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q-Analogue of a New Subclass of Harmonic Univalent Functions Defined by Fractional Calculus Operator

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Abstract
The purpose of the present paper is to introduce a new subclass of harmonic univalent functions by using fractional calculus operator associated with q-calculus. Coefficient condition, extreme points, distortion bounds, convolution and convex combination are obtained for this class. Finally, we discuss a class preserving integral operator for this class.

Keywords: *Harmonic functions; Fractional calculus; q-calculus.*

AMS Subject Classification (2020): *Primary: 30C45 ; Secondary: 30C50.*

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$. For detailed study one may refer to Clunie and Sheil-Small [3] and Duren [5], (see also [9]).

Let S_H represent the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

Note that the class S_H reduces to the class S of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

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Further, we let V_H^n be the subclass of S_H consisting of functions of form $f = h + \overline{g_n}$, where

$$h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k,\ g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k,\ |b_1| < 1. \tag{1.3}$$

The following definitions of fractional derivatives and fractional integrals are due to Owa [11] and Srivastava and Owa [18].

Definition 1.1. The fractional integral of order λ is defined for a function $f(z)$ of the form (1.2) by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi,$$

where $\lambda > 0, f(z)$ is an analytic functions in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 1.2. The fractional derivative of order λ is defined for a function $f(z)$ of the form (1.2) by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi,$$

where $0 \leq \lambda < 1, f(z)$ is an analytic functions in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 1.1 above.

Definition 1.3. Under the hypothesis of Definition 1.2 the fractional derivative of order $n + \lambda$ is defined for a function $f(z)$ by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{1.4}$$

where $0 \leq \lambda < 1$ and $n \in N_0 = \{0, 1, 2.....\}$.

In 2011, Dixit and Porwal [4] introduce a new fractional derivative operator for function of the form (1.2) as follows

$$\begin{aligned} \Omega^0 f(z) &= f(z) \\ \Omega^1 f(z) &= \Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Omega^n f(z) &= \Omega(\Omega^{n-1} f(z)). \end{aligned}$$

Thus, we note that

$$\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k, \tag{1.5}$$

where

$$\phi(k, \lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}.$$

It is worthy to note that for $\lambda = 0, \Omega^n f(z)$ reduces to familiar Salagean operator introduced by Salagean in [16]. They define the above operator for function of the form $f = h + \overline{g}$, where h and g are the form (1.1) as follows

$$\Omega^n f(z) = \Omega^n\ (h(z)) + (-1)^n \overline{\Omega^n\ (g(z))}$$

where

$$\Omega^n\ (h(z)) = z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k$$

and

$$\Omega^n\ (g(z)) = \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^k.$$

The applications of $q-$ calculus is a current and interesting topic of research in Geometric Function Theory. Very recently, Srivastava [17] gave definitions and properties of $q-$ calculus and fractional $q-$ calculus in detail and its applications in his survey-cum-expository review article. Several researchers e.g. see the work of Arif et al. [1], Ahuja *et al.* [2], Jahangiri [8], Najafzadeh and Makinde [10], Porwal and Gupta [12] and Ravindar et al. [14, 15] investigated various subclasses of univalent functions and obtain interesting results.

Now, we recall the concept of q-calculus which was first introduced by Jackson [6, 7]. For $k \in N$, the $q-$ number is defined as follows:

$$[k]_q = \frac{1-q^k}{1-q}, \quad 0<q<1. \tag{1.6}$$

Hence, $[k]_q$ can be expressed as a geometric series $\sum_{i=0}^{k-1} q^i$, when $k \to \infty$ the series converges to $\frac{1}{1-q}$. As $q \to 1$, $[k]_q \to k$ and this is the bookmark of a $q-$ analogue the limit as $q \to 1$ recovers the classical object.
The $q-$ derivative of a function f is defined by

$$D_q(f(z)) = \frac{f(qz)-f(z)}{(q-1)z}, \quad q \neq 1, \ z \neq 0$$

and $D_q(f(0)) = f'(0)$ provided $f'(0)$ exists.
For a function $h(z) = z^k$ observe that

$$D_q(h(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}.$$

Then

$$lim_{q \to 1} D_q(h(z)) = lim_{q \to 1} [k]_q z^{k-1} = k z^{k-1} = h'(z)$$

where h' is the ordinary derivative.

The $q-$ Jackson definite integral of the function f is defined by

$$\int_0^z f(t) d_q t = (1-q) z \sum_{n=0}^{\infty} f(zq^n) q^n, \quad z \in C.$$

Now, we let $R_H(n,q,\beta,\lambda)$ denote the subclass S_H consisting of functions $f = h + \overline{g}$ of the form (1.1) that satisfy the condition

$$\Re \left\{ \frac{\Omega^n \left(D_q(h(z))\right) + (-1)^n \overline{\Omega^n \left(D_q(g(z))\right)}}{z} \right\} < \beta, \tag{1.7}$$

for some $\beta(1<\beta\leq 2)$, $0<q<1$, $\lambda(0\leq\lambda\leq 1)$, $n \in N$ and $z \in U$.

We further let $\overline{R_H}(n,q,\beta,\lambda)$ denote the subclass of $R_H(n,q,\beta,\lambda)$ consisting of functions $f = h + \overline{g_n} \in S_H$ such that h and g_n are of the form (1.3).

If $f(z)$ is of the form (1.2) then the classes $R_H(n,q,\beta,\lambda)$ and $\overline{R_H}(n,q,\beta,\lambda)$ reduce to the classes $R(n,q,\beta,\lambda)$ and $\overline{R}(n,q,\beta,\lambda)$ By specializing the parameter we obtain the following known subclasses studied earlier by various researchers.

1. $R_H(n,0,\beta,\lambda) \equiv R_H(n,\beta,\lambda)$ and $\overline{R_H}(n,0,\beta,\lambda) \equiv \overline{R_H}(n,\beta,\lambda)$ studied by Porwal and Aouf [13].
2. $R(1,0,\beta,0) \equiv R(\beta)$ studied by Uralegaddi *et al.* [19].

In the present paper, we study the coefficient bounds, distortion bounds, extreme points, convolution condition, convex combinations and discuss a class preserving integral operator.

2. Main Results

First, we give a sufficient coefficient condition for functions in $R_H(n,q,\beta,\lambda)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be such that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty} \left[\phi\left(k,\lambda\right)\right]^n [k]_q |a_k| + \sum_{k=1}^{\infty} \left[\phi\left(k,\lambda\right)\right]^n [k]_q |b_k| \leq \beta - 1. \tag{2.1}$$

Then f is sense-preserving, harmonic univalent in U and $f \in R_H\left(n,q,\beta,\lambda\right)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum\limits_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right| \\
&> 1-\frac{\sum\limits_{k=1}^{\infty} k|b_k|}{1-\sum\limits_{k=2}^{\infty} k|a_k|} \\
&\geq 1-\frac{\sum\limits_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_k|}{1-\sum\limits_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_k|} \\
&\geq 0,
\end{aligned}$$

which proves univalence.

Note that f is sense-preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\
&> 1-\sum_{k=2}^{\infty} k|a_k| \\
&\geq 1-\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_k| \\
&\geq \sum_{k=1}^{\infty} k|b_k| \\
&> \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \\
&\geq |g'(z)|.
\end{aligned}$$

Now, we show that $f \in R_H(n,q,\beta,\lambda)$. Using the fact that $Re\,\omega < \beta$, if and only if, $|\omega-1| < |\omega+1-2\beta|$, it suffices to show that

$$\left|\frac{\frac{\Omega^n(D_q(h(z)))+(-1)^n\overline{\Omega^n(D_q(g(z)))}}{z}-1}{\frac{\Omega^n(D_q(h(z)))+(-1)^n\overline{\Omega^n(D_q(g(z)))}}{z}-(2\beta-1)}\right| < 1, \quad z \in U.$$

We have

$$\left|\frac{\frac{z+\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}a_{k}z^{k}+(-1)^{n}\sum_{k=1}^{\infty}\overline{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}b_{k}z^{k}}}{z}-1}{\frac{z+\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}a_{k}z^{k}+(-1)^{n}\sum_{k=1}^{\infty}\overline{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}b_{k}z^{k}}}{z}-(2\beta-1)}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}a_{k}z^{k-1}+(-1)^{n}\frac{\overline{z}}{z}\sum_{k=1}^{\infty}\overline{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}b_{k}z^{k-1}}}{2\left(\beta-1\right)-\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}a_{k}z^{k-1}-(-1)^{n}\frac{\overline{z}}{z}\sum_{k=1}^{\infty}\overline{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}b_{k}z^{k}}}\right|$$

$$\leq \frac{\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|a_{k}|\,|z|^{k-1}+\sum_{k=1}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|b_{k}|\,|z|^{k-1}}{2\left(\beta-1\right)-\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|a_{k}|\,|z|^{k-1}-\sum_{k=1}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|b_{k}|\,|z|^{k-1}}$$

$$\leq \frac{\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|a_{k}|+\sum_{k=1}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|b_{k}|}{2\left(\beta-1\right)-\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|a_{k}|-\sum_{k=1}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|b_{k}|}$$

which is bounded above by 1 by using (2.1) and so the proof is complete.

The harmonic univalent functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}\frac{\beta-1}{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}}x_{k}z^{k}+\sum_{k=1}^{\infty}\frac{\beta-1}{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}}\overline{y_{k}z^{k}}, \tag{2.2}$$

where $1<\beta\leq 2, 0<q<1,\ 0\leq\lambda\leq 1,\ n\in N$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.2) belongs to the class $R_H(n,\beta,\lambda)$ for all $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\leq 1$ because coefficient inequality (2.1) holds.

Theorem 2.2. *Let f_n be given by (1.3). Then $f_n\in\overline{R_H}(n,q,\beta,\lambda)$ if and only if*

$$\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|a_{k}|+\sum_{k=1}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|b_{k}|\leq\beta-1.$$

Proof. Since $\overline{R_H}(n,q,\beta,\lambda)\subset R_H(n,q,\beta,\lambda)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_n of the form (1.3), we notice that the condition

$$\Re\left\{\frac{\Omega^{n}\left(D_q(h(z))\right)+(-1)^{n}\overline{\Omega^{n}\left(D_q(g(z))\right)}}{z}\right\}<\beta$$

is equivalent to

$$\Re\left\{1+\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}a_{k}z^{k-1}+(-1)^{n}\frac{\overline{z}}{z}\sum_{k=1}^{\infty}\overline{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}b_{k}z^{k-1}}\right\}$$

$$\leq\ 1+\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|a_{k}|\,|z|^{k-1}+\sum_{k=1}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|b_{k}|\,|z|^{k-1}<\beta,\ \ z\in U.$$

The above condition must hold for all values of z, $|z|=r<1$. Upon choosing the values of z to be real and let $z\to 1^{-}$, we obtain

$$\sum_{k=2}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|a_{k}|+\sum_{k=1}^{\infty}\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}|b_{k}|\leq\beta-1,$$

which is the required condition.

The harmonic univalent functions of the form

$$f_n(z)=z+\sum_{k=2}^{\infty}\frac{\beta-1}{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}}x_{k}z^{k}+(-1)^{n}\sum_{k=1}^{\infty}\frac{\beta-1}{\left[\phi\left(k,\lambda\right)\right]^{n}\left[k\right]_{q}}y_{k}\overline{z^{k}}, \tag{2.3}$$

where $1<\beta\leq 2,\ 0<q<1,\ 0\leq\lambda\leq 1,\ n\in N,\ x_k\geq 0,\ y_k\geq 0$ and $\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k\leq 1$ belongs to the class $\overline{R_H}(n,q,\beta,\lambda)$.

Theorem 2.3. *If* $f \in \overline{R_H}(n, q, \beta, \lambda)$, *then*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)}(\beta - 1 - |b_1|)r^2, \ |z| = r < 1$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)}(\beta - 1 - |b_1|)r^2, \ |z| = r < 1.$$

Proof. Let $f \in \overline{R_H}(n, q, \beta, \lambda)$. Taking the absolute value of f, we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\
&\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2 \\
&\leq (1+|b_1|)r + \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)} \sum_{k=2}^{\infty}\left(\frac{2}{1-\lambda}\right)^n (1+q)(|a_k| + |b_k|)r^2 \\
&\leq (1+|b_1|)r + \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)} \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n [k]_q (|a_k| + |b_k|)r^2 \\
&\leq (1+|b_1|)r + \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)}(\beta - 1 - |b_1|)r^2
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\
&\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2 \\
&\geq (1-|b_1|)r - \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)} \sum_{k=2}^{\infty}\left(\frac{2}{1-\lambda}\right)^n (1+q)(|a_k| + |b_k|)r^2 \\
&\geq (1-|b_1|)r - \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)} \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n [k]_q (|a_k| + |b_k|)r^2 \\
&\geq (1-|b_1|)r - \left(\frac{1-\lambda}{2}\right)^n \frac{1}{(1+q)}(\beta - 1 - |b_1|)r^2.
\end{aligned}$$

Theorem 2.4. *Let* $f \in clco\overline{R_H}(n, q, \beta, \lambda)$, *if and only if*

$$f(z) = \sum_{k=1}^{\infty}(\lambda_k h_k(z) + \gamma_k g_k(z)), \tag{2.4}$$

where $h_1(z) = z$

$$\begin{aligned}
h_k(z) &= z + \frac{\beta - 1}{[\phi(k,\lambda)]^n [k]_q} z^k, \qquad (k = 2, 3, ...) \\
g_k(z) &= z + (-1)^n \frac{\beta - 1}{[\phi(k,\lambda)]^n [k]_q} \overline{z}^k, \qquad (k = 1, 2, 3, ...)
\end{aligned}$$

and $\sum_{k=1}^{\infty}(\lambda_k + \gamma_k) = 1, \qquad \lambda_k \geq 0$ *and* $\gamma_k \geq 0.$

In particular the extreme points of $\overline{R_H}(n, q, \beta, \lambda)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. For functions f of the form (2.4) we may write

$$f(z) = \sum_{k=1}^{\infty}\{\lambda_k h_k(z) + \gamma_k g_k(z)\}$$
$$= z + \sum_{k=2}^{\infty}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n\,[k]_q}\right)\lambda_k z^k + (-1)^n\sum_{k=1}^{\infty}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n\,[k]_q}\right)\gamma_k\overline{z}^k.$$

Then

$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n\,[k]_q}\lambda_k\right) + \sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}\left(\frac{\beta-1}{[\phi(k,\lambda)]^n\,[k]_q}\gamma_k\right) = \sum_{k=2}^{\infty}\lambda_k + \sum_{k=1}^{\infty}\gamma_k = 1-\lambda_1 \le 1,$$

and so $f \in \text{clco}\,\overline{R_H}(n,q,\beta,\lambda)$.

Conversely, suppose that $f \in \text{clco}\,\overline{R_H}(n,q,\beta,\lambda)$.

Set
$$\lambda_k = \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|a_k|, \qquad (k = 2,3,4,...)$$
and
$$\gamma_k = \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|b_k|, \qquad (k = 1,2,3,...).$$

Then note that by Theorem 2.2,
$$0 \le \lambda_k \le 1, \quad (k = 2,3,4,\ldots)$$
and
$$0 \le \gamma_k \le 1, \quad (k = 1,2,3,\ldots).$$

We define $\lambda_1 = 1 - \sum_{k=2}^{\infty}\lambda_k - \sum_{k=1}^{\infty}\gamma_k$ and note that by Theorem 2.2, $\lambda_1 \ge 0$. Consequently, we obtain $f(z) = \sum_{k=1}^{\infty}\{\lambda_k h_k(z) + \gamma_k g_k(z)\}$ as required.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic function of the form
$$f_n(z) = z + \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$
and
$$F_n(z) = z + \sum_{k=2}^{\infty}|A_k|z^k + (-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z}^k$$
we define their convolution
$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z + \sum_{k=2}^{\infty}|a_k A_k|z^k + (-1)^n\sum_{k=1}^{\infty}|b_k B_k|\overline{z}^k, \tag{2.5}$$
using this definition, we show that the class $\overline{R_H}(n,q,\beta,\lambda)$ is closed under convolution.

Theorem 2.5. *For $1 < \beta \le \alpha \le 2$, let $f_n \in \overline{R_H}(n,q,\beta,\lambda)$ and $F_n \in \overline{R_H}(n,q,\alpha,\lambda)$. Then $(f_n * F_n)(z) \in \overline{R_H}(n,q,\beta,\lambda) \subseteq \overline{R_H}(n,q,\alpha,\lambda)$.*

Proof. Let $f_n(z) = z + \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k$ be in $\overline{R_H}(n,q,\beta,\lambda)$ and $F_n(z) = z + \sum_{k=2}^{\infty}|A_k|z^k + (-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z}^k$ be in $\overline{R_H}(n,q,\alpha,\lambda)$. Then the convolution $(f_n * F_n)(z)$ is given by (2.5). We wish to show that the coefficients of

$f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n(z) \in \overline{R_H}(n,q,\alpha,\lambda)$, we note that $|A_k| \leq 1$ and $|B_K| \leq 1$. Now, for the convolution function $(f_n * F_n)(z)$ we have

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_k A_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_k B_k| &\leq \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_k| \\ &\leq 1, \qquad \text{(since } f \in \overline{R_H}(n,q,\beta,\lambda)).\end{aligned}$$

Therefore $(f_n * F_n)(z) \in \overline{R_H}(n,q,\beta,\lambda) \subseteq \overline{R_H}(n,q,\alpha,\lambda)$.

Theorem 2.6. *The class $\overline{R_H}(n,q,\beta,\lambda)$ is closed under convex combination.*

Proof. For $i = 1,2,3...$ let $f_{n_i}(z) \in \overline{R_H}(n,q,\beta,\lambda)$ where $f_i(z)$ is given by

$$f_{n_i}(z) = z + \sum_{k=2}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k_i}| \overline{z}^k.$$

Then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_{k_i}| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_{k_i}| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i |a_{k_i}|\right) z^k + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i |b_{k_i}|\right) \overline{z}^k.$$

Then by Theorem 2.2, we have

$$\begin{aligned}&\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}\left(\sum_{i=1}^{\infty} t_i |a_{k_i}|\right) + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}\left(\sum_{i=1}^{\infty} t_i |b_{k_i}|\right) \\ = &\sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_{k_i}| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_{k_i}|\right) \\ \leq &\sum_{i=1}^{\infty} t_i = 1.\end{aligned}$$

Therefore

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{R_H}(n,q,\beta,\lambda).$$

3. A Family of Class Preserving Integral Operator

Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.1) then $F(z)$ defined by relation

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt}, \qquad (c > -1). \tag{3.1}$$

Theorem 3.1. *Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.3) and $f(z) \in \overline{R_H}(n,q,\beta,\lambda)$ then $F(z)$ be defined by (3.1) also belong to $\overline{R_H}(n,q,\beta,\lambda)$.*

Proof. Let
$$f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$
be in $\overline{R_H}(n,q,\beta,\lambda)$ then by Theorem 2.2, we have
$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|b_k|\le 1. \tag{3.2}$$
By definition of $F(z)$ we have
$$F(z)=z+\sum_{k=2}^{\infty}\frac{c+1}{c+k}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}\frac{c+1}{c+k}|b_k|\overline{z}^k.$$
Now
$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}\left(\frac{c+1}{c+k}|a_k|\right)+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}\left(\frac{c+1}{c+k}|b_k|\right)\le\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|b_k|$$
$$\le 1.$$
Thus $F(z)\in\overline{R_H}(n,q,\beta,\lambda)$.

Definition 3.1. Let $f=h+\overline{g}$ be defined by (1.1). Then, the q-Jackson integral operator F_q is defined by the relation
$$F_q(z)=\frac{[c]_q}{z^{c+1}}\int_0^z t^c h(t)d_qt+\overline{\frac{[c]_q}{z^{c+1}}\int_0^z t^c g(t)d_qt}, \tag{3.3}$$
where $[c]_q$ is the q-number defined by (1.6).

Theorem 3.2. *Let $f(z)=h(z)+\overline{g(z)}$ be given by (1.3) and $f(z)\in\overline{R_H}(n,q,\beta,\lambda)$ where $1<\beta\le 2,\ 0<q<1,\ 0\le\lambda<1$. Then F_q defined by (3.3) is also in the class $\overline{R_H}(n,q,\beta,\lambda)$.*

Proof. Let
$$f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$
be in $\overline{R_H}(n,q,\beta,\lambda)$. Then by Theorem 2.2, the condition (3.2) is satisfied.

From the representation (3.3) of F_q, it follows that,
$$F_q(z)=z+\sum_{k=2}^{\infty}\frac{[c]_q}{[k+c+1]_q}|a_k|z^k+(-1)^n\sum_{k=1}^{\infty}\frac{[c]_q}{[k+c+1]_q}|b_k|\overline{z}^k.$$
Since
$$[k+c+1]_q-[c]_q=\sum_{i=0}^{k+c}q^i-\sum_{i=0}^{c-1}q^i=\sum_{i=c}^{k+c}q^i>0$$
$$[k+c+1]_q>[c]_q\quad\text{or}\quad\frac{[c]_q}{[k+c+1]_q}<1.$$
Now
$$\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}\frac{[c]_q}{[k+c+1]_q}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}\frac{[c]_q}{[k+c+1]_q}|b_k|\le\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1}|b_k|\le 1.$$
Thus the proof of Theorem 3.2 is established. □

4. Conclusion

This paper deals with a new class of harmonic univalent functions defined by using $q-$ calculus. Coefficient condition, extreme points, distortion bounds, convolution and convex combination are determined for this class. We also study a class preserving integral operator for this class.

Motivated by a recently-published survey-cum-expository review article by Srivastava [17], the interested reader's attention is drawn toward the possibility of investigating the basic (or $q-$) extensions of the results which are presented in this paper. However, as already pointed out by Srivastava [17], their further extensions using the so-called $(p, q)-$ calculus will be rather trivial and inconsequential variations of the suggested extensions which are based upon the classical $q-$ calculus, the additional parameter p being redundant or superfluous (see, for details, [[17], p. 340]).

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