ISSN: **1304-7981** Number: **7**, Year: **2014**, Pages: **7-19** **http://jnrs.gop.edu.tr**

Gaziosmanpasa University
Graduate School of Natural and Applied Sciences
Journal of New Results in Science

Received: 26.05.2014
Accepted: 06.08.2014

Editors-in-Chief: *Naim Çağman*
Area Editor: *Oktay Muhtaroğlu*

Generalized $\omega\alpha$-Closed Sets in Topological Spaces

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Abstract - The aim of this paper is to introduce a new class of closed sets called g$\omega\alpha$-closed sets using $\omega\alpha$-closed sets in topological spaces. This class is independent of $\omega\alpha$-closed sets. This new class of set lies between the class of α-closed sets and the class of αg-closed sets. Some of their properties are investigated. We also define and study the g$\omega\alpha$-closure and g$\omega\alpha$-interior in topological spaces.

Keywords - *Topological spaces, generalized closed sets, $\omega\alpha$-closed sets, g$\omega\alpha$-closed sets and g$\omega\alpha$-open sets.*

1 Introduction

In 1969 Levine [9] gives the concept and properties of generalized closed (briefly g-closed) sets and the complement of g-closed set is said to be g-open set. In 1982 Mashhour et.al [13] introduced and studied the concept of pre-open set. Later Maki et.al [12], Dontechev [6], Gyanambal [7], Arya and Nour [3] and Bhattacharya and Lahiri [4] introduced and studied the concepts of gp-closed, gsp-closed, gpr-closed, gs-closed, sg-closed and αg-closed and their compliments are respective open sets.

N Jasted [16] introduced and studied the concept of α-sets. Later these sets are called as α-open sets in 1983. Mashhours et.al [14] introduced and studied the concept of α-closed sets, α-closure of set, α-continuous functions, α-open functions and α-closed functions in topological spaces. Maki et.al [10] [11] introduced and studied generalized α-closed sets and α-generalized closed sets. Sundarm and Sheik John [20] defined and studied ω-closed sets in topological spaces and recently S.S.Benchalli et.al [5] studied $\omega\alpha$-closed sets in topological spaces.

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2 Preliminaries

Throughout this paper space (X,τ) and (Y,σ) (or simply X and Y) always denote topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space (X, τ) Cl(A), Int(A) and A^c denote the Closure of A, Interior of A and Compliment of A respectively.

Definition 2.1. *A subset A of a topological space (X, τ) is called,*
(i) Semi-open set [8] if A$\subseteq$Cl(Int(A)) and Semi-closed set if Int(Cl(A))$\subseteq$A.
(ii) Pre-open set [13] if A$\subseteq$Int(Cl(A)) and Pre-closed set if Cl(Int(A))$\subseteq$A.
(iii) α-open set [16] if A$\subseteq$Int(Cl(Int(A))) and α-closed set if Cl(Int(Cl(A)))$\subseteq$A.
(iv) Semi-pre-open set [2] (=β-open set [1]) if A$\subseteq$Cl(Int(Cl(A))) and semi-pre-closed (=β-closed set [1]]) if (Cl(Int(Cl(A)))$\subseteq$A.
(v) Regular-open [7] if A=Int(Cl(A)) and Regular-closed if A=Cl(Int(A)).

The α-closure of A is the smallest α-closed set containing A, and this is denoted by αCl(A). Similarly the semi-closure (resp pre-closure and semi-pre-closure) of a set A in a topological space (X,τ) is the intersection of all semi-closed (resp pre-closed and semi-pre-closed) sets containing A and is denoted by scl(A) (resp pcl(A) and spcl(A)).

Definition 2.2. *A subset of a topological space (X,τ) is called a,*
(i) *Generalized closed (briefly g-closed) set [9] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.*
(ii) *Semi-generalized closed (briefly sg-closed) set [4] if scl(A)$\subseteq U$ whenever A$\subseteq U$ and U is Semi-open in X.*
(iii) *Generalized semi-closed (briefly gs-closed) set [3] if scl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.*
(iv) *Generalized α-closed (briefly gα-closed) set [10] if αcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is α-open in X.*
(v) *α-generalized closed (briefly αg-closed) set [11] if αcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.*
(vi) *Generalized pre-closed (briefly gp-closed) set [12] if pcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.*
(vii) *Generalized semi-pre-closed (briefly gsp-closed) set [6] if spcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.*
($viii$) *Generalized pre-regular-closed (briefly gpr-closed) set [7] if pcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is regular-open in X.*
(ix) *Weakly closed (briefly ω-closed) set [21] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is semi-open in X.*
(x) *Weakly generalized closed (briefly ωg-closed) set [20] if cl(int(A))$\subseteq U$ whenever A$\subseteq U$ and U is open in X.*
(xi) *Strongly generalized closed (briefly g^*-closed) set [18] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is g-open in X.*
(xii) *Regular generalized closed (briefly rg-closed) set [17] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is regular-open in X.*
($xiii$) *α-generalized regular closed (briefly αgr-closed) set [23] if αcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is regular-open in X.*
(xv) *g^*-preclosed (briefly g^*p-closed) [22] if pcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is g-open*

in X.
(xiv) *$\omega\alpha$ closed set [5] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open in X.*

The compliment of the above mentioned closed sets are their open sets respectively.

3 g$\omega\alpha$-closed sets in Topological spaces.

In this section we introduce g$\omega\alpha$-closed sets in topological space and study some of their properties.

Definition 3.1. *A subset A of a topological space (X, τ) is called a generalized $\omega\alpha$-closed (g$\omega\alpha$-closed) set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\omega\alpha$-open in X.*

Theorem 3.2. *Every closed set in X is g$\omega\alpha$-closed set.*

Proof: *Let A be a closed set in a topological space X, let G be any $\omega\alpha$-open sets in X such that $A \subseteq G$, Since A is closed, we have $cl(A) = A$, but $\alpha cl(A) \subseteq cl(A)$ is always true. So $\alpha cl(A) \subseteq cl(A) \subseteq G$. Therefore $\alpha cl(A) \subseteq G$. Hence A is g$\omega\alpha$-closed set.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.3. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ then the set $A = \{a, c\}$ is g$\omega\alpha$-closed but not closed.*

Theorem 3.4. *Every α-closed set in X is g$\omega\alpha$-closed set.*

Proof: *Let A be α-closed set in a topological space X. Let U be $\omega\alpha$-open set in X such that $A \subseteq U$. Since A is α-closed we have $\alpha cl(A) = A \subseteq U$. Therefore $\alpha cl(A) \subseteq U$. Hence A is g$\omega\alpha$-closed set.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.5. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ then the set $A = \{b\}$ is g$\omega\alpha$-closed but not α-closed in X.*

Theorem 3.6. *Every g$\omega\alpha$-closed set in X is αg-closed set in X.*

Proof: *Let A be g$\omega\alpha$-closed set in X. Let U be any open set in X, such that $A \subseteq U$. Since every open set is $\omega\alpha$-open set and A is g$\omega\alpha$-closed, we have $\alpha cl(A) \subseteq U$ and hence A is αg-closed set in X.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.7. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$ then the set $A = \{a, b\}$ is αg-closed but not g$\omega\alpha$-closed in X.*

Remark 3.8. *From the theorem 3.4 and 3.6 it follows that $g\omega\alpha$-closed set properly lies between α-closed set and αg-closed set.*

Theorem 3.9. *Every regular-closed (resp ω-closed, $g\alpha$-closed) set is $g\omega\alpha$-closed set.*

Proof: *The proof is obvious from theorem 3.2.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.10. *In Example 3.3 the set $A = \{a, c\}$ is $g\omega\alpha$-closed but not regular-closed (ω-closed, $g\alpha$-closed) set in X.*

Theorem 3.11. *Every $g\omega\alpha$-closed set in X is gs-closed (resp gp-closed, gsp-closed, gpr-closed, rg-closed, ωg-closed, αgr-closed, g^*p-closed) set in X.*

Proof: *Since every open set is $\omega\alpha$-open [5], the proof follows.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.12. *In Example 3.7, the set $A = \{a, b\}$ is gs-closed (gp-closed, gsp-closed, gpr-closed, rg-closed, ωg-closed, αgr-closed) but not $g\omega\alpha$-closed in X.*

Example 3.13. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ then the set $A = \{a, b\}$ is g^*p-closed but not $g\omega\alpha$-closed set in X.*

Remark 3.14. *The concept of $g\omega\alpha$-closed set is independent of the concept of sets namely p-closed, sp-closed, semi-closed, g-closed, sg-closed, g^*-closed, g^*s-closed, $\omega\alpha$-closed sets as seen from the following example.*

Example 3.15. *In Example 3.10, the set $A = \{a, c\}$ is $g\omega\alpha$-closed but not p-closed, sp-closed, semi-closed, sg-closed, g^*s-closed, and the set $B=\{b\}$ is $g\omega\alpha$-closed but not g-closed and g^*-closed in X.*

Example 3.16. *In Example 3.5, the set $A = \{b\}$ is $g\omega\alpha$-closed but not $\omega\alpha$-closed set in X.*

Example 3.17. *Let $X = \{a, b, c, d\}$ and $\tau = \{ X, \phi, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$ then the set $A = \{b\}$ is p-closed and sp-closed but not $g\omega\alpha$-closed set in X.*

Example 3.18. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ then the set $A = \{a\}$ is semi-closed, sg-closed and g^*s-closed but not $g\omega\alpha$-closed set in X .*

Example 3.19. *In Example 3.13, the set $A = \{a, b\}$ is g-closed, g^*-closed, and $\omega\alpha$-closed but not $g\omega\alpha$-closed set in X.*

Theorem 3.20. *Union of two $g\omega\alpha$-closed sets are a $g\omega\alpha$-closed set.*

Proof: *Let A and B be two $g\omega\alpha$-closed sets in (X, τ), let G be any $\omega\alpha$-open set in (X, τ), such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are $g\omega\alpha$-closed sets, $\alpha cl(A) \subseteq G$ and $\alpha cl(B) \subseteq G$. Therefore $\alpha cl(A) \cup \alpha cl(B) = \alpha cl(A \cup B) \subseteq G$. Hence $A \cup B$ is $g\omega\alpha$-closed set.*

Theorem 3.21. *If a subset A of X is $g\omega\alpha$-closed in (X, τ) then $\alpha cl(A)$-A does not contain any non empty $\omega\alpha$-closed set in (X, τ).*

Proof: *Suppose A is $g\omega\alpha$-closed and F be a non empty $\omega\alpha$-closed subset of $\alpha cl(A)$-A. Then $F \subseteq \alpha cl(A) \cap$ (X-A). Since (X-A) is $\omega\alpha$-open and A is $g\omega\alpha$-closed. $\alpha cl(A) \subseteq$ (X-A), therefore $F \subseteq$ (X-$\alpha cl(A)$). Thus $F \subseteq \alpha cl(A) \cap (X\text{-}\alpha cl(A)) = \phi$. That is $F = \phi$. Thus $\alpha cl(A)$-A does not contain any non-empty $\omega\alpha$-closed set in (X, τ).*

However the converse of the above theorem need not be true as seen from the following example.

Example 3.22. *In Example 3.17, the set A = $\{a, b\}$ then $\alpha cl(A)$-A = $\{c, d\}$ does not contain non empty $\omega\alpha$-closed set. But A is not $g\omega\alpha$-closed set in (X, τ).*

Theorem 3.23. *If A is $g\omega\alpha$-closed set in X and $A \subseteq B \subseteq \alpha cl(A)$ then B is also $g\omega\alpha$-closed set in X.*

Proof: *It is given that A is $g\omega\alpha$-closed set in X. To prove B is also $g\omega\alpha$-closed set of X. Let U be an $\omega\alpha$-open set of X, such that $B \subseteq U$. Since $A \subseteq B$, we have $A \subseteq U$. Since A is $g\omega\alpha$-closed, and $\alpha cl(A) \subseteq U$. Now $\alpha cl(B) \subseteq \alpha cl(\alpha cl(A)) = \alpha cl(A) \subseteq U$. So $\alpha cl(B) \subseteq U$. Hence B is $g\omega\alpha$-closed set in X.*

However the converse of the above theorem need not be true as seen from the following example.

Example 3.24. *In Example 3.5, the set A = $\{a\}$ and B = $\{a, b\}$ such that A and B are $g\omega\alpha$-closed sets but $A \subseteq B \nsubseteq \alpha cl(A)$.*

Theorem 3.25. *For each $x \in X$ either x is $\omega\alpha$-closed or x^c is $g\omega\alpha$-closed in X.*

Proof: *Suppose $\{x\}$ is not $\omega\alpha$-closed in X, then $\{x\}^c$ is not $\omega\alpha$-open and the only $\omega\alpha$-open set containing $\{x\}^c$ is the space X itself. Therefore $\alpha cl(\{x\}^c) \subseteq X$. and hence $\{x\}^c$ is $g\omega\alpha$-closed set in (X, τ).*

Theorem 3.26. *Let A be $g\omega\alpha$-closed in (X, τ). Then A is α-closed if and only if $\alpha cl(A)$-A is $\omega\alpha$-closed.*

Proof: *Necessity: Suppose A be α-closed. Then $\alpha cl(A) = A$ and so $\alpha cl(A)\text{-}A = \phi$, which is $\omega\alpha$-closed.*
Sufficiency: Suppose $\alpha cl(A)$-A is $\omega\alpha$-closed. Then $\alpha cl(A)\text{-}A = \phi$, since A is $g\omega\alpha$-closed. That is $\alpha cl(A)$-A or A is α-closed.

Theorem 3.27. *Let $A \subseteq Y \subseteq X$, and suppose that A is $g\omega\alpha$-closed set in X. Then A is $g\omega\alpha$-closed relative to Y.*

Proof: *Let $A \subseteq Y \cap G$ where G is $\omega\alpha$-open. Then $A \subseteq G$ and hence $\alpha cl(A) \subseteq G$. This implies that $Y \cap \alpha cl(A) \subseteq Y \cap G$. Thus A is $g\omega\alpha$-closed relative to Y.*

Now we introduce the following.

Definition 3.28. *A subset A of a topological space (X, τ) is called g$\omega\alpha$-open set if its compliment A^c is g$\omega\alpha$-closed.*

Theorem 3.29. *A subset A of (X, τ) is g$\omega\alpha$-open set if and only if U $\subseteq$ α int(A) whenever U is $\omega\alpha$-closed and U $\subseteq$ A.*

Proof: *Assume that A is g$\omega\alpha$-open in X and U is $\omega\alpha$-closed set of (X, τ) such that U $\subseteq$ A. Then X-A is a g$\omega\alpha$-closed set in (X, τ). Also X-A $\subseteq$ X-U and X-U is $\omega\alpha$-open set of (X, τ). This implies that αcl(X-A) $\subseteq$ X-U. But αcl(X-A) = X-αint(A). Thus X-αint(A) $\subseteq$ X-U. So U $\subseteq$ αint(A).*

Conversely: Suppose U $\subseteq$ αint(A) whenever U is $\omega\alpha$-closed and U $\subseteq$ A, To prove that A is g$\omega\alpha$-open. Let G be $\omega\alpha$-open set of (X, τ) such that X-A $\subseteq$ G. Then X-G $\subseteq$ A. Now X-G is $\omega\alpha$-closed set containing A. So X-G $\subseteq$ αint(A), X-αint(A) $\subseteq$ G, But αcl(X-A) = X-αint(A). Thus αcl(X-A) $\subseteq$ G. That is X-A is g$\omega\alpha$-closed set and hence A is g$\omega\alpha$-open.

Theorem 3.30. *If A is $\omega\alpha$-open and g$\omega\alpha$-closed set then A is α-closed.*

Proof: *Since A $\subseteq$ A and A is $\omega\alpha$-open and g$\omega\alpha$-closed, we have αcl(A) $\subseteq$ A. Thus αcl(A) = A. Hence A is α-closed set of (X, τ).*

Theorem 3.31. *A regular open g$\omega\alpha$-closed set is preclosed and hence clopen.*

Proof: *Let A be regular open g$\omega\alpha$-closed. Since regular open set is $\omega\alpha$-open, αcl(A) $\subseteq$ A. This implies A is α-closed. Since every α-closed (regular) open set is (regular) closed, A is clopen.*

Theorem 3.32. *A set A is g$\omega\alpha$-open in (X, τ) if and only if F $\subseteq$ αint(A) whenever F is $\omega\alpha$-closed in (X, τ) and F $\subseteq$ A.*

Proof: *Suppose F $\subseteq$ αint(A) where F is $\omega\alpha$-closed and F $\subseteq$ A. Let X-A $\subseteq$ G where G is $\omega\alpha$-open in (X, τ). Then G $\subseteq$ X-G and X-G $\subseteq$ αint(A). Thus X-A is g$\omega\alpha$-closed in (X, τ). Hence A is g$\omega\alpha$-open in (X, τ).*

Conversely: Suppose that A is g$\omega\alpha$-open. F $\subseteq$ A and F is $\omega\alpha$-closed in (X, τ). Then X-F is $\omega\alpha$-open and X-A $\subseteq$ X-F. Therefore αcl(X-A) $\subseteq$ X-F. But αcl(X-A) = X-αint(A). Hence F $\subseteq$ αint(A).

Theorem 3.33. *A subset A is g$\omega\alpha$-open in (X, τ) if and only if G = X whenever G is $\omega\alpha$-open and αint(A) $\cup$ (X-G) $\subseteq$ G.*

Proof: *Let A be g$\omega\alpha$-open. G be $\omega\alpha$-open and αint(A) $\cup$ (X-A) $\subseteq$ G. This gives X-G $\subseteq$ (X-αint(A)) $\cap$ (X-(X-A)) = X-αint(A)-(X-A) = αcl(X-A)-(X-A). Since X-A is g$\omega\alpha$-closed and X-G is $\omega\alpha$-closed. Then by theorem 3.32 it follows that X-G = ϕ. Therefore X = G.*

Conversely: Suppose F is $\omega\alpha$-closed and F $\subseteq$ A. Then αint(A) $\cup$ (X-A) $\subseteq$ αint(A) $\cup$ (X-F). It follows that αint(A) $\cup$ (X-F) = X and hence F $\subseteq$ αint(A). Therefore A is g$\omega\alpha$-open in (X, τ).

4 g$\omega\alpha$-Closure and g$\omega\alpha$-Interior

In this section the notion of g$\omega\alpha$-closure and g$\omega\alpha$-interior is defined and some of its basic properties are studied.

Definition 4.1. *For a subset A of (X, τ) g$\omega\alpha$-closure of A is denoted by g$\omega\alpha$cl(A) and is defined as g$\omega\alpha$cl(A) = $\cap\{G;\ A \subseteq G$, G is g$\omega\alpha$-closed in (X, τ)$\}$.*

Theorem 4.2. *For an $x \in X$, $x \in$ g$\omega\alpha$cl(A) if and only if $A \cap V \neq \phi$ for every g$\omega\alpha$-open set V containing x.*

Proof: *Let $x \in$ g$\omega\alpha$cl(A). Suppose there exists a g$\omega\alpha$-open set V containing x such that $V \cap A = \phi$. Then $A \subseteq$ X-V, g$\omega\alpha$cl(A) $\subseteq$ X-V. This implies $x \notin$ g$\omega\alpha$cl(A) which is a contradiction. Hence $A \cap V \neq \phi$.*

Conversely, Suppose $x \notin$ g$\omega\alpha$cl(A) then there exists g$\omega\alpha$-closed set G containing A such that $x \notin G$. Then $x \in$ X-G and X-G is g$\omega\alpha$-open. Also (X-G) $\cap A = \phi$ which is a contradiction to the hypothesis, Hence $x \in$ g$\omega\alpha$cl(A).

Theorem 4.3. *If $A \subseteq X$, then $A \subseteq$ g$\omega\alpha$cl(A) $\subseteq$cl(A).*

Proof: *Since every closed set is g$\omega\alpha$-closed, the proof follows.*

Remark 4.4. *Both containment relations in the theorem 4.3 may be proper as seen from the following example.*

Example 4.5. *In Example 3.10, the set A = $\{a\}$ then g$\omega\alpha$cl(A) = $\{a,\ c\}$ and cl(A) = X, and so $A \subseteq$ g$\omega\alpha$cl(A) $\subseteq$cl(A).*

Theorem 4.6. *If A is g$\omega\alpha$-closed, then g$\omega\alpha$cl(A) = A.*

Proof: *Let A be g$\omega\alpha$-closed set in (X, τ). Since $A \subseteq A$ and A is g$\omega\alpha$-closed set, $A \in \{G;\ A \subseteq G$, G is g$\omega\alpha$-closed set $\}$ which implies that A = $\cap\{G;\ A \subseteq G$, G is g$\omega\alpha$-closed set $\} \subseteq A$, that is g$\omega\alpha$cl(A) $\subseteq A$. But $A \subseteq$ g$\omega\alpha$cl(A) is always true. Hence A = g$\omega\alpha$cl(A).*

Theorem 4.7. *If $A \subseteq X$ and A is g$\omega\alpha$-closed, then g$\omega\alpha$cl(A) is the smallest g$\omega\alpha$-closed subset of X containing A.*

Proof: *Let A be g$\omega\alpha$-closed set in (X, τ). Then g$\omega\alpha$cl(A) = $\cap\{$ G; $A \subseteq G$, G is g$\omega\alpha$-closed in (X, τ)$\}$ Since $A \subseteq A$ and A is g$\omega\alpha$-closed set, g$\omega\alpha$cl(A) = A is the smallest g$\omega\alpha$-closed subset of X containing A.*

However the converse of the above theorem need not be true as seen from the following example.

Example 4.8. *In Example 3.13, the set A = $\{a,\ c\}$ then g$\omega\alpha$cl(A) = X, which is the smallest g$\omega\alpha$-closed set in X containing A but A is not g$\omega\alpha$-closed in (X, τ).*

Remark 4.9. *The following example shows that for any two subsets A and B of X, A$\subseteq$B implies g$\omega\alpha$cl(A) $\neq$ g$\omega\alpha$cl(B).*

Example 4.10. *In example 3.13, the set* $A = \{c\}$ *and* $B = \{a, c\}$ *then* $A \subseteq B$. *Now* $g\omega\alpha cl(A) = \{c\}$ *and* $g\omega\alpha cl(B) = X$. *Hence* $g\omega\alpha cl(A) \neq g\omega\alpha cl(B)$.

Remark 4.11. *For a subset A of* (X, τ) $g\omega\alpha cl(A) \neq cl(A)$ *as seen from the following example.*

Example 4.12. *In Example 3.13, the set* $A = \{c\} \subseteq X$, $g\omega\alpha cl(A) = \{c\}$ *and* $cl(A) = \{b, c\}$ *Therefore* $g\omega\alpha cl(A) \neq cl(A)$.

Remark 4.13. *For any two subsets A and B of* (X, τ), $g\omega\alpha cl(A) = g\omega\alpha cl(B)$ *does not imply that* $A = B$. *This is shown by the following example.*

Example 4.14. *In Example 3.7, the set* $A = \{a\}$ *and* $B = \{a, c\}$ *then* $g\omega\alpha cl(A) = g\omega\alpha cl(B)$. *But* $A \neq B$.

Theorem 4.15. *Let A and B be the subsets of* (X, τ), *Then,*
1. $g\omega\alpha cl(\phi) = \phi$.
2. $g\omega\alpha cl(X) = X$.
3. $g\omega\alpha cl(A)$ *is* $g\omega\alpha$*-closed set in* (X, τ).
4. *If* $A \subseteq B$ *then* $g\omega\alpha cl(A) \subseteq g\omega\alpha cl(B)$.
5. $g\omega\alpha cl(A \cup B) = g\omega\alpha cl(A) \cup g\omega\alpha cl(B)$.
6. $g\omega\alpha cl(g\omega\alpha cl(A)) = g\omega\alpha cl(A)$.

Proof: *Proof of (1), (2), (3) and (4) are obvious from definition 4.1.*
(5). We know that $g\omega\alpha cl(A) \subseteq g\omega\alpha cl(A \cup B)$ *and* $g\omega\alpha cl(B) \subseteq g\omega\alpha cl(A \cup B) \Rightarrow g\omega\alpha cl(A) \cup g\omega\alpha cl(B) \subseteq g\omega\alpha cl(A \cup B)$*–(i). Now we prove* $g\omega\alpha cl(A \cup B) \subseteq g\omega\alpha cl(A) \cup g\omega\alpha cl(B)$. *let x be any point such that* $x \notin g\omega\alpha(A) \cup g\omega\alpha cl(B)$, *then there exists* $g\omega\alpha$*-closed sets P and Q such that* $A \subseteq P$ *and* $B \subseteq Q$, $x \notin P$ *and Q, then* $x \notin P \cup Q$, $A \cup B \subseteq P \cup Q$ *and* $P \cup Q$ *is* $g\omega\alpha$*-closed set by Theorem 3.20, thus* $x \notin g\omega\alpha cl(A \cup B) \Rightarrow g\omega\alpha cl(A \cup B) \subseteq g\omega\alpha cl(A) \cup g\omega\alpha cl(B)$*–(ii). From (i) and (ii)* $g\omega\alpha cl(A \cup B) = g\omega\alpha cl(A) \cup g\omega\alpha(B)$.
(6). Let P be $g\omega\alpha$*-closed set containing A. Then by definition 4.1* $g\omega\alpha cl(A) \subseteq P$. *Since P is* $g\omega\alpha$*-closed set and contains* $g\omega\alpha cl(A)$ *and is contained in every* $g\omega\alpha$*-closed set containing A, it follows* $g\omega\alpha cl(g\omega\alpha cl(A)) \subseteq g\omega\alpha cl(A)$. *Therefore* $g\omega\alpha cl(g\omega\alpha cl(A)) = g\omega\alpha cl(A)$.

Theorem 4.16. *Let A and B be subset of* (X, τ) *then* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(A) \cap g\omega\alpha cl(B)$.

Proof: *Since* $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$, *by theorem 4.15 (4),* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(A)$ *and* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(B)$. *Thus* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(A) \cap g\omega\alpha cl(B)$.

In general $\text{g}\omega\text{cl(A)} \cap \text{g}\omega\alpha\text{cl(B)} \subseteq \text{g}\omega\alpha\text{cl(A} \cap \text{B)}$ as seen from the following example.

Example 4.17. *In Example 3.18, the set* $A = \{a\}$ *and* $B = \{b\}$ *then* $g\omega\alpha cl(A) = \{a, c\}$ *and* $g\omega\alpha cl(B) = \{b, c\}$ *and* $g\omega\alpha cl(A \cap B) = \phi$. *Hence* $g\omega\alpha cl(A) \cap g\omega\alpha cl(B) \subseteq g\omega\alpha cl(A \cap B)$.

Now we introduce the following.

Definition 4.18. *For a subset A of (X, τ) $g\omega\alpha$-interior of A is denoted by $g\omega\alpha int(A)$ and is defined as $g\omega\alpha int(A) = \cup\{$ G; $G \subseteq A$ and G is $g\omega\alpha$-open in $(X, \tau)\}$. that is $g\omega\alpha int(A)$ is the union of all $g\omega\alpha$-open sets contained in A.*

Theorem 4.19. *Let A be subset of (X, τ) then $g\omega\alpha int(A)$ is the largest $g\omega\alpha$-open subset of X contained in A if A is $g\omega\alpha$-open.*

Proof: *Let $A \subseteq X$ be $g\omega\alpha$-open, then $g\omega\alpha int(A) = \cup\{$ G; $G \subseteq A$ and G is $g\omega\alpha$-open in (X, τ) $\}$ Since $A \subseteq A$ and A is $g\omega\alpha$-open, $A = g\omega\alpha int(A)$ is the largest $g\omega\alpha$-open subset of X contained in A.*

The converse of the above theorem need not be true as seen from the following example.

Example 4.20. *In Example 3.18, the set $A = \{b, c\}$, then $g\omega\alpha int(A) = \{b\}$ is $g\omega\alpha$-open in (X, τ), but A is not $g\omega\alpha$-open in (X, τ).*

Remark 4.21. *For any subset A of X, $int(A) \subseteq g\omega\alpha int(A) \subseteq A$.*

Remark 4.22. *For a subset A of X, $g\omega\alpha int(A) \neq int(A)$ as seen from the following example.*

Example 4.23. *In Example 3.5, the set $A = \{b\}$, then $g\omega\alpha int(A) = \{b\}$ and $int(A) = \phi$ hence $g\omega\alpha int(A) \neq int(A)$.*

Remark 4.24. *For any two subsets A and B of X $g\omega\alpha int(A) = g\omega\alpha int(B)$ does not imply that $A = B$. That is shown by the following example.*

Example 4.25. *In Example 3.7, the set $A = \{b\}$ and $B = \{c\}$ then $g\omega\alpha int(A) = \phi = g\omega\alpha int(B)$. But $A \neq B$.*

Remark 4.26. *For any two subsets A and B of X, $g\omega\alpha int(A) \cup g\omega\alpha int(B) \neq g\omega\alpha int(A \cup B)$.*

Example 4.27. *In Example 3.18 the set $A = \{b, c\}$ and $B = \{a, c\}$ now $g\omega\alpha int(A) = \{b\}$ and $g\omega\alpha int(B) = \{a\}$ and $g\omega\alpha int(A \cup B) = g\omega\alpha intX = X$. Hence $g\omega\alpha int(A) \cup g\omega\alpha int(B) \neq g\omega\alpha int(A \cup B)$.*

Theorem 4.28. *For any subset A of X $[X\text{-}g\omega\alpha int(A)] = [g\omega\alpha cl(X\text{-}A)]$.*

Proof: *Let $X \in X\text{-}g\omega\alpha int(A)$, then X is not in $g\omega\alpha int(A)$, that is every $g\omega\alpha$-open set G containing x is such that $G \subseteq A$. This implies every $g\omega\alpha$-open set G containing x intersects X-A. That is $G \cap (X\text{-}A) \neq \phi$. Then by theorem 4.2 $x \in g\omega\alpha cl(X\text{-}A)$ and therefore $[X\text{-}g\omega\alpha int(A)] \subseteq [g\omega\alpha cl(X\text{-}A)]$.*
Conversely; Let $x \in g\omega\alpha cl(X\text{-}A)$, then every $g\omega\alpha$-open set G containing x intersects X-A, That is, $G \cap (X\text{-}A) \neq \phi$. That is every $g\omega\alpha$-open set G containing x is such that $G \subseteq A$. Then by definition 4.18, x not in $g\omega\alpha int(A)$, that is $x \in [X\text{-}g\omega\alpha int(A)]$; and so $[g\omega\alpha cl(X\text{-}A)] \subseteq [X\text{-}g\omega\alpha int(A)]$. Thus $[X\text{-}g\omega\alpha int(A)] = [g\omega\alpha cl(X\text{-}A)]$.

5 g$\omega\alpha$-Neighborhoods and g$\omega\alpha$-Limit points

In this section we define the notion of g$\omega\alpha$-neighborhood, g$\omega\alpha$-limit point and g$\omega\alpha$-derived set of a set and show some of their basic properties and analogous to those for open sets.

Definition 5.1. *Let (X, τ) be a topological space and let $x \in X$. A subset N of X is said to be g$\omega\alpha$-neighborhood of a point $x \in X$ if there exists an g$\omega\alpha$-open set G such that $x \in G \subseteq N$.*

Definition 5.2. *Let (X, τ) be a topological space and A be a subset of X, A subset N of X is said to be g$\omega\alpha$-neighborhood of A if there exists an g$\omega\alpha$-open set G such that A $\in G \subseteq N$.*

The collection of all g$\omega\alpha$-neighborhood of x $\in$ X is called the g$\omega\alpha$-neighborhood system at x and shall be denoted by g$\omega\alpha$N(x).

Theorem 5.3. *A subset A of a topological space is g$\omega\alpha$-open if it is a g$\omega\alpha$-neighborhood of each of its points.*

Proof: *Let a subset G of a topological space be g$\omega\alpha$-open. Then for every $x \in X$, x $\in G \subseteq G$, and therefore G is a g$\omega\alpha$-neighborhood of each of its points.*

The converse of the above theorem need not be true as seen from the following example.

Example 5.4. *In Example 3.7 the set $A = \{b, c\}$ is g$\omega\alpha$-neighborhood of each of its points b and c but A is not g$\omega\alpha$-open.*

Theorem 5.5. *Let (X, τ) be a topological space. If A is g$\omega\alpha$-closed subset of X and x $\in$ g$\omega\alpha$cl(A) if and only if for any g$\omega\alpha$-neighborhood N of x in (X, τ), $N \cap A \neq \phi$.*

Proof: *Let us assume that there is a g$\omega\alpha$-neighborhood N of the point x in (X, τ) such that $N \cap A = \phi$. There exist an g$\omega\alpha$-open set G of X such that $X \in G \subseteq N$. Therefore we have $G \cap A = \phi$ and so $x \in$ X-G. Then g$\omega\alpha$cl(A) $\in$ X-G and therefore x $\notin$ g$\omega\alpha$cl(A), which is the contradiction to the hypothesis $x \in$ g$\omega\alpha$cl(A). Therefore $N \cap A \neq \phi$.*

Conversely: Suppose that $x \notin$ g$\omega\alpha$cl(A). Then there exists a g$\omega\alpha$-closed set G of (X, τ) such that $A \subseteq G$ and $x \notin G$. Thus $x \in$ X-G and X-G is g$\omega\alpha$-open in (X, τ) and hence X-G is a g$\omega\alpha$-neighborhood of x in (X, τ). But $A \cap$ (X-G) $= \phi$ which is a contradiction. Hence $x \in$ g$\omega\alpha$cl(A).

Theorem 5.6. *Let (X, τ) be a topological space and $x \in X$. Let g$\omega\alpha$N(x) be the collection of all g$\omega\alpha$-neighborhood of x. Then,*
1. g$\omega\alpha$N(x) $\neq \phi$ and $x \in$ each member of g$\omega\alpha$N(x).
2. The intersection of the any two members of g$\omega\alpha$N(x) is again a member of g$\omega\alpha$N(x).
3. If N $\in$ g$\omega\alpha$N(x) and $M \subseteq N$, then M $\in$ g$\omega\alpha$N(x).
4. Each member N $\in$ g$\omega\alpha$N(x) is a superset of a member G$\in$g$\omega\alpha$N(x) where G is a g$\omega\alpha$-open set.

Proof: *(1). Since X is gωα-open set containing p, it is a gωα-neighborhood of every p ∈ X. Hence there exists atleast one gωα-neighborhood namely X for each p ∈ X there is gωαN(p) ≠ ϕ. Let N ∈ gωαN(p), N is a gωα-neighborhood of p, then there exists a gωα-open set G such that p ∈ G ⊆ N so p ∈ N. Therefore p ∈ every member N of gωαN(p).*
(2). Let N ∈ gωαN(p)and M ∈gωαN(p). Then by definition 5.1, there exists gωα-open set G and F such that p ∈ G ⊆ N and p ∈ F ⊆ M. Hence p ∈ G ∩ F ⊆ M ∩ N. Note that G ∩ F is a gωα-open set. Therefore it follows that N ∩ M is a gωα-neighborhood of p. Hence N ∩ M ∈ gωαN(p).
(3). If N ∈ gωαN(p) then there is an gωα-open set G such that p ∈ G ⊆ N. Since M ⊆ N, M is gωα-neighborhood of p. Hence M ∈ gωαN(p).
(4). Let N ∈ gωαN(p) then there exists a gωα-open set G, such that p ∈ G ⊆ N. Since G is gωα-open and p ∈ G, G is gωα-neighborhood of P.therefore G ∈ gωαN(p) and also G ⊆ N.

Definition 5.7. *Let (X, τ) be a topological space and A be a subset of X. Then a point x ∈ X is called a gωα-limit point of A if and only if every gωα-neighborhood of x contains a point of A distinct from x. That is [N-{x}] ∩ A ≠ ϕ for each gωα-neighborhood N of x. Also equivalently if and only if every gωα-open set G containing x contains a point of A other then x.*

In a topological space (X, τ) the set of all gωα-limit points of a given subset A of X is called a gωα-derived set of A and is denoted by gωαd(A).

Theorem 5.8. *Let A and B be subset of a topological space (X, τ). Then,*
1. gωαd(ϕ) = ϕ.
2. If A ⊆ B, then gωαd(A) ⊆ gωαd(B).
3. If x ∈ gωαd(A), then x ∈ gωαd[A-{x}].
4. gωαd(A ∪ B) = gωαd(A) ∪ gωαd(B).
5. gωαd(A ∩ B) ⊆ gωαd(A) ∩ gωαd(B).
Proof: *(1). Let x be any point of X and x ∈ gωαd(ϕ). That is x is a gωα-limit point of ϕ. Then for every gωα-open set G containing x, we should have [G-{x}] ∩ϕ ≠ ϕ which is impossible. Hence gωαd(ϕ) = ϕ.*
(2). If x ∈ gωαd(A), that is if x is gωα-limit point of A, then by Definition 5.7 [G-{x}] ∩ A ≠ ϕ for every gωα-open set G containing x. Since A ⊆ B implies [G-{x}] ∩ A ⊆ [G-{x}] ∩ B. Thus if x is a gωα-limit point of A it is also a gωα-limit point of B, that is x ∈ gωαd(B). Hence gωαd(A) ⊆ gωαd(B).
(3). If x ∈ gωαd(A), by definition 5.7 every gωα-open set G containing x contains at least one point other than x of A-{x}. Hence x is gωα-limit point of A-{x} and it belongs to gωαd[A-{x}]. Therefore x ∈ gωαd(A) ⇒ x ∈ gωαd[A-{x}].
(4). Since A ⊆ A ∪ B and B ⊆ A ∪ B, from (1) gωαd(A) ∪ gωαd(B) ⊆ gωαd(A ∪ B). To prove other way If x ∉ gωαd(A) ∪ gωαd(B), then x ∉ gωαd(A) and x ∉ gωαd(B). Hence there exists gωα-neighborhoods G_1 and G_2 of x such that G_1 ∩(A-{x}) = ϕ and G_2 ∩ (B-{x}) = ϕ Since G_1 ∩ G_2 is gωα-neighborhood of x, we have (G_1 ∩ G_2) ∩ [(A ∪ B)-{x}] = ϕ. Therefore x ∉ gωαd(A ∪ B). Hence gωαd(A ∪ B) = gωαd(A) ∪ gωαd(B).
(5). Since A ∩ B ⊆ A and A ∩ B ⊆ B, by (2) gωαd(A ∩B) ⊆ gωαd(A) and gωαd(A ∩ B) ⊆ gωαd(B). Consequently gωαd(A ∩ B) ⊆ gωαd(A) ∩ gωαd(B).

Theorem 5.9. *Let (X, τ) be a topological space and A be a subset of X. If A is $g\omega\alpha$-closed,then $g\omega\alpha d(A) \subseteq A$.*

Proof: *Let A be $g\omega\alpha$-closed, Now we will show that $g\omega\alpha d(A) \subseteq A$. Since A is $g\omega\alpha$-closed, X-A is $g\omega\alpha$-open. To each $x \in$ X-A there exists $g\omega\alpha$-neighborhood G of x such that $G \subseteq$ X-A. Since A $\cap$ (X-A) = ϕ, the $g\omega\alpha$-neighborhood G contains no point of A and so X is not a $g\omega\alpha$-limit point of A. Thus no point of X-A can be $g\omega\alpha$-limit point of A that is, A contains all its $g\omega\alpha$-limit points. that is $g\omega\alpha d(A) \subseteq A$.*

Acknowledgement

The authors are greatfull to the referees for their valuable and constructive comments which lead to significant improvement of the paper and to the University Grants Commission, New Delhi, India for its financial support under UGC-SAP-II DRS to the Department of Mathematics, Karnatak University, Dharwad, India. Also this research was supported by the University Grants Commission, New Delhi, India. under No.F.4-1/2006(BSR)/7-101/2007(BSR) dated: 20th June, 2012.

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