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On A New Almost Convergent Sequence Space Defined By The Matrix Δ_u^λ

Δ_u^λ *Matrisi Yardımıyla Tanımlanan Yeni Bir Hemen Hemen Yakınsak Dizi Uzayı Üzerine*

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Abstract
In this study, it is defined almost sequence spaces $f(\hat{\Lambda}), f_0(\hat{\Lambda})$ and $fs(\hat{\Lambda})$ as domain of the matrix Δ_u^λ. Some topological properties of these spaces are investigated and determined $\beta-$, $\gamma-$duals of aforementioned sequence space. Futhermore, it is characterized the class of matrices $(f(\hat{\Lambda}):\mu)$, $(fs(\hat{\Lambda}):\mu)$, $\left(\mu:f(\hat{\Lambda})\right)$ and $\left(\mu:fs(\hat{\Lambda})\right)$, where μ is any given sequence space.

Keywords: Almost Convergent, Dual Spaces, Matrix Transformations, Matrix Domain of a Sequence Space, Sequence Spaces

Öz
Bu çalışmada Δ_u^λ matrisinin etki alanları olarak $f(\hat{\Lambda}), f_0(\hat{\Lambda})$ ve $fs(\hat{\Lambda})$ hemen hemen yakınsak dizi uzayları tanımlandı. Bu uzayların bazı topolojik özellikleri incelendi ve $\beta-$, $\gamma-$dualleri belirlendi. Ayrıca, $(f(\hat{\Lambda}):\mu)$, $(fs(\hat{\Lambda}):\mu)$, $\left(\mu:f(\hat{\Lambda})\right)$ ve $\left(\mu:fs(\hat{\Lambda})\right)$ matris sınıfları karakterize edildi.

Anahtar kelimeler: *Hemen Hemen Yakınsaklık, Dual Uzaylar, Matris Dönüşümleri, Bir Dizi Uzayının Matris Etki Alanı, Dizi Uzayları*

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1. Introduction

Let us denote space of functionals from $\mathbb{N}$ to $\mathbb{C}$, by w, where $\mathbb{N}$ and $\mathbb{C}$ show sets of natural numbers and complex numbers, respectively. When the sequence space is called, it is understood a linear subspace of w. The famous classic sequence spaces are $l_\infty, c, c_{0,} l_p$. These symbols represents sequence space all bounded, convergent, null and absolutely $p-$summable sequences, respectively. Also, we denote the spaces of all bounded and convergent series by bs and cs.

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers, ϑ and σ optional sequence spaces. If $x \in \vartheta$ implies that sequence $Ax = \{(Ax)_n\} \in \sigma$, where sequence Ax is the A-transform of the sequence x and the general term of this sequence is

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (1)$$

in this case, for each $n \in \mathbb{N}$, the series on the right side of the above equation is convergent. Then we say that the matrix A is a matrix transformation from ϑ to σ and denote it by $A: \vartheta \to \sigma$. The class of such matrices is showed by $(\vartheta: \sigma)$.

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞.

A matrix E is called triangle, if main diagonal's elements aren't zero and elements on the top of the main diagonal are zero. For triangle matrices E, F and a sequence y, the equality $E(Fy) = (EF)y$ holds. Further, a triangle matrix W uniquely has an inverse $W^{-1} = Z$, also a triangle matrix. The equality $y = W(Zy) = Z(Wy)$ yields for talked about matrices.

If there exists a single sequence (t_k) of scalars satisfied the following equation, then the sequence (t_k) is known a *Schauder basis* (or shortly *basis*) for a normed sequence space ϑ, where mentioned above equation is, for every $y \in \vartheta$,

$$lim\left\|y - \sum_{n=0}^{k} \alpha_n t_n\right\| = 0 \quad (2)$$

The series $\sum_n \alpha_n t_n$ which has the sum y is called the enlargement of y according to (t_k) and written as $y = \sum_n \alpha_n t_n$. *Schauder basis* and *algebraic basis* coincide for finite sequence spaces.

The matrix domain ϑ_A of an infinite matrix A in a sequence space ϑ is defined by

$$\vartheta_A = \{y = (y_k) \in w: Ay \in \vartheta\} \quad (3)$$

which is a sequence space. Although in the most cases, the new sequence space is the expansion or the contraction of the original space ϑ, in some cases, these spaces are overlap.

Combined with a linear topology a sequence space ϑ is denominated a K −space, if for each $\vartheta \in \mathbb{N}$, coordinate maps $p_i: \vartheta \to \mathbb{C}$, described by $p_i(y) = y_i$ are continuous. A K −space which is a complete linear metric space is entitled an $FK-$ *space*. An FK −space whose topology is normable is called a $BK-$ space (Lorentz, 1948) which comprises Φ, the set of all finitely nonzero sequences.

Let us assume that E −is a triangle matrix, in that case, we can obviously say that the sequence spaces ϑ_E and ϑ are linearly isomorphic, i.e., $\vartheta_E \cong \vartheta$ and if ϑ is a $BK-$ space, then ϑ_E is also a BK −space with the norm given by $\|y\|_{\vartheta_E} = \|Ey\|_\vartheta$, for all $y \in \vartheta_E$. As well as above mentioned sequence spaces $l_\infty, c, c_{0,}$ and almost convergent sequence space f are BK −spaces with the ordinary supnorm described by

$$\|y\|_\infty = \sup_{k\in\mathbb{N}} |y_k|. \quad (4)$$

Also l_p are $BK-$ spaces with the ordinary norm defined by

$$\|y\|_p = \left(\sum_k |y_k|^p\right)^{1/p}, (1 \le p < \infty). \quad (5)$$

Since the sequence space to be defined is almost convergent sequence space in this study, let's first remember the definition of almost convergent sequence space.

A continuous linear functional ψ on l_∞ is said a *Banach limit*, if

i) For every $y = (y_k)$, $\psi(y) \ge 0$,
ii) $\psi(y_{\rho(k)}) = \psi(y_k)$, where ρ is shift operatör which is described onto w with $\rho(k) = k + 1$,
iii) $\psi(e) = 1$, where $e = (1,1, \dots, 1, \dots)$.

A sequence $y = (y_k) \in l_\infty$ is entitled to be almost convergent to generalized limit l, if all Banach limits y are l (Lorentz, 1948), and denoted $f - limy = l$. In other words, $f - limy = l$ iff uniformly in n

$$\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} y_{k+n} = l. \tag{6}$$

We indicate the sets of all almost convergent sequences by f and series by fs and define as follow:

$$f = \left\{y = (y_k) \in w: \lim_{m\to\infty} s_{mn}(y) = l, \ uniformly\ in\ n\right\} \tag{7}$$

where l exists uniformly in n,

$$s_{mn}(y) = \frac{1}{m+1}\sum_{k=0}^{m} y_{k+n}, \tag{8}$$

and

$$fs = \left\{y = (y_k) \in w: \exists l \in C \ni \lim_{m\to\infty} \sum_{k=0}^{m}\sum_{j=0}^{n+k} \frac{y_j}{m+1} = l\ uniformly\ in\ n\right\}. \tag{9}$$

As known that the containments $c \subset f \subset l_\infty$ are precisely acquired. Owing to these containments, norms $\|.\|_f$ and $\|.\|_\infty$ of the spaces f and l_∞ are equivalent. Therefore the sets f and f_0 are *BK*-spaces having the following norm

$$\|y\|_f = \sup_{m,n} |s_{mn}(y)| \tag{10}$$

When we look according to summability theory perspective, we can see that to define new *Banach spaces* by the matrix domain of triangle and investigate their algebraical, geometrical and topological properties is well-known. Therefore, many authors were interested in this subject and by using some known matrices, they did many studies by using some known matrices. Some of them are here:
(Başar et al., 2011; Candan, 2014, 2018; Candan et al., 2015; Karaisa et al., 2015; Kayaduman et al., 2012a,b; Kirisçi, 2012, 2014).

The matrix to be used to construct sequence spaces in this paper is below:
Let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be strictly increasing sequence of positive reals tending to infinity, i.e.

$$0 < \lambda_1 < \lambda_2 < \cdots \text{ and } \lambda_k \to \infty, \text{ as } k \to \infty.$$

Let $u = (u_k)$ be a sequence such that $u_k \neq 0$, for all $k \in \mathbb{N}$. We define the matrix $\hat{\Lambda} = \Delta_u^\lambda = (\hat{\lambda}_{nk})$ as

$$\hat{\lambda}_{nk} = \begin{cases} \frac{(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)}{\lambda_n} u_k, & if\ k < n, \\ \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} u_n, & if\ k = n, \\ 0, & if\ k > n, \end{cases} \tag{11}$$

Where

$$\hat{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k (x_k - x_{k-1}) \tag{12}$$

and if $y = (y_k)$ is $\hat{\Lambda}$-transform of a sequence $x = (x_k)$, where for all $k \in \mathbb{N}$

$$y_k = \sum_{i=0}^{k} \frac{(\lambda_i - \lambda_{i-1})}{\lambda_k} u_i (x_i - x_{i-1}). \tag{13}$$

In (Ganie et al., 2013), using the matrix above, the sequence spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$ were defined and investigated. Using the same matrix, we also define the following sequence spaces.
Firstly, let us define sequence spaces $f(\hat{\Lambda})$ and $f_0(\hat{\Lambda})$:

$$f(\hat{\Lambda}) = \{x = (x_k) \in w: y = (y_k) = \hat{\Lambda}(x) \in f\}. \tag{14}$$

If $y = (y_k) \in \hat{\Lambda}(x) \in f$, it means that $\exists l \in \mathbb{C}$ such that uniformly in n,

$$\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m} y_{k+n} = \lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m}\left(\sum_{i=0}^{k+n}\left(\frac{\lambda_i - \lambda_{i-1}}{\lambda_{k+n}}\right) u_i(x_i - x_{i-1})\right) = l. \tag{15}$$

If $l = 0$, $y = (y_k) \in \hat{\Lambda}(x) \in f_0$, and we can define

$$f_0(\hat{\Lambda}) = \{x = (x_k) \in w: y = (y_k) = \hat{\Lambda}(x) \in f_0\}, \tag{16}$$

The other sequence space is $fs(\hat{\Lambda})$:

$$fs(\hat{\Lambda}) = \{x = (x_k) \in w: y = (y_k) = \hat{\Lambda}(x) \in fs\}, \tag{17}$$

i.e. If $y = (y_k) \in \hat{\Lambda}(x) \in fs$, then $\exists l \in \mathbb{C} \ni$ uniformly in n,

$$\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k+n}\sum_{i=0}^{j} \frac{(\lambda_i - \lambda_{i-1})}{\lambda_j} u_i(x_i - x_{i-1}) = l. \tag{18}$$

We can redefine the spaces $fs(\hat{\Lambda}), f(\hat{\Lambda})$ and $f_0(\hat{\Lambda})$ by the notation of (3), $fs(\hat{\Lambda}) = (fs)_{\hat{\Lambda}}$, $f(\hat{\Lambda}) = (f)_{\hat{\Lambda}}$ and $f_0(\hat{\Lambda}) = (f_0)_{\hat{\Lambda}}$.

This paper is organized as following: In chapter 2: some topological properties of defined sequence spaces; in chapter 3: dual spaces of these spaces;

in chapter 3: the characterization of some matrix classes between these spaces and some classical sequence spaces are given.

2. Some Topological Properties Of These Spaces

Theorem 2.1:

i) The sequence space $f(\hat{\Lambda})$ is normed space with

$$\|x\|_{f(\hat{\Lambda})} = \sup_{m,n}\left|\frac{1}{m+1}\sum_{k=0}^{m}\left(\sum_{i=0}^{k+n}\frac{(\lambda_i-\lambda_{i-1})}{\lambda_{k+n}}\right)u_i(x_i-x_{i-1})\right| \tag{19}$$

ii) The sequence space $fs(\hat{\Lambda})$ is normed space with with

$$\|x\|_{fs(\hat{\Lambda})} = \sup_{m,n}\left|\frac{1}{m+1}\sum_{k=0}^{m}\left(\sum_{j=0}^{k+n}\sum_{i=0}^{j}\frac{(\lambda_i-\lambda_{i-1})}{\lambda_j}u_i(x_i-x_{i-1})\right)\right| \tag{20}$$

Theorem 2.2: The spaces $f(\hat{\Lambda})$, $f_0(\hat{\Lambda})$ and $fs(\hat{\Lambda})$ are linearly isomorphic to the spaces f, f_0 and fs, respectively, i.e. $f(\hat{\Lambda}) \cong f$, $f_0(\hat{\Lambda}) \cong f_0$, and $fs(\hat{\Lambda}) \cong fs$.

Proof: We show that there is a linear transformation between $f(\hat{\Lambda})$ and f. Therefore we have to define a transformation from $f(\hat{\Lambda})$ to f. Using the matrix $\hat{\Lambda}$, it can be described the transformation T as $T(x) = \hat{\Lambda}(x)$, for each $x \in f(\hat{\Lambda})$. It is easy to see that T is linear. If $T(x) = 0$, then $x = 0$, so T is one-to-one. Finally, we need to show that T is surjective.

Let us assume $y = (y_k) \in f$ and describe $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k}\left(\sum_{m=j-1}^{j}(-1)^{j-m}\frac{\lambda_m}{u_j(\lambda_j-\lambda_{j-1})}y_m\right) \tag{21}$$

From here, we have

$$\sum_{i=0}^{k}\left(\frac{\lambda_i-\lambda_{i-1}}{\lambda_k}\right)u_i(x_i-x_{i-1}) =$$
$$\sum_{i=0}^{k}\left(\frac{\lambda_i-\lambda_{i-1}}{\lambda_k}\right)u_i\left(\sum_{j=0}^{i}\sum_{m=j-1}^{j}(-1)^{j-m}.\frac{\lambda_m}{u_j(\lambda_j-\lambda_{j-1})}y_m\right)-\sum_{j=0}^{i-1}\sum_{m=j-1}^{j}(-1)^{j-m}\frac{\lambda_m}{u_j(\lambda_j-\lambda_{j-1})}y_m$$
$$=\sum_{i=0}^{k}\frac{(\lambda_i-\lambda_{i-1})}{\lambda_k}u_i\left(\sum_{m=i-1}^{i}(-1)^{i-m}.\frac{\lambda_m}{u_i(\lambda_i-\lambda_{i-1})}y_m\right)$$
$$= y_k \tag{22}$$

For all $k \in \mathbb{N}$, which leads us to the truth that uniformly in m

$$f_{\hat{\Lambda}} - lim x = f - lim y \tag{23}$$

which implies that $x \in f_{\hat{\Lambda}}$, consequently, we see that T is surjective. Hence, T is a linear bijection that therefore shows that the spaces $f(\hat{\Lambda})$ and f are linearly isomorphic, as desired. This completes the proof. The fact $f_0(\hat{\Lambda}) \cong f_0$ can be analogously attested.

Due to the well known fact that the matrix domain λ_A of the normed sequence space denoted by λ has got a base iff the matrix domain λ_A of the normed sequence space denoted by λ has got a base, whenever a matrix $A = (a_{nk})$ is a triangle (Jarrah, et al., 1990). (Remark 2.4) and since the space f has no Schauder basis, we have;

Corollary 2.1: The space $f_{\hat{\Lambda}}$ has no Schauder Basis.

3. The $\alpha-$, $\beta-$, $\gamma-$Duals Of These Spaces

The $\alpha-, \beta-, \gamma-$duals of the *sequence space* X are defined by

$$X^{\alpha} = \left\{\begin{matrix} a = (a_k) \in w: ax = (a_k x_k) \in l_1, \\ \forall x = (x_k) \in X \end{matrix}\right\} \tag{24}$$

$$X^{\beta} = \left\{\begin{matrix} a = (a_k) \in w: ax = (a_k x_k) \in cs, \\ \forall x = (x_k) \in X \end{matrix}\right\} \tag{25}$$

$$X^{\gamma} = \left\{\begin{matrix} a = (a_k) \in w: ax = (a_k x_k) \in bs, \\ \forall x: (x_k) \in X \end{matrix}\right\} \tag{26}$$

here cs and bs are defined to be *sequence spaces* of all convergent and bounded series, respectively.

Lemma 3.1: (Sıddıqi, 1971) So as to the matrix A belongs to the matrix class from f to l_∞ is necessary and sufficient condition

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty \tag{27}$$

is satisfied.

Lemma 3.2: (Sıddıqi, 1971) So as to the matrix A belongs to the matrix class from f to c is necessary and sufficient conditions:

i) $$sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty \tag{28}$$

ii) for each $k \in \mathbb{N}$ $$\lim_{n\to\infty} a_{nk} = \alpha_k \tag{29}$$

iii) $$\lim_{n\to\infty} \sum_k a_{nk} = \alpha \tag{30}$$

iv) $$\lim_{n\to\infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0 \tag{31}$$

are satisfied.

Theorem 3.1: The γ−dual of the space $f_{\hat{\Lambda}}$ is the intersection of the sets

$$b_1 = \left\{a = (a_k) \in w: \sup_{n\in N} \sum_{k=0}^{n-1} |\hat{a}_k(n)| < \infty\right\}, \tag{32}$$

$$b_2 = \left\{a = (a_k) \in w: \sup_{n\in N} \left|\frac{\lambda_n}{u_n(\lambda_n-\lambda_{n-1})} a_n\right| < \infty\right\}. \tag{33}$$

Proof: For an optional sequence $a = (a_k) \in w$ and take into consideration the following equality.

$$\sum_{k=0}^{n} a_k x_k =$$

$$\sum_{k=0}^{n} \left\{\sum_{j=0}^{k}\left[\sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{u_j(\lambda_j - \lambda_{j-1})} y_i\right]\right\} a_k =$$

$$\sum_{k=0}^{n-1} \frac{\lambda_k}{u_k}\left[\frac{a_k}{\lambda_k - \lambda_{k-1}} \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k}\right) \sum_{j=k+1}^{n-1} a_j\right] y_k$$

$$+ \frac{\lambda_n}{u_n(\lambda_n - \lambda_{n-1})} a_n y_n$$

$$= \sum_{k=0}^{n-1} \hat{a}_k(n) y_k + \frac{\lambda_n}{u_n(\lambda_n - \lambda_{n-1})} a_n y_n$$

$$= D_n(y) \tag{34}$$

where the general term d_{nk} of the matrix D is determined as follows,

$$D = (d_{nk}) = \begin{cases} \hat{a}_k(n), & k < n, \\ \frac{\lambda_n}{u_n(\lambda_n - \lambda_{n-1})} a_n, & k = n, \\ 0, & k > n, \end{cases} \tag{35}$$

for all $k, n \in \mathbb{N}$, where

$$\hat{a}_k(n) = \frac{\lambda_k}{u_k}\left[\frac{a_k}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k}\right) \sum_{j=k+1}^{n-1} a_j\right]. \tag{36}$$

Thus, we deduce from (4), that $a_k x_k \in bs$ whenever $x = (x_k) \in f_{\hat{\Lambda}}$ iff $Dy \in l_\infty$ whenever $y = (y_k) \in f$, where $D = (d_{nk})$ is described in (35). That's why with assistance of Lemma 3.1, ${f_{\hat{\Lambda}}}^{\gamma} = b_1 \cap b_2$.

Theorem 3.2: The β −dual of the space $f_{\hat{\Lambda}}$ is the intersection of the sets

$$b_3 = \left\{a = (a_k) \in w: \lim_{n\to\infty} d_{nk} \ exists\right\}, \tag{37}$$

$$b_4 = \left\{a = (a_k) \in w: \lim_{n\to\infty} \sum_k d_{nk} \ exists\right\}, \tag{38}$$

$$b_5 = \left\{a = (a_k) \in w: \lim_{n\to\infty} \sum_k \Delta(d_{nk} - \alpha_k) < \infty\right\}, \tag{39}$$

where $\alpha_k = \lim_{n\to\infty} d_{nk}$. Then ${f_{\hat{\Lambda}}}^{\beta} = \cap_{k=1}^{5} b_k$.

Proof: Let us take any sequence $a \in w$. By (4), $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f_{\hat{\Lambda}}$ iff $Dy \in c$ whenever $y = (y_k) \in f$, where $D = (d_{nk})$ is designated in (35). We derive the consequence by Lemma 3.2 that $\{f_{\hat{\Lambda}}\}^{\beta} = \cap_{k=1}^{5} b_k$.

Theorem 3.3: The γ −dual of the space $fs_{\hat{\Lambda}}$ is the intersection of the sets

$$b_6 = \left\{a = (a_k) \in w: \sup_{n} \sum_k \Delta(d_{nk}) < \infty\right\} \tag{40}$$

$$b_7 = \left\{a = (a_k) \in w: \lim_{k\to\infty} d_{nk} = 0\right\}, \tag{41}$$

That is, $\{fs_{\widehat{\Lambda}}\}^{\gamma} = b_6 \cap b_7$.

Proof: This might be acquired in a similar concept as talk about in the proof of theorem 3.1 with lemma 3.1 instead of Lemma 4.2 (iii). So, we neglect details.

Theorem 3.4: Defined the set

$$b_8 = \left\{a = (a_k) \in w: \lim_{n\to\infty} \sum_k |\Delta^2(d_{nk})| < \infty\right\}, \tag{42}$$

Then, $\{fs_{\widehat{\Lambda}}\}^{\beta} = b_3 \cap b_6 \cap b_7 \cap b_8$.

Proof: This, might be acquired in a similar concept as talk about in the proof of theorem 3.2 with Lemma 3.2 instead of lemma 4.2 (iv). So, we disregard details.

4. Characterization of Some Matrix Classes

For shortness, let us write

$$a_{nk} = \sum_{j=0}^{n} a_{jk} \tag{43}$$

$$a(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k} \tag{44}$$

$$\Delta a_{nk} = a_{nk} - a_{n,k+1}. \tag{45}$$

Theorem 4.1: (Başar, 2012) Let μ be an *FK*-space, U be a triangle matrix, $P = U^{-1}$ and η be optional subset of w. Then, we have $A = (a_{nk}) \in (\mu_U : \eta)$ iff for all $n \in \mathbb{N}$,

$$C^{(n)} = \left(c_{mk}^{(n)}\right) \in (\mu, c) \tag{46}$$

and

$$C = (c_{nk}) \in (\mu, \eta), \tag{47}$$

where

$$c_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} a_{nj} p_{jk}, & 0 \le k \le m \\ 0, & k > m, \end{cases} \tag{48}$$

and for all $k, m, n \in \mathbb{N}$,

$$c_{nk} = \sum_{j=k}^{\infty} a_{nj} p_{jk}. \tag{49}$$

Lemma 4.1: $A \in (f:f)$ iff

i) $sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty$ (50)

ii) $f - lim\, a_{nk} = \alpha_k$, exist, for each fixed $k \in \mathbb{N}$ (51)

iii) $f - lim \sum_k a_{nk} = \alpha$ (52)

iv) uniformly in n $\lim_{m\to\infty} \sum_k |\Delta[a(n,k,m) - \alpha_k]| = 0$, (53)

are satisfied.

For an infinite matrix $A = (a_{nk})$, we shall write for shortness that:

$$d_{mk}^{n} = \tilde{a}_{nk}(m) = \frac{\lambda_k}{u_k}\left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k}\right) \sum_{j=k+1}^{m} a_{nj}\right] \tag{54}$$

where $k < m$.

$$d_{nk} = \tilde{a}_{nk} = \frac{\lambda_k}{u_k}\left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k}\right) \sum_{j=k+1}^{\infty} a_{nj}\right] \tag{55}$$

$$\hat{a}_{nk} = \sum_{i=0}^{n} \left(\frac{\lambda_i - \lambda_{i-1}}{\lambda_k}\right) u_i (a_{ik} - a_{i-1,k}) \tag{56}$$

Theorem 4.2: Let us assume that the entries of the infinite matrices given by $A = (a_{nk})$ and

$H = (h_{nk})$ are related by the following relation

$$h_{nk} = \tilde{a}_{nk} \tag{57}$$

for all $k, n \in \mathbb{N}$, μ is an orbitrary sequence space. Then $A \in (f_{\widehat{\Lambda}} : \mu)$ iff for all $n \in \mathbb{N}$, $\{a_{nk}\}_{k\in\mathbb{N}} \in (f_{\widehat{\Lambda}})^{\beta}$ and $H \in (f:\mu)$.

Proof: Let us take an orbitrary sequence space μ and it is satisfied the condition (56) and recall that $f_{\widehat{\Lambda}}$ and f are linearly isomorphic. We take $A \in (f_{\widehat{\Lambda}} : \mu)$ and $y = (y_k) \in f$.

Thus, $H.\hat{\Lambda}$ does exist and $\{a_{nk}\}_{k\in\mathbb{N}} \in\cap_{k=1}^{5} b_k$ which satisfies that $\{h_{nk}\}_{k\in\mathbb{N}} \in l_1$, for each $n \in \mathbb{N}$. Therefore, Hy exists and thus for all $n \in \mathbb{N}$

$$\sum_k h_{nk}y_k = \sum_k a_{nk}x_k. \tag{58}$$

We have by (56) that $Hy = Ax$, which leads us to consequence $H \in (f:\mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in (f_{\hat{\Lambda}})^{\beta}$, for each $n \in \mathbb{N}$ and $H \in (f:\mu)$ satisfy, and take any $x = (x_k) \in f_{\hat{\Lambda}}$. Then, Ax exists. Thus, we acquire from the following equality for each $n \in \mathbb{N}$,

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m}\left[\sum_{j=0}^{k}\left(\sum_{i=j-1}^{j}(-1)^{j-i}.\frac{\lambda_i}{u_j(\lambda_j-\lambda_{j-1})}y_i a_{nj}\right)\right]. \tag{59}$$

As $m \to \infty$ that $Ax = Hy$ and this shows that $A \in (f_{\hat{\Lambda}}:\mu)$.

Theorem 4.3: $A \in (f_{\hat{\Lambda}}:c)$ iff $D^{(n)} = \left(d_{mk}^{(n)}\right) \in (f:c)$ and $D = (d_{nk}) \in (f:c)$.

Theorem 4.4: $A \in (f_{\hat{\Lambda}}:l_\infty)$ iff $D^{(n)} = \left(d_{mk}^{(n)}\right) \in (f:c)$ and $D = (d_{nk}) \in (f:l_\infty)$.

If we change the roles for the spaces $f_{\hat{\Lambda}}$ and f with μ, we have following theorems.

Theorem 4.5: Assume that the entries of the infinite matrices $A = (a_{nk})$ and $L = (l_{nk})$ are related by the following relation $l_{nk} = \hat{a}_{nk}$ in (56), for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then, $A \in (\mu:f_{\hat{\Lambda}})$ iff $L \in (\mu:f)$.

Proof: Let $x = (x_k) \in \mu$ and take into account the following equality

$$\begin{aligned}\left\{\hat{\Lambda}(Ax)\right\}_n &= \sum_{i=0}^{n}\frac{(\lambda_i-\lambda_{i-1})}{\lambda_k}u_i[(Ax)_i-(Ax)_{i-1}]\\ &= \sum_{i=0}^{n}\frac{(\lambda_i-\lambda_{i-1})}{\lambda_k}u_i\sum_j\left(a_{ij}-a_{i-1,j}\right)x_j\\ &= \sum_j\left(\sum_{i=0}^{n}\frac{(\lambda_i-\lambda_{i-1})}{\lambda_k}u_i\left(a_{ij}-a_{i-1,j}\right)\right)x_j\\ &= (Lx)_n\end{aligned} \tag{60}$$

which leads us to consequence that $Ax \in f_{\hat{\Lambda}}$ iff $Lx \in f$. Thus, proof is completed.

At this time, we are going to denote the following conditions:

for each fixed $k \in \mathbb{N}$

$$lim\, a_{nk} = \alpha_k, \text{ exist} \tag{61}$$

$$\lim_{n\to\infty}\sum_k a_{nk} = \alpha, \tag{62}$$

$$\lim_{n\to\infty}\sum_k|\Delta(a_{nk}-\alpha_k)| = 0, \tag{63}$$

$$sup_{n\in\mathbb{N}}\sum_k|\Delta(a_{nk})| < \infty, \tag{64}$$

for each fixed $n \in \mathbb{N}$,

$$\lim_{k\to\infty} a_{nk} = 0, \tag{65}$$

$$\lim_{n\to\infty}\sum_k|\Delta^2 a_{nk}| = \alpha, \tag{66}$$

$$f - lim\, a_{nk} = \alpha_k \text{ exists,} \tag{67}$$

uniformly in n

$$\lim_{m\to\infty}\sum_k|a(n,k,m)-\alpha_k| = 0, \tag{68}$$

uniformly in n

$$f - \ lim\sum_k a_{nk} = \alpha, \tag{69}$$

$$\lim_{m\to\infty}\sum_k|\Delta[a(n,k,m)-\alpha_k]| = 0, \tag{70}$$

uniformly in n

$$\lim_{q\to\infty}\sum_k\frac{1}{q+1}\left|\sum_{i=0}^{q}\Delta[a(n+i,k)-\alpha_k]\right| = 0, \tag{71}$$

$$sup_{n\in\mathbb{N}}\sum_k|\Delta a(n,k)| < \infty, \tag{72}$$

for each fixed $k \in \mathbb{N}$

$$f - lim\, a(n,k) = \alpha_k \text{ exists,} \tag{73}$$

uniformly in n

$$\lim_{q\to\infty}\sum_k\frac{1}{q+1}\left|\sum_{i=0}^{q}\Delta^2[a(n+i,k)-\alpha_k]\right| = 0, \tag{74}$$

$$sup_{n\in\mathbb{N}}\sum_k|a(n,k)| < \infty, \tag{75}$$

for each fixed $k \in \mathbb{N}$

$$\sum_n a_{nk} = \alpha_k, \tag{76}$$

$$\sum_n\sum_k a_{nk} = \alpha, \tag{77}$$

$$\lim_{n\to\infty} \sum_k |\Delta a(n,k) - \alpha_k| = 0, \quad (78)$$

Lemma 4.2: Let $A = (a_{nk})$ be an infinite matrix. In that case, the following expressions hold:

i) $A = (a_n) \in (l_\infty : f)$ iff conditions (50), (67) and (68) hold. (Duran, 1972).

ii) $A = (a_{nk}) \in (f : f)$ iff conditions (50), (67) and (69) hold. (Duran, 1972).

iii) $A = (a_{nk}) \in (fs : l_\infty)$ iff conditions (64) and (65) hold. (Başar, 2012).

iv) $A = (a_{nk}) \in (fs : c)$ iff conditions (61), (64) and (66) hold. (Öztürk, 1983).

v) $A = (a_{nk}) \in (c : f)$ iff conditions (50), (67) and (69) hold. (King, 1966).

vi) $A = (a_{nk}) \in (bs : f)$ iff conditions (64), (65), (67) and (71) hold. (Başar et al, 1991).

vii) $A = (a_{nk}) \in (fs : f)$ iff conditions (65), (67) (70) and (71) hold (Başar, 1991).

viii) $A = (a_{nk}) \in (cs : f)$ iff conditions (64) and (67) hold (Başar et al., 1989).

ix) $A = (a_{nk}) \in (bs : fs)$ iff conditions (65), (71) and (73) hold (Başar et al., 1991).

x) $A = (a_{nk}) \in (fs : fs)$ iff conditions (71) and (74) hold (Başar, 1991).

xi) $A = (a_{nk}) \in (cs : fs)$ iff conditions (72) and (73) hold (Başar et al., 1989).

xii) $A = (a_{nk}) \in (f : cs)$ iff conditions (75) and (78) hold (Başar, 1989).

Corollary 4.1: The following statements hold:

i) $A = (a_{nk}) \in (f_{\hat{\Lambda}} : l_\infty)$ iff $\{a_{nk}\}_{k\in\mathbb{N}} \in (f_{\hat{\Lambda}})^\beta$ for all $n \in \mathbb{N}$ and (50) hold with $\tilde{a}_{nk}$ instead of a_{nk}.

ii) $A = (a_{nk}) \in (f_{\hat{\Lambda}} : c)$ iff $\{a_{nk}\}_{k\in\mathbb{N}} \in (f_{\hat{\Lambda}})^\beta$ for all $n \in \mathbb{N}$ and (50), (61), (63) hold with $\tilde{a}_{nk}$ instead of a_{nk}.

iii) $A = (a_{nk}) \in (f_{\hat{\Lambda}} : bs)$ iff $\{a_{nk}\}_{k\in\mathbb{N}} \in (f_{\hat{\Lambda}})^\beta$ for all $n \in \mathbb{N}$ and (75) hold with $\tilde{a}_{nk}$ instead of a_{nk}.

iv) $A = (a_{nk}) \in (f_{\hat{\Lambda}} : cs)$ iff $\{a_{nk}\}_{k\in\mathbb{N}} \in (f_{\hat{\Lambda}})^\beta$ for all $n \in \mathbb{N}$ and (75), (78) hold with $\tilde{a}_{nk}$ instead of a_{nk}.

Corollary 4.2: The following statements hold:

i) $A = (a_{nk}) \in (l_\infty : f_{\hat{\Lambda}})$ iff (50), (67) and (68) hold with $\hat{a}_{nk}$ instead of a_{nk}.

ii) $A = (a_{nk}) \in (f : f_{\hat{\Lambda}})$ iff (50), (67), (69) and (70) hold with $\hat{a}_{nk}$ instead of a_{nk}.

iii) $A = (a_{nk}) \in (c : f_{\hat{\Lambda}})$ iff (50), (67) and (69) hold with $\hat{a}_{nk}$ instead of a_{nk}.

Corollary 4.3: The following statements hold:

i) $A = (a_{nk}) \in (bs : f_{\hat{\Lambda}})$ iff (64), (65), (67) and (71) hold with $\hat{a}_{nk}$ instead of a_{nk}.

ii) $A = (a_{nk}) \in (fs : f_{\hat{\Lambda}})$ iff (65), (67) and (71) hold with $\hat{a}_{nk}$ instead of a_{nk}.

iii) $A = (a_{nk}) \in (cs : f_{\hat{\Lambda}})$ iff (64) and (67) hold with $\hat{a}_{nk}$ instead of a_{nk}.

Corollary 4.4: The following statements hold:

i) $A = (a_{nk}) \in (bs : fs_{\hat{\Lambda}})$ iff (65), (71) and (73) hold with $\hat{a}_{nk}$ instead of a_{nk}.

ii) $A = (a_{nk}) \in (fs : fs_{\hat{\Lambda}})$ iff (71) and (74) hold with $\hat{a}_{nk}$ instead of a_{nk}.

iii) $A = (a_{nk}) \in (cs : fs_{\hat{\Lambda}})$ iff (72) and (73) hold with $\hat{a}_{nk}$ instead of a_{nk}.

5. Conclusions

The purpose of this paper is to define some new almost sequence spaces, to give some properties of these spaces and to determine $\beta-$, $\gamma-$ duals of these spaces, also to characterize some matrix classes between these spaces and some classical sequence spaces. Studying the domain of generalized difference matrix Δ_u^λ in the spaces f, f_0, fs and determining the $\beta-$, $\gamma-$ duals of these spaces, characterizing the infinite matrices belongs to the class of matrices $(f(\hat{\Lambda}) : \mu), (fs(\hat{\Lambda}) : \mu), (\mu : f(\hat{\Lambda}))$ and $(\mu : fs(\hat{\Lambda}))$- where μ is any given sequence space-are significant in terms of filling up a gap in the existing literatüre of summability theory.

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