{v+1} \) on \( T(M^n) \), then we get the following results.

\[
\begin{align*}
\phi_{(F^{v+1})H} X^H &= -\lambda^2 \{ -((L_Y F^{v-1})X)^H + (\hat{R}(Y, F^{v-1}X)u)^v \} \\
\phi_{(F^{v+1})H} Y^v &= \lambda^2 \{ -((L_Y F^{v-1})Y)^v - (\nabla_{F^{v-1}X} Y)^v + (F^{v-1}(\nabla_X Y))^v \} \\
\end{align*}
\]

where \( X, Y \in \mathfrak{g}_0(T(M^n)) \), the horizontal lifts \( X^H, Y^H \in \mathfrak{g}_0(T(M^n)) \) and the vertical lift \( X^v, Y^v \in \mathfrak{g}_0(T(M^n)) \)

Proof. i)

\[
\phi_{(F^{v+1})H} X^H = -(L_Y H X^{(F^{v+1})H}) X^H
= -\lambda^2 [Y, F^{v-1}X]^H + \lambda^2 \gamma \hat{R} [Y, F^{v-1}X]
+ \lambda^2 (F^{v-1} Y, X)^H - \lambda^2 (F^{v-1} \hat{R}(Y, X) u)^v
= \lambda^2 \{ -((L_Y F^{v-1})X)^H + (\hat{R}(Y, F^{v-1}X)u)^v \}
\]

ii)

\[
\phi_{(F^{v+1})H} Y^v = -(L_Y V (F^{v+1})H) X^H
\]
\[
\begin{align*}
&= -\lambda^2 [Y, F^{u+1}X]^v + \lambda^2 (\nabla_Y F^{u+1}X)^v \\
&\quad + \lambda^2 (F^{u+1} [Y, X])^v - \lambda^2 (F^{u-1} (\nabla_Y X))^v \\
&= \lambda^2 \left\{ - \left[ (L_Y F^{u-1}) X + (\nabla_Y F^{u-1}X)^v \right] \right. \\
&\quad \left. + \lambda^2 (F^{u-1} (\nabla_X Y))^v \right\}
\end{align*}
\]

\[
\phi_{(F^{u+1})^H} X^v = -(L_Y^H (F^{u+1})^H) X^v \\
= -\lambda^2 \nabla_Y (F^{u-1}X)^v + \lambda^2 (F^{u-1})^H L_Y^v X^v \\
= 0
\]

**Theorem 30.** The horizontal lift \( Y^H \) is an holomorphic vector field with respect to \((F^{u+1})^H\), if \( L_Y F^{u-1} = 0 \) and \( F^{u-1} = \lambda^2 I \) for \( Y \in M \).

**Proof.** i)

\[
(L_Y^H (F^{u+1})^H) X^H = L_Y^H (F^{u+1})^H X^H - (F^{u+1})^H L_Y^H X^H \\
= \lambda^2 [Y, F^{u+1}X]^H - \lambda^2 \hat{R} (Y, F^{u+1}X) \\
- \lambda^2 (F^{u-1} [Y, X])^H + \lambda^2 (F^{u-1} (\hat{R} (Y, X) u))^v \\
= \lambda^2 \left\{ ((L_Y F^{u-1}) X)^H - (\hat{R} (Y, F^{u-1}X u))^v \right. \\
\quad \left. + (F^{u-1} (\hat{R} (Y, X) u))^v \right\}
\]

ii)

\[
(L_Y^H (F^{u+1})^H) X^v = L_Y^H (F^{u+1}X)^v - (F^{u+1})^H L_Y^H X^v \\
= \lambda^2 [Y, F^{u-1}X]^v - \lambda^2 (\nabla_Y F^{u-1}X)^v - \lambda^2 (F^{u-1} [Y, X])^v \\
- \lambda^2 (F^{u-1} (\nabla_X Y))^v \\
= \lambda^2 \left\{ ((L_Y F^{u-1}) X)^H + (\nabla_Y F^{u-1}X)^v - (F^{u-1} (\nabla_X Y))^v \right\}
\]

**Declaration of Competing Interests** The authors declare that they have no competing interests.
Some Notes on Lifts of the $F((v+1), \lambda^2(v-1))$-Structure on Bundles

References


[23] Sahin, B., Semi-invariant Riemannian submersions from almost Hermitian manifolds, 
