Süleyman Demirel Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 24, Sayı 1, 230-236, 2020

Süleyman Demirel University
Journal of Natural and Applied Sciences
Volume 24, Issue 1, 230-236, 2020

DOI: 10.19113/sdufenbed.546847

Lucas Polynomial Approach for Second Order Nonlinear Differential Equations

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(Alınış / Received: 29.03.2019, Kabul / Accepted: 15.02.2020, Online Yayınlanma / Published Online: 20.04.2020)

Keywords
Lucas polynomial,
Operational matrices,
Collocation points

Abstract: This paper presents the Lucas polynomial solution of second-order nonlinear ordinary differential equations with mixed conditions. Lucas matrix method is based on collocation points together with truncated Lucas series. The main advantage of the method is that it has a simple structure to deal with the nonlinear algebraic system obtained from matrix relations. The method is applied to four problems. In the first two problems, exact solutions are obtained. The last two problems, Bratu and Duffing equations are solved numerically; the results are compared with the exact solutions and some other numerical solutions. It is observed that the application of the method results in either the exact or accurate numerical solutions.

İkinci Mertebeden Doğrusal Olmayan Diferansiyel Denklemler için Lucas Polinom Yaklaşımı

Anahtar Kelimeler
Lucas polinomu,
İşlevsel matrisler,
Sıralama noktaları

Özet: Bu makale, karışık koşullar altında ikinci mertebeden doğrusal olmayan adi diferansiyel denklemlerin Lucas polinom çözümünü oluşturur. Lucas matris yöntemi sıralama noktaları ile birlikte sınırlandırılmış Lucas serisine dayanmaktadır. Yöntemin en büyük avantajı matris bağıntılarında elde edilen doğrusal olmayan cebirsel sistemi ele almak için basit bir yapıya sahip olmasıdır. Yöntem dört probleme uygulanır. İlk iki problemde, tam çözümler elde edilir. Son iki problemde Bratu ve Duffing denklemleri sayısal olarak çözülür; sonuçlar, tam çözümler ve diğer bazı sayısal çözümler ile karşılaştırılır. Yöntemin uygulanması, tam ve doğru sayısal çözümler vermesine yol açtığı gözlemlenmektedir.

1. Introduction

Nonlinear ordinary differential equations play an important role in many physical phenomena such as chemical reactions [1], spring-mass systems [2, 3], quantum physics [4], analytical chemistry [5], astronomy [6] and biology [7]. From last decade, researchers pay attention towards analytical and numerical solutions of nonlinear ordinary differential equations. However there is a stiff problem while solving nonlinear equations analytically. Thus, the importance of numerical solutions increase day by day.
The Bratu equation arises in the fuel ignition of the thermal combustion theory, radiative heat transfer, and the Chandrasekhar model of the expansion of the universe [10, 11, 13]. Wazwaz [13] studied Adomian decomposition method for a reliable treatment of the Bratu-type equations; in this study, exact solutions of Bratu-type equations are presented. In another study, restarted Adomian's decomposition method is used the approximate the analytical solution by Vahidi and Hasanzade [14]. Several numerical techniques, such as the finite difference method [15], weighted residual method [12], the shooting method [10], decomposition technique [16], Legendre wavelets [17], wavelet analysis method [18], B-spline method [19], Jacobi-Gauss collocation method [20], Laplace transform decomposition method [21], He's variational iteration [22, 23], have been implemented to solve the Bratu model numerically.
On the other hand, Duffing model [24] arises in several scientific fields such as classical oscillator in chaotic systems, non-uniformity caused by an infinite domain, nonlinear mechanical oscillators, magnetic-pliancy mechanical systems, nonlinear vibration of beams and plates, prediction of diseases. Duffing equation occurs as a result of the motion of a body subjected to a nonlinear spring power, linear sticky damping, and periodic powering [25]. A variety of numerical methods such as the improved Taylor matrix method [26], generalized differential quadrature rule [27], shifted Chebyshev polynomials [28], Daftardar-Jafari (DJM) method [25], Runge-Kutta-Fehlenberg algorithm

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[29], Laplace decomposition algorithm [30] are used to obtain numerical solutions of this equation.
In this study, Lucas matrix method is applied for solving the second-order ordinary differential equations with nonlinear terms, namely the Ricatti, Bratu and Duffing equations. To the best knowledge of author's this is the first application of the Lucas polynomial solution for both Bratu and Duffing equations.

2. Fundamentals of the Numerical Method

A general class of second-order nonlinear ordinary differential equation can be expressed as

$$\sum_{k=0}^{2} P_k(x)\, y^{(k)}(x) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x)\, y^{(p)}(x) y^{(q)}(x) = g(x), \tag{1}$$

with the initial conditions

$$\sum_{k=0}^{1} a_{kj} y^{(k)}(a) = \lambda_j,\ j = 0,1. \tag{2}$$

where the functions $P_k(x)$, $Q_{pq}(x)$, $y(x)$ and $g(x)$ are defined on $[a,b]$.
The function $y(x)$ is approximated by an N^{th} order polynomial such as

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n L_n(x) \tag{3}$$

where a_n's are the unknown coefficients to be determined and $L_n(x)$'s are the Lucas polynomials defined as follows [31, 32].

$$L_n(x) = \sum_{n=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \begin{pmatrix} n-k \\ k \end{pmatrix} t^{n-2k}$$

Our aim is to seek the approximate Lucas solution of Eq. 1, which can be expressed in matrix form (see [8, 9])

$$y(x) \cong y_N(x) = \mathbf{L}(x)\mathbf{A} \tag{4}$$

where

$$\mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T.$$

The matrix form $\mathbf{L}(x)$ is

$$\mathbf{L}(x) = \mathbf{T}(x)\mathbf{M}^T. \tag{5}$$

where $\mathbf{T}(x) = \begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix}$ and $\mathbf{M}$ is a nonsingular matrix given as in [8, 9].
Substituting Eq. 5 into Eq. 4 yields

$$y_N(x) = \mathbf{T}(x)\mathbf{M}^T\mathbf{A}.$$

In order to find the k^{th} derivative of the above equation, we use the relation

$$\mathbf{L}^{(k)}(x) = \mathbf{L}(x)\mathbf{S}^k \tag{6}$$

where $\mathbf{S} = (\mathbf{M}^T)^{-1}\mathbf{B}\mathbf{M}^T$,

$$\mathbf{S}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The relation in Eq. 6 enables to approximate the derivatives of the solution as

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \mathbf{L}(x)\mathbf{S}^k\mathbf{A}\, k = 0,1,2,\cdots,m. \tag{7}$$

In addition, by Eq. 7, we can easily present the matrix forms of the expressions

$$\left(y^{(0)}(x)\right)^2, y^{(1)}(x)y^{(0)}(x), \left(y^{(1)}(x)\right)^2, y^{(2)}(x)y^{(1)}(x), \\ y^{(2)}(x)y^{(0)}(x), \left(y^{(2)}(x)\right)^2$$

similarly as follows

$$\begin{aligned} \left(y^{(0)}(x)\right)^2 &= \mathbf{L}(x)\,\overline{\mathbf{L}(x)}\,\overline{\mathbf{A}} \\ y^{(1)}(x)\,y^{(0)}(x) &= \mathbf{L}(x)\,\mathbf{S}\,\overline{\mathbf{L}(x)}\,\overline{\mathbf{A}} \\ \left(y^{(1)}(x)\right)^2 &= \mathbf{L}(x)\,\mathbf{S}\,\overline{\mathbf{L}(x)}\,\overline{\mathbf{S}}\,\overline{\mathbf{A}} \\ y^{(2)}(x)\,y^{(1)}(x) &= \mathbf{L}(x)\,\mathbf{S}^2\,\overline{\mathbf{L}(x)}\,\overline{\mathbf{S}}\,\overline{\mathbf{A}} \\ y^{(2)}(x)\,y^{(0)}(x) &= \mathbf{L}(x)\,\mathbf{S}^2\,\overline{\mathbf{L}(x)}\,\overline{\mathbf{A}} \\ \left(y^{(2)}(x)\right)^2 &= \mathbf{L}(x)\,\mathbf{S}^2\,\overline{\mathbf{L}(x)}\,\overline{\mathbf{S}^2}\,\overline{\mathbf{A}} \end{aligned} \tag{8}$$

where

$$\overline{\mathbf{L}}(x) = \operatorname{diag}\left[\mathbf{L}(x)\right]_{(N+1)\times(N+1)^2},$$

$$\overline{\mathbf{S}} = \operatorname{diag}\left[\mathbf{S}\right]_{(N+1)^2\times(N+1)^2},$$

$$\overline{\mathbf{S}^2} = \operatorname{diag}\left[\mathbf{S}^2\right]_{(N+1)^2\times(N+1)^2}$$

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T,$$

$$\overline{\mathbf{A}} = \begin{bmatrix} a_0 A & a_1 A & \cdots & a_N A \end{bmatrix}^T,.$$

Substituting the collocation points

$$x_i = a + \frac{b-a}{N} i,\ i = 0, 1, \cdots, N$$

into Eq. 1 gives

$$\sum_{k=0}^{2} P_k(x_i) y^{(k)}(x) + \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq}(x_i) y^{(p)}(x_i) y^{(q)}(x_i) = g(x_i),$$

which can be written in matrix form as

$$\sum_{k=0}^{2} \mathbf{P}_k \mathbf{Y}^{(k)} + \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq} \mathbf{Y}^{(p,q)} = \mathbf{G} \tag{9}$$

where

$$\mathbf{P}_k = \mathrm{diag}\begin{bmatrix} P_k(x_0) & P_k(x_1) & \cdots & P_k(x_N) \end{bmatrix}$$

and

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix}, \mathbf{Y}^{(p,q)} = \begin{bmatrix} y^{(p)}(x_0) y^{(q)}(x_0) \\ y^{(p)}(x_1) y^{(q)}(x_1) \\ \vdots \\ y^{(p)}(x_N) y^{(q)}(x_N) \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

When we insert the collocation points into each equation in Eq. 9, we can write the matrices $\mathbf{Y}^{(0,0)}, \mathbf{Y}^{(1,0)}, \mathbf{Y}^{(1,1)}, \mathbf{Y}^{(2,0)}, \mathbf{Y}^{(2,1)}$ and $\mathbf{Y}^{(2,2)}$ as follows

$$\begin{aligned} &\mathbf{Y}^{(0,0)} = \mathbf{L}^*_{0,0}\overline{\mathbf{A}},\ \mathbf{Y}^{(1,0)} = \mathbf{L}^*_{1,0}\overline{\mathbf{A}},\ \mathbf{Y}^{(1,1)} = \mathbf{L}^*_{1,1}\overline{\mathbf{A}} \\ &\mathbf{Y}^{(2,0)} = \mathbf{L}^*_{2,0}\overline{\mathbf{A}},\ \mathbf{Y}^{(2,1)} = \mathbf{L}^*_{2,1}\overline{\mathbf{A}},\ \mathbf{Y}^{(2,2)} = \mathbf{L}^*_{2,2}\overline{\mathbf{A}} \end{aligned} \tag{10}$$

where

$$\mathbf{L}^*_{0,0} = \begin{bmatrix} L(x_0)\overline{L(x_0)} \\ L(x_1)\overline{L(x_1)} \\ \vdots \\ L(x_N)\overline{L(x_N)} \end{bmatrix}, \mathbf{L}^*_{1,0} = \begin{bmatrix} L(x_0)S\overline{L(x_0)} \\ L(x_1)S\overline{L(x_1)} \\ \vdots \\ L(x_N)S\overline{L(x_N)} \end{bmatrix},$$

$$\mathbf{L}^*_{1,1} = \begin{bmatrix} L(x_0)S\,\overline{L(x_0)}\,\overline{S} \\ L(x_1)S\,\overline{L(x_1)}\,\overline{S} \\ \vdots \\ L(x_N)S\,\overline{L(x_N)}\,\overline{S} \end{bmatrix}, \mathbf{L}^*_{2,0} = \begin{bmatrix} L(x_0)S^2\,\overline{L(x_0)} \\ L(x_1)S^2\,\overline{L(x_1)} \\ \vdots \\ L(x_N)S^2\,\overline{L(x_N)} \end{bmatrix},$$

$$\mathbf{L}^*_{2,1} = \begin{bmatrix} L(x_0)S^2\,\overline{L(x_0)}\,\overline{S} \\ L(x_1)S^2\,\overline{L(x_1)}\,\overline{S} \\ \vdots \\ L(x_N)S^2\,\overline{L(x_N)}\,\overline{S} \end{bmatrix},$$

$$\mathbf{L}^*_{2,2} = \begin{bmatrix} L(x_0)S^2\,\overline{L(x_0)}\,\overline{S^2} \\ L(x_1)S^2\,\overline{L(x_1)}\,\overline{S^2} \\ \vdots \\ L(x_N)S^2\,\overline{L(x_N)}\,\overline{S^2} \end{bmatrix}$$

3. Application of the Numerical Method

The fundamental matrix equation derived from Eq. 1 can be stated by substituting the matrix relations in Eq. 7 and Eq. 10 into Eq. 9 as

$$\sum_{k=0}^{2} \mathbf{P}_k \mathbf{L} \mathbf{S}^k \mathbf{A} + \sum_{p=0}^{2} \sum_{q=0}^{p} \mathbf{Q}_{pq} \mathbf{L}^*_{p,q} \overline{\mathbf{A}} = \mathbf{G}$$

which can also be expressed as

$$\mathbf{W}\mathbf{A} + \mathbf{V}\overline{\mathbf{A}} = \mathbf{G} \tag{11}$$

where

$$\mathbf{W} = [w_{ij}] = \sum_{k=0}^{2} \mathbf{P}_k \mathbf{L} \mathbf{S}^k;\ i, j = 0, 1, \cdots, N$$

$$\mathbf{V} = [v_{ij}] = \sum_{p=0}^{2} \sum_{q=0}^{p} \mathbf{Q}_{pq} \mathbf{L}^*_{p,q};\ i = 0, 1, \cdots, N,$$
$$j = 0, 1, \cdots, (N+1)^2$$

The augmented matrix of the equation 11 is

$$[\mathbf{W}; \mathbf{V}; \mathbf{G}] \tag{12}$$

Following the same procedure the matrix equation for the initial conditions given in Eq. 2 can be written as

$$\mathbf{U}\mathbf{A} + \mathbf{O}^*\overline{\mathbf{A}} = \lambda$$

where

$$\begin{aligned} \mathbf{U} &= \begin{bmatrix} u_{j0} & u_{j1} & \cdots & u_{jN} \end{bmatrix} \\ &= \sum_{k=0}^{1} \left(a_{kj}\mathbf{L}(a) + b_{kj}\mathbf{L}(b) + c_{kj}\mathbf{L}(c) \right) \mathbf{S}^k,\ j = 0, 1, \end{aligned}$$

$\lambda = \begin{bmatrix} \lambda_0 & \lambda_1 \end{bmatrix}^T$ and $\mathbf{0}^*$ is the zero matrix in the form

$$\mathbf{0}^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}_{2\times(N+1)^2}.$$

The augmented form of the above equation is

$$\begin{bmatrix} \mathbf{U} & ; & \mathbf{O}^* & ; & \lambda \end{bmatrix} \tag{13}$$

Consequently, to find Lucas coefficients a_n, $(n = 0, 1, \cdots, N)$, by replacing the two row matrices 12 by the last 2 rows (or any two rows) of the augmented matrix 13, we obtain the resulting matrix

$$\left[\widetilde{\mathbf{W}};\ \widetilde{\mathbf{V}} : \widetilde{\mathbf{G}}\right]$$

4. Numerical examples and Discussion

In this section, four stiff problems are considered. In the first two problems the exact solutions are obtained. The third problem is the Bratu equation and the last one is the Duffing equation which are both solved numerically and compared with other numerical results where available.

Problem 1: As a first example, we consider the following nonlinear ordinary differential equation

$$y''(x)+2y'(x)+y(x)+y^2(x)-y''(x)y'(x)=x^2+x+2$$

with the initial conditions $y(0)=0$ and $y'(0)=1$.
In this problem, $P_0(x)=1$, $P_1(x)=2$, $P_2(x)=1$, $Q_{00}(x,y)=1$, $Q_{21}(x,y)=-1$, $g(x)=x^2+x+2$ when $a=0$, $b=1$ and $N=2$, the collocation points are computed as $\{x_0=0,\ x_1=1/2,\ x_2=1\}$. Application of the method gives the fundamental matrix equation of the considered problem as

$$\mathbf{W}=\sum_{k=o}^{2}\mathbf{P}_k\mathbf{L}\,\mathbf{S}^k\mathbf{A}=\mathbf{P}_0\mathbf{L}\,\mathbf{S}^0\mathbf{A}+\mathbf{P}_1\mathbf{L}\,\mathbf{S}^1\mathbf{A}+\mathbf{P}_2\mathbf{L}\,\mathbf{S}^2\mathbf{A}$$

where

$$\mathbf{L}=\begin{bmatrix}2&0&2\\2&\frac{1}{2}&\frac{9}{4}\\2&1&3\end{bmatrix},\mathbf{P}_0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$\mathbf{P}_1=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix},\mathbf{P}_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$\mathbf{S}^0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\mathbf{S}^1=\begin{bmatrix}0&\frac{1}{2}&0\\0&0&2\\0&0&0\end{bmatrix},$$

$$\mathbf{S}^2=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

and hence

$$\mathbf{W}=\begin{bmatrix}2&2&4\\2&\frac{5}{2}&\frac{25}{2}\\2&3&9\end{bmatrix}$$

Let us now prepare

$$\mathbf{V}=\mathbf{Q}_{00}\mathbf{L}^*_{0,0}+\mathbf{Q}_{21}\mathbf{L}^*_{2,1}$$

Recall that

$$\mathbf{L}^*_{0,0}=\begin{bmatrix}4&0&4&0&0&0&4&0&4\\4&1&\frac{9}{2}&1&\frac{1}{4}&\frac{9}{8}&\frac{9}{2}&\frac{9}{8}&\frac{81}{16}\\4&2&6&2&1&3&6&3&9\end{bmatrix},$$

and

$$\mathbf{L}^*_{2,1}=\begin{bmatrix}0&0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&2&2\\0&0&0&0&0&0&0&2&4\end{bmatrix}.$$

Thus, the matrix $\mathbf{V}$ is obtained as

$$\mathbf{V}=\begin{bmatrix}4&0&4&0&0&0&4&-2&4\\4&1&\frac{9}{2}&1&\frac{1}{4}&\frac{9}{8}&\frac{9}{2}&-\frac{7}{8}&\frac{49}{16}\\4&2&6&2&1&3&6&1&5\end{bmatrix}$$

The augmented matrix of the equation $\mathbf{WA}+\mathbf{V}\overline{\mathbf{A}}=\mathbf{G}$ is

$$\begin{bmatrix}\widetilde{\mathbf{W}}&;&\widetilde{\mathbf{V}}&;&\widetilde{\mathbf{G}}\end{bmatrix}=$$

$$\begin{bmatrix}2&2&4&;&4&0&4&0&0&0&4&-2&4&:&2\\2&0&2&;&0&0&0&0&0&0&0&0&0&:&0\\0&1&0&;&0&0&0&0&0&0&0&0&0&:&1\end{bmatrix}$$

where

$$y(0)=0\Rightarrow y(0)=\mathbf{L}(0)\mathbf{A}=0$$

$$y'(0)=1\Rightarrow y'(0)=\mathbf{L}(0)\mathbf{SA}=1$$

and

$$\begin{bmatrix}2&0&2\end{bmatrix}\mathbf{A}=0\Rightarrow$$
$$\begin{bmatrix}2&0&2&;&0&0&0&0&0&0&0&0&0&;&0\end{bmatrix}$$

$$\begin{bmatrix}2&0&2\end{bmatrix}\begin{bmatrix}0&\frac{1}{2}&0\\0&0&2\\0&0&0\end{bmatrix}A=1\Rightarrow$$
$$\begin{bmatrix}0&1&0&;&0&0&0&0&0&0&0&0&0&;&1\end{bmatrix}$$

Solving this system, $\mathbf{A}$ is obtained as $\mathbf{A}=\begin{bmatrix}0&1&0\end{bmatrix}^T$
From Eq. 4, $y(t)$ is obtained as

$$y(x)=\ \mathbf{L}(t)\ \mathbf{A}=\begin{bmatrix}2&x&x^2+2\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Thus, the solution of the problem appears as

$$y(x)=x,$$

which is the exact solution which shows that the present method is accurate, efficient and applicable.

Problem 2: The second example is the well known Riccati equation

$$\begin{aligned}&y'(x)-3y(x)+e^{-x}y^2(x)=-e^x\\&y(0)=0.\end{aligned}$$

In this problem,

$$\begin{cases}P_0(x)=-3,\quad P_1(x)=1,\\Q_{00}(x,y)=e^{-x},\\g(x)=-e^x\end{cases}$$

and the collocation points when $a=0$, $b=1$ and $N=2$ are computed as $\{x_0=0,\ x_1=1/2,\ x_2=1\}$.
The fundamental matrix equation of the given equation is

$$\mathbf{W}=\sum_{k=o}^{1}\mathbf{P}_k\mathbf{L}\,\mathbf{S}^k\mathbf{A}=\mathbf{P}_0\mathbf{L}\,\mathbf{S}^0\mathbf{A}+\mathbf{P}_1\mathbf{L}\,\mathbf{S}^1\mathbf{A}$$

where

$$\mathbf{L}=\begin{bmatrix} 2 & 0 & 2 \\ 2 & \frac{1}{2} & \frac{9}{4} \\ 2 & 1 & 3 \end{bmatrix}, \mathbf{P}_0=\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

$$\mathbf{P}_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{S}^0=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{S}^1=\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{S}^2=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence, $\mathbf{W}$ is computed as

$$\mathbf{W}=\begin{bmatrix} -6 & 1 & -6 \\ -6 & \frac{-1}{2} & \frac{-23}{4} \\ 6 & -2 & -7 \end{bmatrix}.$$

$\mathbf{V}$ is easily obtained via

$$\mathbf{V}=\mathbf{Q}_{00}\mathbf{L}^*_{0,0}$$

$$\mathbf{Q}_{00}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{1}{2}} & 0 \\ 0 & 0 & e^{-1} \end{bmatrix},$$

$$\mathbf{L}^*_{0,0}=\begin{bmatrix} 4 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 4 \\ 4 & 1 & \frac{9}{2} & 1 & \frac{1}{4} & \frac{9}{8} & \frac{9}{2} & \frac{9}{8} & \frac{81}{16} \\ 4 & 2 & 6 & 2 & 1 & 3 & 6 & 3 & 9 \end{bmatrix}$$

After solving the augmented matrix of the equation, $\mathbf{A}$ yields $\mathbf{A}=\begin{bmatrix} 0 & 1 & 1/2 \end{bmatrix}^T$. From Eq. 4, $y(t)$ is obtained as

$$y(x)=\mathbf{L}(t)\ \mathbf{A}=\begin{bmatrix} 2 & x & x^2+2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$=1+x+\frac{x^2}{2}\cong e^x$$

Thus, the solution of the problem becomes $y(x)=e^x$, which is the exact solution.

Problem 3: We considered the classical Bratu's problem [16]

$$y''(x)+\lambda\, e^{y(x)}=0, \quad 0<x<1$$
$$y(0)=y(1)=0$$

which has the exact solution for $\lambda>0$, $y(x)=-2\ln\left[\frac{\cosh((x-1/2)\theta/2)}{\cosh(\theta/4)}\right]$, where θ is the solution of $\theta=\sqrt{2\lambda}\cosh(\theta/4)$.

The problem is solved by taking $N=4$ and $\lambda=1$. The nonlinear term e^y is approximated by $1+y+\frac{y^2}{2}$. Table 1 presents the comparison of our method with the exact solution and Decomposition Method (DM) with $N=6$, [16] at some particular points. One can see that the numerical solutions obtained by the present method using a fourth order polynomial has an accuracy up to four decimal places. On the other hand, the solutions obtained from Adomian Decomposition method using a sixth order polynomial has an accuracy only up to third order decimal place except for two points. Thus, we can say that the present method is more effective since it enables us to solve the problem with a better accuracy by using a less order polynomial.

Table 1. Comparison of the present method (PM) with the exact solution (ES) and Decomposition method (DM) [16]

x_i	ES	PM. $N=4$	DM, $N=6$ [16]	$\lvert e_4(x_i)\rvert$	$\lvert e_6(x_i)\rvert$ [16]
0.1	0.049847	0.049517	0.047162	3.30e-04	2.69e-03
0.2	0.089190	0.088587	0.87168	6.03e-04	2.02e-03
0.3	0.117609	0.117694	0.117761	8.15e-04	1.52e-04
0.4	0.134790	0.133838	0.136992	9.53e-04	2.20e-03
0.5	0.140539	0.139540	0.143555	9.99e-04	3.02e-03
0.6	0.134790	0.133840	0.136992	9.50e-04	2.20e-03
0.7	0.117609	0.116798	0.117761	8.11e-04	1.52e-04
0.8	0.089190	0.088592	0.087168	5.98e-04	2.02e-03
0.9	0.049847	0.049520	0.047161	3.26e-04	2.69e-03

Fig. 1 shows the numerical and exact solutions of the problem. It can be seen that the numerical solution matches well with the exact solution. This shows that the present method is effective even using a fourth order polynomial.

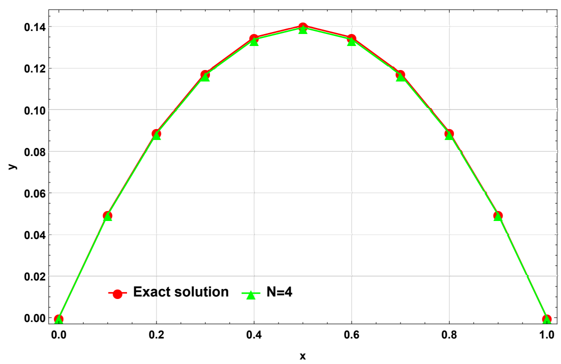

Figure 1. Exact solution and computed solution for $N=4$

Problem 4: The final example is the Duffing type equation

$$y''(x)+y'(x)-y(x)y'(x)=2e^x-e^{2x}, \quad x\in[0,0.5]$$
$$y(0)=y'(0)=1$$

The exact solution of the problem is $y(x)=e^x$. The problem is solved by taking $N=5$ and $N=6$ to find out the order of the polynomial which is adequate to approximate the exact solution. The numerical results and the exact solution are presented in Fig. 2. From this figure, one can see that the numerical solution for $N=5$ does not match with the exact solution for increasing values of t, but $N=6$ matches well with the exact solution. In order to emphasize this difference, the absolute errors for both values of

N are given in Fig. 3. It can be seen that when $N = 6$ the absolute errors are quite smaller. This shows that taking $N = 6$ is adequate to solve this problem.

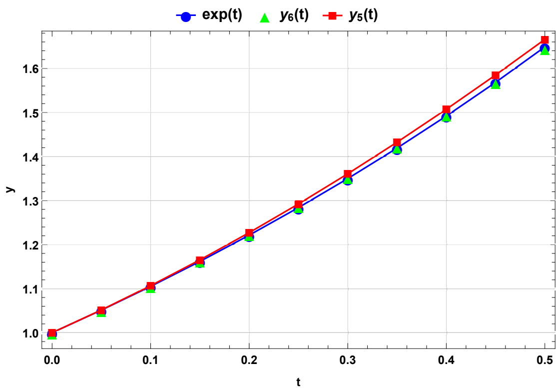

Figure 2. Comparison of the exact solution and the approximated solutions for $N = 5$ and $N = 6$.

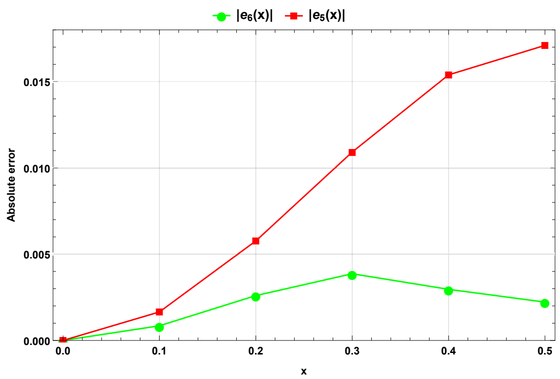

Figure 3. Error analysis of the results for $N = 5$ and $N = 6$.

5. Conclusion

In this paper, the Lucas matrix method is applied to solve nonlinear ordinary differential equations. The method has a simple and efficient structure, which makes it eligible to solve systems of nonlinear algebraic equations, thus greatly simplifying the problems. The proposed method provides several prominent advantages compared to other numerical methods. Main advantage is that the method converts the nonlinear algebraic functions to a matrix system then solves directly the system by eliminating rows of the resulting matrix. Thus, the numerical solution is immediately reached by inserting the obtained coefficients into the solution form. This issue makes the method practical without using any iteration techniques for differential equations of nonlinear type.
Four test problems are considered. In the first two problems, the application of the method results in the exact solutions. The third and fourth problems are the Bratu and Duffing equations, which are solved numerically and compared with the analytical solutions and other numerical solutions when available. One can see that the numerical results have high accuracy even for quite small values of N.
It is well known that many nonlinear ordinary differential equations either do not have an analytical solutions or have complex solutions which require numerical methods. In order to solve such kind of problems, one has to verify the numerical technique on equations where the analytical solutions are known. Hence, the applications of the Lucas matrix method can be extended to solve indicated type of problems as it is eligible to solve stiff problems.

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