



A Different Interpretation on Magnetic Surfaces Generated by Special Magnetic

Curve in $Q^2 \subset E_1^3$

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Abstract

By thinking the magnetic flow connected by the Killing magnetic field, the magnetic field on the setting out particle orbit is investigated in $Q^2 \subset E_1^3$. Clearly, dealing with the Killing magnetic field of α -magnetic curve, the rotational surface generated by α -magnetic is expressed in $Q^2 \subset E_1^3$, and the variant kinds of axes of rotation in lightlike cone $Q^2 \subset E_1^3$ is given. Furthermore, the specific kinetic energy, specific angular momentum and conditions being geodesic on rotational surface generated by α -magnetic curve are expressed with the help of Clairaut's theorem.

Keywords: Magnetic curve; Null cone; Killing vector field; Specific kinetic energy; Specific angular momentum.



$Q^2 \subset E_1^3$ 'de Özel Manyetik Eğri Tarafından Oluşturulan Manyetik Yüzeyler Üzerine Farklı Bir Yorum

Öz

Killing manyetik alanı ile bağlantılı manyetik akış düşünülerek, parçacık yörüngesindeki manyetik alan $Q^2 \subset E_1^3$ uzayında incelendi. Açıkçası, $Q^2 \subset E_1^3$ Lightlike cone uzayında farklı tipten dönme eksenini verilerek α -manyetik eğrinin Killing manyetik alanı kullanılıp α -manyetik dönelel yüzey $Q^2 \subset E_1^3$ uzayında ifade edildi. Ayrıca, elde edilen manyetik yüzey üzerinde spesifik kinetik enerji, spesifik açısal momentum ve dönelel yüzey üzerinde α -manyetik eğrilerin jeodezik olma koşulları Clairaut's teoremi yardımı ile ifade edildi.

Anahtar Kelimeler: Manyetik eğri; Null koni; Killing vektör alanı; Spesifik kinetik enerji; Spesifik açısal momentum.

1. Introduction

The geodesics have been commonly studied in Riemannian geometry, metric geometry and general relativity by a lot of mathematicians. More definitely, a curve on a surface is called to be geodesic if its geodesic curvature is zero. The geodesic equations are given by constant of motion due to energy, many approaches that reflect serious use of energy idea are introduced in many books according to concerned topics. However, it seems attractive to use the relativistic energy in describing the central force problem. Furthermore, the equation of action including the energy and angular momentum are a natural topic using by many applications. Though we consider about the submanifolds of the pseudo-Riemannian space forms, also we can obtain less studying on submanifolds of the pseudo-Riemannian lightlike cone than we think.

In [1], different magnetic curves were found in the 2 –dimensional lightlike cone using the Killing magnetic field of magnetic curves by the authors. Also, some characterizations and definitions and examples of these curves with their shapes were given. Studying on the degenerate submanifolds of Lorentzian manifolds with degenerate metric was studied by a lot of mathematicians finding out significant connection between null submanifolds and spacetime [2]. In [3, 4], the authors gave some knowledge about magnetic curves corresponding to a Killing magnetic fields. The magnetic curves on a Riemannian manifold (M, d) were defined orbits of charged particles setting out on M under the motion of a magnetic field F . Namely, each trajectory δ is obtained by solving the Lorentz equation $\nabla_{\delta'} \delta' = \psi(\delta')$, where ψ is the Lorentz force as to F and ∇ is the Levi Civita connection of d , [4]. A magnetic field F on M is a closed 2 –form on

(M, d) and it is related to F with a (1,1)-tensor field ψ , is said to be the Lorentz force. They are associated as to $d(\psi(X), Y) = F(X, Y)$, for any vector fields X, Y on M , [5]. In [6], the magnetic flow combination by the Killing magnetic field was examined by Bozkurt et al. in a three-dimensional orientated Riemannian manifold (M^3, d) . For the study of the magnetic curves associated to magnetic fields on arbitrary dimensional spaces, we also refer the reader to [7-9]. References [10-13] contain detailed information about surfaces and curves. In [14], the authors expressed a precise classification of the magnetic curves of the resembling magnetic field for an discretionary 3-dimensional normal paracontact metric structure equipped by a Killing characteristic vector field. In [15], magnetic curves as to the Killing magnetic field W in the \mathbb{R}_1^3 were examined by the authors. In [16, 17], the authors expressed some characterizations about curves in 3-Dimensional null cone. In [18], the expressions of the cone curvature function and cone curves were investigated by the author. In [19], the functions of the cone curves that was defined and the formulas of the curves were also given by the author in \mathbf{Q}^2 and \mathbf{Q}^3 .

The physical properties as energy and momentum are replaced by the specific quantities found by partitioning out the mass, and the term of the motion is very considerable in terms of its specific energy and specific angular momentum. In an evident sense, it is considered that use of relativistic energy is required and considerable.

Hence, we can say that the specific energy of the particle is constant because of the point of view of its motion in space as the physical approach according to references [20, 21], it is only accelerated perpendicular to the surface. If a force is accountable for this acceleration, that is to say the normal force which supplies the particle on the surface, since it is perpendicular to the velocity of the particle. Therefore, we can say that its energy and specific energy E must be constant. Resembling the speed must be constant along a geodesic according to this cause.

In this study, the specific energy and specific angular momentum on rotated surface generated by α -magnetic curve are tried to express in Galilean space and that the speed is constant along a geodesic is shown using Clairaut's theorem. Furthermore, using some parameters geodesic formulas are given.

2. Preliminaries

Let E_1^3 be the 3-dimensional pseudo-Euclidean space as follows

$$d(X, Y) = \langle X, Y \rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

for all $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in E_1^3$. E_1^3 is a smooth pseudo-Riemannian manifold pointing out by (2,1).

Let M be a submanifold of E_1^3 . If the pseudo-Riemannian metric d of E_1^3 is reduced a pseudo-Riemannian metric \tilde{d} (in turn in order, a Riemannian metric, a degenerate quadratic form) on M , then M is named a timelike (in turn in order, spacelike, lightlike) submanifold of E_1^3 .

The lightlike cone is given by

$$\mathbf{Q}^2 = \{\delta \in E_1^3 : d(\delta, \delta) = 0\}.$$

A vector $X \neq 0$ in E_1^3 is called spacelike, timelike, null, if $\langle X, X \rangle > 0, \langle X, X \rangle < 0, \langle X, X \rangle = 0$, in turn in order. A frame field $\{\delta, \alpha, y\}$ on E_1^3 is called an asymptotic orthonormal frame field, if following equals hold

$$\langle \delta, \delta \rangle = \langle y, y \rangle = \langle \delta, \alpha \rangle = \langle y, \alpha \rangle = 0, \langle \delta, y \rangle = \langle \alpha, \alpha \rangle = 1.$$

Let the curve $\delta: I \rightarrow \mathbf{Q}^2 \subset E_1^3$ be a regular curve in \mathbf{Q}^2 for $\xi \in I$ and for $\delta'(\xi) = \alpha(\xi)$, using an asymptotic orthonormal frame along the curve $\delta(\xi)$ and the cone Frenet formulas of $\delta(\xi)$ are written as follows

$$\begin{aligned} \delta'(\xi) &= \alpha(\xi) \\ \alpha'(\xi) &= \kappa(\xi)\delta(\xi) - y(\xi) \\ y'(\xi) &= -\kappa(\xi)\alpha(\xi), \end{aligned} \tag{1}$$

where cone curvature function of the curve $\delta(\xi)$ is expressed by the function $\kappa(\xi)$, [18].

Let $\delta: I \rightarrow \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in \mathbf{Q}^2 with arc length parameter s . Then the curve $\delta = \delta(s) = (\delta_1, \delta_2, \delta_3)$ can be taken down by

$$\delta(s) = \frac{g_s^{-1}}{2}(g^2 - 1, 2g, g^2 + 1),$$

for some non constant function $g(s)$ and $g_s = g'$, [19].

The Lorentzian cross-product $\times: E_1^3 \times E_1^3 \rightarrow E_1^3$ is expressed with following formula

$$\beta \times \varsigma = \begin{bmatrix} i & j & -k \\ \beta_1 & \beta_2 & \beta_3 \\ \varsigma_1 & \varsigma_2 & \varsigma_3 \end{bmatrix},$$

where $\beta = (\beta_1, \beta_2, \beta_3)$, $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in E_1^3$. Here i, j, k indices are used as common meaning. We can express that this product has resembling algebra properties as the cross product in E^3 . Thus, it is antisymmetric and $\beta \times \zeta$ is orthogonal on both β and ζ .

The Lorentz force ψ of a magnetic field F on \mathbf{Q}^2 is defined to be a skew-symmetric operator given by

$$d(\psi(X), Y) = F(X, Y),$$

for all $X, Y \in \mathbf{Q}^2$, [5].

The α –magnetic trajectories of F are δ on \mathbf{Q}^2 that satisfy the Lorentzian equation [5]

$$\nabla_{\delta'} \delta' = \psi(\delta').$$

In addition, the mixed product of the vector fields $X, Y, Z \in \mathbf{Q}^2$ is defined by

$$d(X \times Y, Z) = dv_d(X, Y, Z),$$

where a volume on \mathbf{Q}^2 is denoted by dv_d and if W is a Killing vector in \mathbf{Q}^2 and let $F_W = \iota_W vol_d$ be the Killing magnetic field and the inner product is expressed by ι . Thus, the equation Lorentz force of F_W is given by

$$\psi(X) = W \times X, \forall X \in \mathbf{Q}^2.$$

Clearly, the Lorentz equation is expressed as [5]

$$\nabla_{\delta'} \delta' = \psi(\delta') = W \times \delta'.$$

In E_1^3 , to think over the Killing vector field $W = a\partial_\delta + b\partial_\alpha + c\partial_y$, $a, b, c \in \mathbb{R}_0$, solutions of the Lorentz equation given by

$$\delta'' = W \times \delta',$$

are the magnetic trajectories $\delta: I \rightarrow \mathbf{Q}^2 \subset E_1^3$ determined by W , [5].

Definition 1. Let γ be a curve given by

$$\gamma(s) = (x(w(s), v(s)), y(w(s), v(s)), z(w(s), v(s)))$$

which is an arc-length parametrized geodesic on a surface of revolution. We need the differential equations satisfied by $(w(s), v(s))$. Denote the differentiation with respect to s by an overdot. From the Lagrangian

$$L = \dot{w}^2 + \rho^2 \dot{v}^2, \tag{2}$$

we obtain the Euler-Lagrange equations

$$\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{w}} \right) = \frac{\partial L}{\partial w}; \frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{v}} \right) = \frac{\partial L}{\partial v} \text{ or } \ddot{u} = \rho \rho' \dot{v}^2; \frac{d}{ds} (\rho \dot{v}^2) = 0, \tag{3}$$

so that is a constant of the motion [5, 15].

Theorem 1. (Clairaut’s Theorem) Let γ be a geodesic on a surface of revolution S , let ρ be the distance function of a point of S from the axis of rotation, and let θ be the angle between γ and the meridians of S . The $\rho \sin \theta$ is constant along γ . On the contrary, if $\rho \sin \theta$ is constant along some curve γ on the surface, and if no part of γ is part of some parallel of S , then γ is a geodesic [5].

Definition 2. A one-parameter group of diffeomorphisms of a manifold M is a smooth map $\psi: M \times \mathbb{R} \rightarrow M$, such that $\varphi_t(x) = \varphi(x, t)$, where

1. $\varphi_t: M \rightarrow M$ is a diffeomorphism,
2. $\varphi_0 = id$.
3. $\varphi_{s+t} = \varphi_s \circ \varphi_t$.

This group is associated with a vector field W given by $\frac{d}{dt} \varphi_t(x) = W(x)$, and the group of diffeomorphisms is called the flow of W [22].

If a one-parameter group of isometries is generated by a vector field W , then this vector field is called a Killing vector field [22].

3. The Expression of α –Magnetic Curves in $\mathbb{Q}^2 \subset E_1^3$

In this section, a new kind of a magnetic curve called α –magnetic curves in $\mathbb{Q}^2 \subset E_1^3$ and some theorems are given.

Definition 3. Let $\delta: I \rightarrow Q^2 \subset E_1^3$ be a spacelike curve in Q^2 and F_W be a magnetic field on $Q^2 \subset E_1^3$, the curve δ is called as α –magnetic curve if its W_α vector field satisfies the Lorentz force equation [1]

$$\nabla_\alpha \alpha = \psi^\alpha(\alpha) = W_\alpha \times \alpha.$$

Theorem 2. Let $\delta(s)$ be a spacelike α –magnetic curve in the $Q^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{\delta, \alpha, y\}$. Hence, the Lorentz force is expressed by

$$\psi^\alpha = \begin{bmatrix} w_1 & 1 & 0 \\ \kappa & 0 & -1 \\ 0 & -\kappa & -w_1 \end{bmatrix}, \tag{4}$$

where w_1 is a function defined by $w_1 = d(\psi^\alpha(\delta), y)$ [1].

Theorem 3. Let δ be a spacelike curve in the $Q^2 \subset E_1^3$. The curve δ is an α –magnetic trajectory of α –magnetic field W_α if and only if the vector field W_α is written by

$$W_\alpha = \mp w_1 \vec{\alpha}, \tag{5}$$

and δ is a geodesic curve, where $w_1 = d(\psi^\alpha(\delta), y)$, the cone curvature function $\kappa(\xi) = -1$ [1].

Theorem 4. Let δ be an α –magnetic trajectory generated by the Killing vector field $W_\alpha = \mp w_1 \vec{\alpha}$ in $Q^2 \subset E_1^3$. Then the curve δ is written by

$$\delta_\alpha(\xi) = \delta(0) + c\xi, \tag{6}$$

where $c = \mp \frac{1}{w_1} \in \mathbb{R}_0$. Remark that, if $W_\alpha = \mp w_1 \vec{\alpha}$ holds, the magnetic curve δ is a straight line in the direction of W_α [1].

4. The Surface of Rotation Formed by α –magnetic Curve in $Q^2 \subset E_1^3$

In this section, using α –magnetic trajectory, a new kind of a magnetic surface of rotated by α –magnetic curve is defined, and some characterizations are given in $Q^2 \subset E_1^3$.

Theorem 5. Let δ be an α –magnetic trajectory as to the killing vector field W_α in $Q^2 \subset E_1^3$. Then

i) The rotational surface $A^\alpha(\xi, t)$ formed by the ψ^α is given by

$$\Lambda^\alpha(\xi, t) = \left(\begin{array}{l} \left(\frac{2\kappa \sinh(w_1 \xi)}{w_1^3} + \frac{2\kappa \cosh(w_1 \xi)}{w_1} \right) (at + b), \\ \left(-1 - \frac{\xi}{w_1^2} - \frac{\xi^2}{2!} + w_1 e^{w_1 \xi} + \left(\frac{4\kappa^2 \xi^5}{5!} + \dots \right) \right) (at + b), \\ (\cosh(\sqrt{2\kappa} \xi) + \frac{2\kappa w_1^2 \xi^4}{4!} + \dots)(at + b), \\ \left(\frac{\kappa e^{w_1 \xi}}{w_1} - 2\kappa^2 \sinh(\sqrt{2\kappa} \xi) + \left(\frac{-4(\kappa w_1)^2 \xi^5}{5!} + \dots \right) \right) (at + b) \end{array} \right).$$

ii) The Gaussian and mean curvatures of the rotated surface Λ_1^α generated by $\delta_\alpha(t)$ are given by

$$K = \frac{- \left(\left(\frac{\kappa' + \kappa w_1}{w_1} + \frac{\sqrt{2\kappa^2 \kappa'}}{\sqrt{\kappa}} \right) \left(\frac{2}{w_1^2} \right) \right)^2}{b \zeta_1},$$

$$H = \frac{-a}{2 \zeta_1} \left(\begin{array}{l} \left(\frac{\kappa' + \kappa w_1}{w_1} + \frac{\sqrt{2\kappa^2 \kappa'}}{\sqrt{\kappa}} \right) \\ \left(\frac{4\kappa'}{w_1^2} + \frac{2\kappa'' + 2\kappa w_1^2}{w_1} \right) \\ -1 + w_1^3 \\ -\kappa' \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) \\ + \frac{\kappa' \kappa}{w_1} \left(\frac{2\kappa - 1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \end{array} \right) \left(\begin{array}{l} \left(\frac{2\kappa}{w_1} \right)^2 \\ -1 \\ + w_1 \\ 1 - \left(\frac{\kappa}{w_1} \right)^2 \end{array} \right) +$$

$$+ \frac{2a}{w_1^2 \xi_1} \left(\begin{array}{l} \left(\frac{\kappa' + \kappa w_1}{w_1} \right) \\ \left(\frac{2\kappa - 1}{w_1} + w_1 \right) \\ + \frac{\sqrt{2\kappa^2 \kappa'}}{\sqrt{\kappa}} \end{array} \right) \left(\begin{array}{l} \left(\frac{2\kappa - 1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \\ \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) \\ -\kappa \left(\frac{\kappa' + \kappa w_1}{w_1} \right) \\ + \frac{\sqrt{2\kappa^2 \kappa'}}{\sqrt{\kappa}} \end{array} \right),$$

where the previous equations are consisted without loss of generality for $\xi, t = 0$.

iii) If $\kappa = \text{constant}$, the Gaussian and mean curvatures of the rotated surface Λ_1^α generated by $\delta_\alpha(t)$ are given by

$$K = \frac{\xi_3}{\zeta_2}, \quad H = \frac{-\xi_4}{2\zeta_2},$$

where

$$\xi_2 = \begin{pmatrix} b^2 \left(\left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right)^2 - \kappa^2 (1 - 2\kappa\sqrt{2\kappa})^2 \right) \left(\left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa^2}{w_1} \right) \\ - \left(\left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa^2}{w_1} (1 - 2\kappa\sqrt{2\kappa})b \right)^2 \end{pmatrix},$$

$$\xi_3 = -8b^2\kappa^5 \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right)^2,$$

$$\xi_4 = \begin{pmatrix} b^2\kappa(2\kappa w_1 - 1 + w_1^3)(1 - 2\kappa\sqrt{2\kappa}) \\ + 2\kappa^2 b^2 \begin{pmatrix} \frac{1}{w_1} \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \\ -(1 - 2\kappa\sqrt{2\kappa}) \left(\frac{2\kappa}{w_1} + w_1 - 1 \right) \end{pmatrix} \\ - \kappa b^2 w_1 \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \end{pmatrix} \left(\left(\frac{2\kappa}{w_1} + w_1 - 1 \right)^2 + 1 - \left(\frac{\kappa}{w_1} \right)^2 \right)$$

$$- 2 \left(\left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa^2}{w_1} (1 - 2\kappa\sqrt{2\kappa})b \right) \left(2b\kappa^2\sqrt{2\kappa} \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \right),$$

where the curvature of the curve δ is κ and $\delta_\alpha(t) = (0, at + b, 0)$, $a \neq 1$, $b \in \mathbb{R}_0$.

Proof. We will research one parameter group of Lorentz of transformation which is unchangeable all points on the α –axis. It necessitates the Killing vector field to supply $W_\alpha(\xi) = w_1 \overrightarrow{\alpha(\xi)}$. Hence, we can use 3×3 matrix ψ^α , and we can write the one parameter group of homomorphism $\varphi_\xi(\delta, \alpha, y)$ expressed as $\varphi'_\xi(\delta) = \psi^\alpha \varphi_\xi(\delta)$. Therefore, we find $\varphi_\xi = e^{\xi \psi_\xi}$ and calculating the matrix exponential, we have

$$\Pi^\alpha(\xi) = \begin{bmatrix} \frac{2\kappa \sinh(w_1 \xi)}{w_1^3} + \frac{2\kappa \cosh(w_1 \xi)}{w_1} & & \\ w_1 \sinh(\sqrt{2\kappa} \xi) - 1 + \cosh(\sqrt{2\kappa} \xi) - w_1 e^{w_1 \xi} + \left(\frac{4\kappa w_1 \xi^5}{5!} + \dots\right) & + w_1 e^{w_1 \xi} + \left(\frac{4\kappa^2 \xi^5}{5!} + \dots\right) & \frac{\cosh(\sqrt{w_1} \xi)}{w_1} + \frac{2\kappa \xi^4}{4!} + \dots \\ \frac{\kappa(\cosh(w_1 \xi) + \cosh(\sqrt{w_1} \xi))}{w_1} + 2\kappa^2 \left(\frac{\xi^3}{3!} + \dots\right) + \left(\frac{2w_1 \kappa^2 \xi^4}{4!} + \dots\right) & \cosh(\sqrt{2\kappa} \xi) + \frac{2\kappa w_1^2 \xi^4}{4!} + \dots & \frac{\sinh(w_1 \xi) - \cosh(w_1 \xi)}{w_1} + \frac{2\kappa w_1 \xi^4}{4!} + \dots \\ \frac{\kappa^2 \cos(w_1 \xi)}{w_1} + \frac{2\kappa^3 \xi^4}{4!} + \dots & \frac{\kappa e^{w_1 \xi}}{w_1} - 2\kappa^2 \sinh(\sqrt{2\kappa} \xi) + \left(\frac{-4(\kappa w_1)^2 \xi^5}{5!} + \dots\right) & \kappa \cosh(\sqrt{2\kappa} \xi) e^{-w_1 \xi} - \frac{\xi \sinh(\sqrt{2\kappa} \xi)}{\sqrt{2\kappa}} + \left(\frac{4\kappa w_1^3 \xi^5}{5!} + \dots\right) \end{bmatrix}, \tag{7}$$

where $-\infty < \xi < \infty$. Here, we deal with the Lorentz force ψ^α rotation for α –magnetic curve. Also, we say rotating a curve taking the rotation matrix $\Pi^\alpha(\xi)$, and here the axis of rotation is written as $W_\alpha(\xi) = w_1 \overrightarrow{\alpha(\xi)}$. Barely, we can carry any point in \mathbf{Q}^2 to the α –axis using some expressions, we can suppose that the curve δ_α lies on α –axis. Therefore, we can give one of its parametrizations as follows

$$\delta_\alpha(t) = (0, at + b, 0), a \neq 1, b \in \mathbb{R}_0.$$

Namely, the rotated surface $\Lambda_{W_\alpha}^\alpha$ around W_α can be parametrized by

$$\begin{aligned} \Lambda^\alpha(\xi, t) &= \Pi^\alpha(\xi) \times \begin{bmatrix} 0 \\ at + b \\ 0 \end{bmatrix} \\ &= \left(\left(\frac{2\kappa \sinh(w_1 \xi)}{w_1^3} + \frac{2\kappa \cosh(w_1 \xi)}{w_1} - 1 - \frac{\xi}{w_1^2} - \frac{\xi^2}{2!} + w_1 e^{w_1 \xi} + \left(\frac{4\kappa^2 \xi^5}{5!} + \dots\right) \right) (at + b), \right. \\ &\quad \left. \left(\cosh(\sqrt{2\kappa} \xi) + \frac{2\kappa w_1^2 \xi^4}{4!} + \dots \right) (at + b), \right. \\ &\quad \left. \left(\frac{\kappa e^{w_1 \xi}}{w_1} - 2\kappa^2 \sinh(\sqrt{2\kappa} \xi) + \left(\frac{-4(\kappa w_1)^2 \xi^5}{5!} + \dots\right) \right) (at + b) \right); \quad -\infty < \xi < \infty, t \in I. \end{aligned}$$

Hence, we have researched the rotated surface Λ^α without loss of generality we suppose that $\xi, t = 0$. Then, we get the first and second fundamental forms as follows

$$\begin{aligned}
 E &= b^2 \left(\left(\frac{2\kappa-1}{w_1^2} + \frac{2\kappa'}{w_1} \right)^2 - \left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right)^2 \right), \\
 F &= ab \left(\left(\frac{2\kappa-1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa}{w_1} \left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right) \right), \\
 G &= a^2 \left(\left(\frac{2\kappa}{w_1} - 1 + w_1 \right)^2 + 1 - \left(\frac{\kappa}{w_1} \right)^2 \right), \\
 L &= ab^2 \left(\left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right) \cdot \left(\frac{4\kappa'}{w_1^2} + \frac{2\kappa''+2\kappa w_1^2}{w_1} - 1 + w_1^3 - \kappa' \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) \right) \right. \\
 &\quad \left. + \frac{\kappa'\kappa}{w_1} \left(\frac{2\kappa-1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \right), \\
 M &= ba^2 \left(\left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right) \left(\frac{2}{w_1^2} \right) \right), \quad N = 0; \\
 n_{\Lambda_1^\alpha} &= ab \left(\begin{array}{c} \left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right), \\ \left(\frac{\kappa}{w_1} \left(\frac{2\kappa-1}{w_1^2} + \frac{2\kappa'}{w_1} \right) - \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) \left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right) \right), \\ \left(\frac{2\kappa-1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \end{array} \right). \tag{8}
 \end{aligned}$$

Hence, these results in the first and second fundamental form are given by

$$\begin{aligned}
 I_{\Lambda_1^\alpha} &= a^2 b^2 \left(\left(\left(\frac{2\kappa-1}{w_1^2} + \frac{2\kappa'}{w_1} \right)^2 - \left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right)^2 \right) \cdot \left(\left(\frac{2\kappa}{w_1} - 1 + w_1 \right)^2 + 1 - \left(\frac{\kappa}{w_1} \right)^2 \right) \right. \\
 &\quad \left. - \left(\left(\frac{2\kappa-1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa}{w_1} \left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right) \right)^2 \right) = a^2 b^2 \zeta_1 \\
 II_{\Lambda_1^\alpha} &= -b^2 a^4 \left(\left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right) \left(\frac{2}{w_1^2} \right) \right)^2.
 \end{aligned}$$

So, by using formulas, we obtain the Gaussian and the mean curvatures as follows

$$K = \frac{- \left(\left(\frac{\kappa'+\kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2\kappa'}{\sqrt{\kappa}} \right) \left(\frac{2}{w_1^2} \right) \right)^2}{b\zeta_1};$$

$$H = \frac{-a}{2\xi_1} \left(\begin{array}{l} \left(\frac{\kappa' + \kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2 \kappa'}{\sqrt{\kappa}} \right) \\ \left(\frac{4\kappa'}{w_1^2} + \frac{2\kappa'' + 2\kappa w_1^2}{w_1} - 1 + w_1^3 - \kappa' \left(\frac{2\kappa}{w_1} - 1 \right) \right) \\ + \frac{\kappa' \kappa}{w_1} \left(\frac{2\kappa - 1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \end{array} \right) \left(\left(\frac{2\kappa}{w_1} - 1 + w_1 \right)^2 + 1 - \left(\frac{\kappa}{w_1} \right)^2 \right) + \frac{4a}{w_1^2 \xi_1} \left(\frac{\kappa' + \kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2 \kappa'}{\sqrt{\kappa}} \right) \left(\left(\frac{2\kappa - 1}{w_1^2} + \frac{2\kappa'}{w_1} \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa}{w_1} \left(\frac{\kappa' + \kappa w_1}{w_1} + \frac{\sqrt{2}\kappa^2 \kappa'}{\sqrt{\kappa}} \right) \right).$$

If $\kappa = \text{constant}$, we can give the following equations for K and H ,

$$E = b^2 \left(\left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right)^2 - \kappa^2 (1 - 2\kappa\sqrt{2\kappa})^2 \right),$$

$$F = \left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa^2}{w_1} (1 - 2\kappa\sqrt{2\kappa})b,$$

$$G = \left(\frac{2\kappa}{w_1} + w_1 - 1 \right)^2 + 1 - \left(\frac{\kappa}{w_1} \right)^2,$$

$$L = b^2 \kappa (2\kappa w_1 - 1 + w_1^3) (1 - 2\kappa\sqrt{2\kappa}) + 2\kappa^2 b^2 \left(\frac{1}{w_1} \left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right) - (1 - 2\kappa\sqrt{2\kappa}) \left(\frac{2\kappa}{w_1} + w_1 - 1 \right) - \kappa b^2 w_1 \left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right) \right); M = 2b\kappa^2 \sqrt{2\kappa} \left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right); N = 0;$$

$$n_{\Lambda_2^\alpha} = b \left(\begin{array}{l} \kappa (1 - 2\kappa\sqrt{2\kappa}), \\ \kappa \left(\frac{1}{w_1} \left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right) - (1 - 2\kappa\sqrt{2\kappa}) \left(\frac{2\kappa}{w_1} + w_1 - 1 \right) \right), \\ \left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right) \end{array} \right).$$

And, these results in the first and the second fundamental form are given by

$$I_{\Lambda_2^\alpha} = \left(\begin{array}{l} b^2 \left(\left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right)^2 - \kappa^2 (1 - 2\kappa\sqrt{2\kappa})^2 \right) \cdot \left(\left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa^2}{w_1} \right) \\ - \left(\left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) - \frac{\kappa^2}{w_1} (1 - 2\kappa\sqrt{2\kappa})b \right)^2 \end{array} \right) = \zeta_2$$

$$II_{\Lambda_2^\alpha} = -8b^2 \kappa^5 \left(\frac{2\kappa - 1}{w_1^2} + w_1^2 \right)^2 = \xi_3.$$

By using formulas $K = \frac{LN-M^2}{EG-F^2}$, $H = -\frac{1}{2} \frac{LG-2MF+NE}{EG-F^2}$, we have the Gaussian and mean curvatures as follows

$$K = \frac{\xi_3}{\zeta_2},$$

$$H = \frac{-1}{2\zeta_2} \left(\begin{array}{l} \left(\begin{array}{l} b^2\kappa(2\kappa w_1 - 1 + w_1^3)(1 - 2\kappa\sqrt{2\kappa}) \\ + 2\kappa^2 b^2 \left(\frac{1}{w_1} \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \right. \\ \left. - (1 - 2\kappa\sqrt{2\kappa}) \left(\frac{2\kappa}{w_1} + w_1 - 1 \right) \right) \\ - \kappa b^2 w_1 \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \end{array} \right) \\ \left(\frac{2\kappa}{w_1} + w_1 - 1 \right)^2 + 1 - \left(\frac{\kappa}{w_1} \right)^2 \\ - 2 \left(\begin{array}{l} \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \left(\frac{2\kappa}{w_1} - 1 + w_1 \right) \\ - \frac{\kappa^2}{w_1} (1 - 2\kappa\sqrt{2\kappa}) b \end{array} \right) \\ \left(2b\kappa^2\sqrt{2\kappa} \left(\frac{2\kappa-1}{w_1^2} + w_1^2 \right) \right) \end{array} \right) = \frac{-\xi_4}{2\zeta_2}.$$

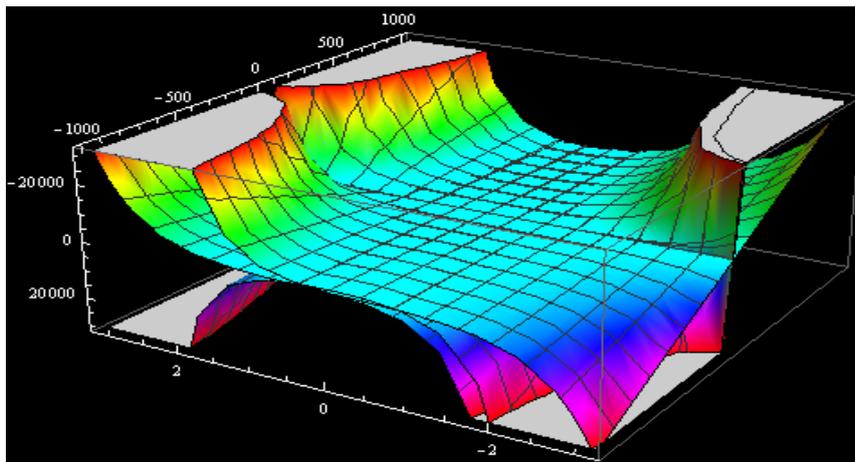


Figure 1: The α – magnetic surface formed by the W_α trajectory

4.1. The Clairaut’s Theorem on magnetic surface generated by α –magnetic curve in $Q^2 \subset E_1^3$

In this section, Clairaut’s theorem is given on magnetic surface generated by α –magnetic curve in $Q^2 \subset E_1^3$. Also, the general equations of geodesics on surface formed by an α –magnetic curve in Q^2 are expressed.

Theorem 6. Let $A^\alpha(\xi, t)$ be the α –magnetic surface generated by α –magnetic curve and let $\delta_\alpha(t): I \subset \mathbb{R} \rightarrow Q^2$ be a regular curve in Q^2 . Then the following statements are held

1. For $h(\xi) = \text{constant}$, the following equation supplies

$$0 = \begin{pmatrix} \frac{4}{w_1^3} (\sinh^2(w_1\xi) + \cosh^2(w_1\xi)) - 2(\frac{f'}{w_1^3} + f)\sinh(w_1\xi) \\ -2\cosh(w_1\xi)(\frac{f'}{w_1} + \frac{f}{w_1^2}) + \frac{\sqrt{2}}{2} \sin(2\sqrt{2}\xi) + g' \cos(\sqrt{2}\xi) \\ -\sqrt{2}g\sin(\sqrt{2}\xi) + 2(\frac{1}{w_1^3} + \frac{1}{w_1})\sinh(2w_1\xi) + ff' + gg' \end{pmatrix}.$$

Hence, the Lagrange equation on the magnetic surface $\Lambda^\alpha(\xi, t)$ is given by

$$E(\xi, t)\dot{\xi}^2 + G(\xi)t^2 = L.$$

2. The curve $\delta(s) = \Lambda^\alpha(\xi(s), t(s))$ is a geodesic on the surface $\Lambda^\alpha(\xi, t)$ if and only if the following equations satisfy

$$\xi = \int \frac{\cos\theta}{at+b} ds + c_8 \text{ (or } \xi = \int \frac{\cos\theta}{at+b} ds), \quad 2 \int E(\xi, t)d\xi = c_5s + c_6,$$

$$0 = 2G(\xi)t - \left\{ \int \frac{\partial E(\xi, t)}{\partial t} \dot{\xi} ds + c_4 \right\},$$

or

$$t = \int \sin\theta ds \text{ (or } t = \int \sin\theta ds + c_3), \quad t = \frac{c_1s}{2G(\xi)} + c_2,$$

$$2E(\xi, t)\ddot{\xi} = G_\xi \dot{t}^2 - E_\xi \dot{\xi}^2,$$

where $c_i \in \mathbb{R}_0$.

Proof. Let $\Lambda^\alpha(\xi, t)$ be the magnetic surface generated by α –magnetic curve and let $\delta_\alpha(t): I \subset \mathbb{R} \rightarrow \mathbf{Q}^2$ be a regular curve in \mathbf{Q}^2 are parametrized by

$$\Lambda^\alpha(\xi, t) = \begin{pmatrix} (\frac{2\kappa\sinh(w_1\xi)}{w_1^3} + \frac{2\kappa\cosh(w_1\xi)}{w_1} + f(\xi))(at + b), \\ (\cosh(\sqrt{2}\kappa\xi) + g(\xi))(at + b), \\ (\frac{\kappa e^{w_1\xi}}{w_1} - 2\kappa^2 \sinh(\sqrt{2}\kappa\xi) + h(\xi))(at + b); \end{pmatrix}$$

where

$$f(\xi) = -1 - \frac{\xi}{w_1^2} - \frac{\xi^2}{2!} + w_1 e^{w_1\xi} + \frac{4\kappa^2\xi^5}{5!} + \dots, \tag{9}$$

$$g(\xi) = \frac{2\kappa w_1^2 \xi^4}{4!} + \dots; h(\xi) = \frac{-4(\kappa w_1)^2 \xi^5}{5!} + \dots$$

Also,

$$\Lambda_{\xi}^{\alpha} = \left\{ \begin{array}{l} \left(\left(\frac{2\kappa'}{w_1^3} + 2\kappa \right) \sinh(w_1\xi) \right. \\ \left. + \left(\frac{2\kappa}{w_1^2} + \frac{2\kappa'}{w_1} \right) \cosh(w_1\xi) + f'(\xi) \right) (at + b), \\ \left(\left(\sqrt{2\kappa} + \frac{\sqrt{2\kappa'}}{2\sqrt{\kappa}} \xi \right) \sinh(\sqrt{2\kappa}\xi) + g'(\xi) \right) (at + b), \\ \left(\begin{array}{l} -4\kappa\kappa' \sinh(\sqrt{2\kappa}\xi) \\ -2\kappa' \left(\sqrt{2\kappa} + \frac{\sqrt{2\kappa'}}{2\sqrt{\kappa}} \xi \right) \cosh(\sqrt{2\kappa}\xi) + h'(\xi) \end{array} \right) (at + b) \end{array} \right\} = (at + b)N_{\xi},$$

$$\Lambda_t^{\alpha} = \left(\begin{array}{l} \left(\frac{2\kappa \sinh(w_1\xi)}{w_1^3} + \frac{2\kappa \cosh(w_1\xi)}{w_1} + f(\xi) \right) a, \\ \left(\cosh(\sqrt{2\kappa}\xi) + g(\xi) \right) a, \\ \left(\frac{\kappa e^{w_1\xi}}{w_1} - 2\kappa^2 \sinh(\sqrt{2\kappa}\xi) + h(\xi) \right) a; \end{array} \right) = N_t.$$

Hence, we have

$$E(\xi, t) = (at + b)^2 \left(\begin{array}{l} A^2 \sinh^2(w_1\xi) + 2A \sinh(w_1\xi) (B \cosh(w_1\xi) + f'(\xi)) \\ + B^2 \cosh^2(w_1\xi) + 2B \cosh(w_1\xi) f'(\xi) + f'^2(\xi) \\ + (C^2 - 16\kappa^2 \kappa') \sinh^2(\sqrt{2\kappa}\xi) + 2C \sinh(\sqrt{2\kappa}\xi) g'(\xi) \\ + g'^2(\xi) + 8\kappa\kappa' \sinh(\sqrt{2\kappa}\xi) (h'(\xi) - 2\kappa' C \cosh(\sqrt{2\kappa}\xi)) \\ - h'^2(\xi) + 4\kappa' C \cosh(\sqrt{2\kappa}\xi) - 4\kappa'^2 C^2 \cosh^2(\sqrt{2\kappa}\xi) \end{array} \right),$$

$$E(\xi, t) = P(t)N(\xi),$$

where $A = \frac{2\kappa'}{w_1^3} + 2\kappa, B = \frac{2\kappa}{w_1^2} + \frac{2\kappa'}{w_1}, C = \sqrt{2\kappa} + \frac{\sqrt{2\kappa'}}{2\sqrt{\kappa}} \xi$.

$$G(\xi) = a \left(\begin{array}{l} \frac{4\kappa^2}{w_1^6} \sinh^2(w_1\xi) + \frac{2\kappa}{w_1^3} \sinh(w_1\xi) \left(\frac{2\kappa}{w_1} \cosh(w_1\xi) + f(\xi) \right) \\ + \frac{4\kappa^2}{w_1^2} \cosh^2(w_1\xi) + \frac{4\kappa}{w_1} f(\xi) \cosh(w_1\xi) + f^2(\xi) \\ + \cosh^2(\sqrt{2\kappa}\xi) + 2g(\xi) \cosh(\sqrt{2\kappa}\xi) + g^2(\xi) \\ - 4\kappa^4 \sinh^2(\sqrt{2\kappa}\xi) + 4\kappa^2 \sinh(\sqrt{2\kappa}\xi) h(\xi) - h^2(\xi) \end{array} \right), \tag{10}$$

for the following equation

$$h(\xi) = \text{constant},$$

$$0 = \left(\begin{array}{l} \frac{4}{w_1^3} (\sinh^2(w_1\xi) + \cosh^2(w_1\xi)) \\ - 2 \left(\frac{f'}{w_1^3} + f \right) \sinh(w_1\xi) - 2 \cosh(w_1\xi) \left(\frac{f'}{w_1} + \frac{f}{w_1^2} \right) \\ + \frac{\sqrt{2}}{2} \sin(2\sqrt{2}\xi) + g' \cos(\sqrt{2}\xi) - \sqrt{2} g \sin(\sqrt{2}\xi) \\ + 2 \left(\frac{1}{w_1^5} + \frac{1}{w_1} \right) \sinh(2w_1\xi) + ff' + gg' \end{array} \right), \tag{11}$$

where for the α –magnetic curve, we have $\kappa = -1, \frac{1}{w_1} = c$, we obtain $F(\xi) = 0$. Thus, the first fundamental form is given by

$$I = \begin{pmatrix} E(\xi, t) & 0 \\ 0 & G(\xi) \end{pmatrix}. \tag{12}$$

Moreover, it is important to note that, the coordinates of parametrization are orthogonal, since the first fundamental form is diagonal. So, from the first fundamental form, we have the Lagrangian equation given by

$$E(\xi, t)d^2\xi + G(\xi)d^2t = L \text{ or } E(\xi, t)\dot{\xi}^2 + G(\xi)\dot{t}^2 = L, \tag{13}$$

and a geodesic on the surface $\Lambda^\alpha(\xi, t)$ is given by the Euler-Lagrangian equations,

$$\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{\xi}} \right) = \frac{\partial L}{\partial \xi}; \frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{t}} \right) = \frac{\partial L}{\partial t}.$$

i) For the equation

$$2E(\xi, t)\ddot{\xi} = G_\xi \dot{t}^2 - E_\xi \dot{\xi}^2, \tag{14}$$

and from the equation $\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{t}} \right) = \frac{\partial L}{\partial t} = 0$, we obtain $\frac{\partial}{\partial s} (2G(\xi)\dot{t}) = 0$, which means $2G(\xi)\dot{t}$ is constant along the geodesic and we have

$$t = \frac{c_1 s}{2G(\xi)} + c_2. \tag{15}$$

Let δ_α be a geodesic on the surface $\Lambda^\alpha(\xi, t)$, so the curve is written as $(\xi(s), t(s))$, also let θ be the angle between $\dot{\delta}_\alpha$ and a meridian and N_ξ is the vector pointing along meridians of Λ^α and N_t is the vector pointing along meridians of Λ^α . We can say that $\{N_\xi, N_t\}$ orthonormal basis and hence a unit vector $\dot{\delta}$ tangent to $\Lambda^\alpha(\xi, t)$ can be written by

$$\dot{\delta} = \dot{\xi}\Lambda_\xi^\alpha + \dot{t}\Lambda_t^\alpha = N_\xi \cos\theta + N_t \sin\theta = \dot{\xi}(at + b)N_\xi + \dot{t}N_t.$$

We see that $\dot{t} = \sin\theta$, hence we write

$$2G(\xi)\dot{t} = 2G(\xi)\sin\theta \tag{16}$$

being a constant along δ_α . On the contrary, let δ be α -magnetic curve with $2G(\xi)\dot{t} = 2G(\xi)\sin\theta$ is a constant. Hence, the second Euler-Lagrange equation is satisfied, differentiating L and substituting this into the first equation yields the first Euler-Lagrange equation. Furthermore, we can also write

$$t = \int \sin\theta ds \text{ or } t = \int \sin\theta ds + c_3. \tag{17}$$

ii) For the equation

$$2G(\xi)\dot{t} - \left\{ \int \frac{\partial E(\xi,t)}{\partial t} \dot{\xi} ds + c_4 \right\} = 0, \tag{18}$$

from the equation $\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial w} \right) = \frac{\partial L}{\partial w} = 0$, we obtain $\frac{\partial L}{\partial w} = 2E(\xi,t)\dot{\xi}$ is constant and which means

$$2 \int E(\xi,t) d\xi = c_5 s + c_6,$$

which has a constant along the geodesic. We see that $(at + b)\dot{\xi} = \cos\theta$, hence we write

$$2E(\xi,t)\dot{\xi} = \frac{2E(\xi,t)}{at+b} \cos\theta, \tag{19}$$

being a constant along δ_α . On the contrary, δ_α is a curve with $2E(\xi,t)\dot{\xi} = \frac{2E(\xi,t)}{at+b} \cos\theta$ that it is a constant. Hence, the first Euler-Lagrange equation is satisfied, differentiating L and substituting this into the second equation yields the second Euler-Lagrange equation. Furthermore, we can write equation as follows

$$\xi = \int \frac{\cos\theta}{at+b} ds \text{ or } \xi = \int \frac{\cos\theta}{at+b} ds + c_8. \tag{20}$$

Theorem 7. The general equations of geodesics on the surface generated by an α -magnetic curve in Q^2 are given by

i) For the parameter $\xi = \int \frac{\cos\theta}{at+b} ds + c_8$ (or $\xi = \int \frac{\cos\theta}{at+b} ds$) and the equations $2 \int E(\xi,t) d\xi = c_5 s + c_6$, $2G(\xi)\dot{t} - \left\{ \int \frac{\partial E(\xi,t)}{\partial t} \dot{\xi} ds + c_4 \right\} = 0$, the following equation holds

$$\frac{dt}{d\xi} = c_{11} \sqrt{\frac{E(\xi,t)}{G(\xi)}} \sqrt{LE(\xi,t) - c_{10}} = \frac{at + b}{\sqrt{G(\xi)} \cos\theta} \sqrt{L - \frac{E(\xi,t) \cos^2\theta}{at + b}}.$$

ii) For the parameters $t = \int \sin\theta ds$ (or $t = \int \sin\theta ds + c_3$) or $t = \frac{c_1 s}{2G(\xi)} + c_2$ and the equation $2E(\xi,t)\ddot{\xi} = G_\xi \dot{t}^2 - E_\xi \dot{\xi}^2$, the following equation holds

$$\frac{d\xi}{dt} = c_1 G(\xi) \sqrt{\frac{G(\xi)\varepsilon - c_2}{E(\xi, t)G(\xi)}} = \frac{1}{\sin\theta} \sqrt{\frac{\varepsilon - G(\xi)\sin^2\theta}{E(\xi, t)}}$$

where $c_i \in \mathbb{R}_0, i \in I$.

Proof. In order to obtain the general equation of geodesics, we should use the Euler Lagrange equations,

i) For the parameters $\xi = \int \frac{\cos\theta}{at+b} ds + c_8$ (or $\xi = \int \frac{\cos\theta}{at+b} ds$) and the equations $2 \int E(\xi, t)d\xi = c_5s + c_6$,

$$2G(\xi)\dot{t} - \left\{ \int \frac{\partial E(\xi, t)}{\partial t} \dot{\xi} ds + c_4 \right\} = 0,$$

we have $\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{s}} \right) = \frac{\partial L}{\partial t} \neq 0$ and $\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{\xi}} \right) = \frac{\partial L}{\partial \xi} = 0$. From the solving of the second Lagrangian equation, we get $\frac{\partial}{\partial s} (2E(\xi, t)\dot{\xi}) = 0$, which means $\frac{d\xi}{ds} = \frac{c_5}{2E(\xi, t)}$.

If we put the value of $\dot{\xi}$ at $E(\xi, t)\dot{\xi}^2 + G(\xi)t^2 = L$,

$$E(\xi, t) \left(\frac{d\xi}{ds} \right)^2 + G(\xi) \left(\frac{dt}{d\xi} \frac{d\xi}{ds} \right)^2 = L, \tag{21}$$

we obtain the general equation of geodesics on $\Lambda^\alpha(\xi, t)$ as follow

$$\frac{dt}{d\xi} = c_{11} \sqrt{\frac{E(\xi, t)}{G(\xi)}} \sqrt{LE(\xi, t) - c_{10}} \quad \text{or} \quad \frac{dt}{d\xi} = \frac{at+b}{\sqrt{G(\xi)\cos\theta}} \sqrt{L - \frac{E(\xi, t)\cos^2\theta}{at+b}}. \tag{22}$$

ii) For the parameters $t = \int \sin\theta ds$ (or $t = \int \sin\theta ds + c_7$) or $t = \frac{c_5s}{2G(\xi)} + c_6$ and the equation

$$2E(\xi, t)\ddot{\xi} = G_\xi \dot{t}^2 - E_\xi \dot{\xi}^2, \tag{23}$$

since $\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{\xi}} \right) = \frac{\partial L}{\partial \xi} \neq 0$ and from the solving of the differential equations in $\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{t}} \right) = \frac{\partial L}{\partial t} = 0$,

$$\dot{t} = \sin\theta \quad \text{or} \quad \dot{t} = \frac{c_1}{2G(\xi)},$$

using equations $\frac{\partial}{\partial s} \left(\frac{\partial L}{\partial \dot{t}} \right) = \frac{\partial L}{\partial t}$, we have $\frac{\partial}{\partial s} (2G(\xi)\dot{t}) = 0$, which means

$$\frac{dt}{ds} = \frac{c_1}{2G(\xi)} \tag{24}$$

If we put the value of \dot{t} at $E(\xi, t)\dot{\xi}^2 + G(\xi)\dot{t}^2 = L$,

$$E(\xi, t) \left(\frac{d\xi}{dt} \frac{dt}{ds}\right)^2 + G(\xi) \left(\frac{dt}{ds}\right)^2 = L, \tag{25}$$

we can obtain the general equation of geodesics on $\Lambda^\alpha(\xi, t)$ as follow

$$\frac{d\xi}{dt} = c_1 G(\xi) \sqrt{\frac{G(\xi)L - c_2}{E(\xi, t)G(\xi)}} \text{ or } \frac{d\xi}{dt} = \frac{1}{\sin\theta} \sqrt{\frac{L - G(\xi)\sin^2\theta}{E(\xi, t)}}. \tag{26}$$

5. The Physical Approach on α –magnetic Surface in $Q^2 \subset E_1^3$

In this section, we try to express as the point of view of a physicist to imagine tracing out a geodesic by determining the affine parameter s with the time, thinking that the picture is now of a point particle that is moving on the surface, tracing out a path called the orbit of the particle.

Let $\Lambda^\alpha(\xi(s), t(s))$ be a parametrized curve on surface as

$$\Lambda^\alpha(\xi(s), t(s)) = \begin{pmatrix} \left(\frac{2\kappa\sinh(w_1\xi)}{w_1^3} + \frac{2\kappa\cosh(w_1\xi)}{w_1} + f(\xi)\right)(at + b), \\ (\cosh(\sqrt{2\kappa}\xi) + g(\xi))(at + b), \\ \left(\frac{\kappa e^{w_1\xi}}{w_1} - 2\kappa^2\sinh(\sqrt{2\kappa}\xi) + h(\xi)\right)(at + b) \end{pmatrix}.$$

Also, the Lagrange equation on $\Lambda^\alpha(\xi, t)$ the magnetic surface is given as follows

$$E(\xi, t)\dot{\xi}^2 + G(\xi)\dot{t}^2 = L.$$

Furthermore, the tangent vector to this curve can be obtained using the chain rule as follow

$$\begin{aligned} \dot{\delta} &= \frac{d\Lambda^\alpha(\xi(s), t(s))}{ds} = \frac{d\xi(s)}{ds} \Lambda^\alpha_\xi + \frac{dt(s)}{ds} \Lambda^\alpha_t = N_\xi \cos\theta + N_t \sin\theta \\ &= \dot{\xi} \Lambda^\alpha_\xi + \dot{t} \Lambda^\alpha_t = \dot{\xi}(at + b)N_\xi + \dot{t}N_t. \end{aligned} \tag{27}$$

Hence, we can write the tangent vector of the geodesic curve as follows

$$\vec{Y} = \frac{d\Lambda^\alpha(\xi(s), t(s))}{ds} = Y^\xi \Lambda^\alpha_\xi + Y^t \Lambda^\alpha_t \tag{28}$$

and we can also write with two component vector notation for components according to the basis vectors $\Lambda^\alpha_t, \Lambda^\alpha_\xi$ and its norm $Y = \langle \vec{Y}, \vec{Y} \rangle^{1/2} = \sqrt{g_{ij} \frac{dz^i}{ds} \frac{dz^j}{ds}}$ is the speed, which is just the time rate

of change of the arc length along the curve δ . Think that $Y^{\xi^*} = \sqrt{E(\xi, t)}Y^\xi = Y\cos\theta$ is just the radial velocity while Y^ξ is the horizontal angular velocity and $Y^{t^*} = \sqrt{G(\xi)}Y^t = Y\sin\theta$ is the vertical component of the velocity vector. Hence, we can give the velocity in terms of polar coordinates in the tangent plane to explain its magnitude and slope angle according to the radial direction on the surface.

The role of the radial variable on this velocity plane is played by the speed, here we can say that the direction of the velocity according to the direction $\Lambda_{t^*}^\alpha$ on this plane is given by the angle θ . Also, we can say that the speed is constant along the geodesic. In [20, 21], to find out the system of two second order geodesic equations it is expressed that a standard physics technique of partially can be used integrating them by anyone and so lessen them to two first order equations by taking two constants of the movement that it comes out from the two independent symmetries of the equations of movement. Those physical properties as energy and momentum are replaced by the specific quantities found by partitioning out the mass. Therefore, we can write the specific kinetic energy as follows

$$E = \frac{1}{2}Y^2 = \frac{1}{2}E(\xi, t) \left(\frac{d\xi}{ds}\right)^2 + \frac{1}{2}G(\xi) \left(\frac{dt}{ds}\right)^2 = \frac{1}{2}(Y^2\cos^2\theta + Y^2\sin^2\theta), \tag{29}$$

using the right side of the previous equations we can say that both the specific energy and speed have to be constant along geodesic.

In the point of view of the physics, the specific kinetic energy of the particle is constant because of its motion in space, and only accelerates perpendicular to the surface. If a force is accountable for this acceleration, that is to say that the normal force that it supplies the particle on the surface, because of perpendicular to the velocity of the particle it wouldn't study on the particle. Therefore, the specific energy E has to be constant. Resembling, we say that the speed $Y = \sqrt{2E}$ is constant along a geodesic in respect of this cause.

Theorem 8. Let $\Lambda^\alpha(\xi, t)$ be the magnetic surface generated by α –magnetic curve. Then the specific kinetic energy of the particle on the α –magnetic surface $\Lambda^\alpha(\xi, t)$ is constant under evident conditions and the following statements are held:

1. For the parameter $t = \int \sin\theta ds$ (or $t = \frac{c_1s}{2G(\xi)} + c_2$) and the equation $2E(\xi, t)\ddot{\xi} = G_\xi \dot{t}^2 - E_\xi \dot{\xi}^2$, the specific angular momentum ℓ_1 and specific kinetic energy E^1 are constants along a geodesic, and are given as following equations

$$\ell_1 = \sqrt{G(\xi)}Y\sin\theta; E^1 = \frac{1}{2}\left(E(\xi, t)\left(\frac{d\xi}{ds}\right)^2 + \frac{\ell_1^2}{G(\xi)}\right). \tag{30}$$

2. For the parameter $\xi = \int \frac{\cos\theta}{at+b} ds$ and the equations $2 \int E(\xi, t)d\xi = c_5s + c_6$, $2G(\xi)\dot{t} - \left\{ \int \frac{\partial E(\xi, t)}{\partial t} \dot{\xi} ds + c_4 \right\} = 0$, the specific angular momentum ℓ_2 and specific kinetic energy E^2 are constant along a geodesic, and are given as follows

$$\ell_2 = -\sqrt{E(\xi, t)}Y\cos\theta; E^2 = \frac{1}{2}\left(\frac{\ell_2^2}{E(\xi, t)} + G(\xi)\left(\frac{dt}{ds}\right)^2\right), \tag{31}$$

where $c_i \in \mathbb{R}_0$, and Y is the tangent vector of the geodesic curve.

Proof. 1) For the equation $t = \int \sin\theta ds$ (or $t = \frac{c_1s}{2G(\xi)} + c_2$) and the equation $2E(\xi, t)\ddot{\xi} = G_\xi \dot{t}^2 - E_\xi \dot{\xi}^2$, we write $2G(\xi)\dot{t} = 2G(\xi)\sin\theta$ being a constant along δ and by using this situation we explain in this physics language. Also, we can explain as to circular motion around an axis with radius $\|\vec{R}_1\| = \sqrt{G(\xi)}$ or $\vec{R}_1 = \sqrt{G(\xi)}\vec{e}_1$, namely the velocity $Y^{t*} = \sqrt{G(\xi)}Y^t = Y\sin\theta = \sqrt{G(\xi)}\frac{dt}{ds}$ in the angular direction multiplied by the radius $\sqrt{G(\xi)}$ of the circle. Physically, the specific angular momentum ℓ_1 can be written as following equation

$$\ell_1 = \vec{e}_3 \cdot (\vec{R}_1 \times_{G_3} \vec{Y}) = \sqrt{G(\xi)}Y\sin\theta, \tag{32}$$

since $Y^{t*} = Y\sin\theta = \sqrt{G(\xi)}\frac{dt}{ds}$, we can write $\sqrt{G(\xi)}Y\sin\theta = G(\xi)\frac{dt}{ds}$ being a constant along $\delta(\xi)$, and we say that the specific angular momentum ℓ_1 is constant along a geodesic and we get

$$\ell_1 = G(\xi)\frac{dt}{ds} \Rightarrow \frac{dt}{ds} = \frac{\ell_1}{G(\xi)}. \tag{33}$$

This expression can be rewritten the changeable angular velocity dt/ds in the specific energy formula according to the constant angular momentum, the specific energy E^1 is given as

$$E^1 = \frac{1}{2}\left(E(\xi, t)\left(\frac{d\xi}{ds}\right)^2 + \frac{\ell_1^2}{G(\xi)}\right). \tag{34}$$

2) For the parameter $\xi = \int \frac{\cos\theta}{at+b} ds$ and the equations $2 \int E(\xi, t)d\xi = c_5s + c_6$, $2G(\xi)\dot{t} - \left\{ \int \frac{\partial E(\xi, t)}{\partial t} \dot{\xi} ds + c_4 \right\} = 0$, we write $2E(\xi, t)\dot{\xi} = 2E(\xi, t)\cos\theta$ being a constant along $\delta(\xi)$ and by using this situation we explain in this physics language. Also, we can express as in the case of circular movement round an axis with radius $\|\vec{R}_2\| = \sqrt{E(\xi, t)}$ or $\vec{R}_2 = \sqrt{E(\xi, t)}\vec{e}_2$, that is to say the velocity $Y^{\xi*} = \sqrt{E(\xi, t)}Y^\xi = Y\cos\theta = \sqrt{E(\xi, t)}\frac{d\xi}{ds}$ in the angular direction

multiplied by the radius $\sqrt{E(\xi, t)}$ of the circle. The first geodesic equation is told that the specific angular momentum is constant along a geodesic, and the specific angular momentum ℓ_2 can be taken down as following equation

$$\ell_2 = \vec{e}_3 \cdot (\vec{R}_2 \times_{G_3} \vec{Y}) = -\sqrt{E(\xi, t)}Y\cos\theta, \tag{35}$$

since $\sqrt{E(\xi, t)}\frac{d\xi}{ds} = Y\cos\theta$, we can write $-E(\xi, t)\frac{d\xi}{ds} = -\sqrt{E(\xi, t)}Y\cos\theta$, and we say that the specific angular momentum is constant along a geodesic. So, we have

$$\ell_2 = -E(\xi, t)\frac{d\xi}{ds} \Rightarrow \frac{d\xi}{ds} = \frac{-\ell_2}{E(\xi, t)}. \tag{36}$$

Hence, this statement can be rewritten the changeable angular velocity $d\xi/ds$ in the specific energy formula according to the constant angular momentum specific energy E^2 is given by

$$E^2 = \frac{1}{2} \left(\frac{\ell_2^2}{E(\xi, t)} + G(\xi) \left(\frac{dt}{ds} \right)^2 \right). \tag{37}$$

6. Conclusion

In this study, the α –magnetic surfaces constituted by using the α –magnetic curves to be geodesics on the surface are expressed. The α –magnetic surfaces generated by the α –magnetic curves are examined, and some certain results of describing the geodesics are given on the surfaces. Our results show that the specific energy and specific angular momentum obtained on the α –magnetic surfaces can be expressed in \mathbf{Q}^2 . The physical meanings of specific energy and specific angular momentum are of course related with the physical meaning itself. First of all, the conditions of being geodesic of the curves selected as magnetic curves are examined, and these geodesic conditions allows us to express the specific energy. We are working on the properties of these surfaces with a view to devising suitable metric in \mathbf{Q}^2 . It is hoped that researches will benefit from this study about the rotated surface.

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