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MILDLY $\star$-NORMAL SPACES AND SOME FUNCTIONS

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Abstract In this paper, mildly $\star$-normal spaces and some new ideal topological functions are introduced. Characterizations and properties of such new notions are studied. Some preservation theorems for mildly $\star$-normal spaces are obtained.

Keywords *mildly $\star$-normal space, $\mathcal{I}rg$-continuous function, completely $\star$-continuous function, almost $\mathcal{I}rg$-continuous function, almost $\mathcal{I}g$-continuous function, ideal topological space.*

1 Introduction

An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

1. A $\in \mathcal{I}$ and B $\subseteq$ A imply B $\in \mathcal{I}$ and
2. A $\in \mathcal{I}$ and B $\in \mathcal{I}$ imply A$\cup$B $\in \mathcal{I}$. [24]

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Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X if $\mathcal{P}$(X) is the set of all subsets of X, a set operator $(\bullet)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [7] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^\star(\bullet)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ [19]. We will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space.

In this paper, we introduce and study a new class of spaces called mildly $\star$-normal spaces. Furthermore, we introduce new types of functions called almost $\mathcal{I}rg$-continuous, almost $\mathcal{I}g$-continuous, $\mathcal{I}rg$-closed, almost $\mathcal{I}rg$-closed, almost $\mathcal{I}g$-closed and $\mathcal{I}rc$-preserving functions in ideal topological spaces. Subsequently, the relationships between mildly $\star$-normal spaces and new ideal topological functions are investigated. Moreover, we obtain characterizations of mildly $\star$-normal spaces, properties of new ideal topological functions and preservation theorems for mildly $\star$-normal spaces in ideal topological spaces.

2 Preliminaries

Definition 2.1. *[19] Let (X, τ) be a topological space. A subset A of X is called*

1. *regular open if A = int(cl(A));*
2. *regular closed if A = cl(int(A)).*

The complement of regular open set is regular closed.
The collection of regular open (resp. regular closed) subsets of X is denoted by $RO(X)$ (resp. $RC(X)$).

Remark 2.2. *In any topological spaces, the following holds.*
Every regular closed set is a closed set.[19]

Definition 2.3. *[18] A function f : (X, τ) $\to$ (Y, σ) is called*

1. *rc-preserving if f(F) is regular closed in Y for every $F \in RC(X)$;*
2. *R-continuous if $f^{-1}(F)$ is regular open in X for every $F \in RO(Y)$;*
3. *almost continuous if $f^{-1}(F)$ is closed in X for every $F \in RC(X)$.*

Definition 2.4. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. A subset A of X is called*

1. *$\mathcal{I}g$-closed [14] if $A^\star \subseteq U$ or $cl^\star(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
2. *$\mathcal{I}rg$-closed [15] if $A^\star \subseteq U$ or $cl^\star(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*
3. *$\mathcal{I}g$-open [14] if $X \backslash A$ is $\mathcal{I}g$-closed.*
4. *$\mathcal{I}rg$-open [15] if $X \backslash A$ is $\mathcal{I}rg$-closed.*
5. *$\star$-closed [7] if $A^\star \subseteq A$.*

Remark 2.5. *We have the following implications for properties of subsets.*

$$\star\text{-closed} \rightarrow \mathcal{I}g\text{-closed} \rightarrow \mathcal{I}rg\text{-closed}.$$

None of the above implications is reversible.[1, 15]

Theorem 2.6.

1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is $\mathcal{I}g$-open in X if and only if $F \subseteq int^\star(A)$ whenever F is closed and $F \subseteq A$.[14]*

2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is $\mathcal{I}rg$-open in X if and only if $F \subseteq int^\star(A)$ whenever F is regular closed and $F \subseteq A$.[15]*

Definition 2.7. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called $\star$-continuous if the inverse image of each open set of Y is an $\star$-open set in X.[1]*

Lemma 2.8. *[24] For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, we have*

1. *$X \backslash int^\star(A) = cl^\star(X \backslash A)$,*

2. *$X \backslash cl^\star(A) = int^\star(X \backslash A)$.*

Definition 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is regular $\mathcal{I}$-closed [9] if $A=(int(A))^\star$.*

Remark 2.10. *In any ideal topological spaces, the following holds.*

1. *Every regular $\mathcal{I}$-closed set is a regular closed set.[8]*

2. *Every closed set is $\star$-closed set.[7]*

Definition 2.11. *A subset A of a topological space (X, τ) is said to be preopen [13] if $A \subseteq int(cl(A))$.*

Definition 2.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\mathcal{I}$ is said to be completely codence [3] if $PO(X) \cap \mathcal{I} = \{\emptyset\}$, where $PO(X)$ is the family of all preopen sets in (X, τ).*

Definition 2.13. *A subset A of a topological space (X, τ) is said to be α-open [17] if $A \subseteq int(cl(int(A)))$. The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ.*

Lemma 2.14. *[23] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^\star \subset \tau^\alpha$.*

Definition 2.15. *[18] A topological space (X, τ) is called mildly normal if for every pair of disjoint H, $K \in RC(X)$, there exist disjoint open sets U and V such that $H \subseteq U$ and $K \subseteq V$.*

3 Characterizations of Mildly $\star$-Normal Spaces

Definition 3.1. *An ideal topological space $(X, \tau, \mathcal{I})$ is called $\star$-normal if for any pair of disjoint closed sets A and B of X, there exist disjoint $\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.[20]*

Definition 3.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is called mildly $\star$-normal if for every pair of disjoint H, $K \in RC(X)$, there exist disjoint $\star$-open sets U and V such that $H \subseteq U$ and $K \subseteq V$.*

Theorem 3.3. *The following are equivalent for an ideal topological space $(X, \tau, \mathcal{I})$.*

1. *X is mildly $\star$-normal;*
2. *for any disjoint H, $K \in RC(X)$, there exist disjoint $\mathcal{I}g$-open sets U and V such that $H \subseteq U$ and $K \subseteq V$;*
3. *for any disjoint H, $K \in RC(X)$, there exist disjoint $\mathcal{I}rg$-open sets U and V such that $H \subseteq U$ and $K \subseteq V$;*
4. *for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists a $\mathcal{I}rg$-open set U of X such that $H \subseteq U \subseteq cl^\star(U) \subseteq V$.*
5. *for any $H \in RC(X)$ and any $V \in RO(X)$ containing H, there exists an $\star$-open set U of X such that $H \subseteq U \subseteq cl^\star(U) \subseteq V$.*

Proof: (1) $\Rightarrow$ (2). Proof is immediate from the fact that any $\star$-open set is $\mathcal{I}g$-open.

(2) $\Rightarrow$ (3). Proof is immediate from the fact that any $\mathcal{I}g$-open set is $\mathcal{I}rg$-open.

(3) $\Rightarrow$ (4). Let H$\in RC(X)$ and V$\in RO(X)$. By (3) there exist disjoint $\mathcal{I}rg$-open sets U and W such that H $\subseteq$ U and X$\setminus$V $\subseteq$ W. By Theorem 2.6(2), we have X$\setminus$V $\subseteq$ int*(W) $\Rightarrow$ X$\setminus$int*(W) $\subseteq$ V. Since U$\cap$W$= \emptyset$, we have U$\cap$int*(W)$=\emptyset$ and so cl*(U)$\subseteq$ X$\setminus$int*(W). Therefore, we obtain H $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V where U is $\mathcal{I}rg$-open.

(4) $\Rightarrow$ (5). Let H and K be disjoint regular closed sets of X. Then H $\subseteq$ X$\setminus$K where X$\setminus$K $\in$ RO(X). By (4) there exists a $\mathcal{I}rg$-open set G of X such that H $\subseteq$ G $\subseteq$ cl*(G) $\subseteq$ X$\setminus$K. By Theorem 2.6(2), we have H $\subseteq$ int*(G). If U = int*(G), U is $\star$-open set such that H $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ cl*(G) $\subseteq$ X$\setminus$K. Therefore H $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ X$\setminus$K.

(5) $\Rightarrow$ (1). Let H and K be disjoint regular closed sets of X. Then H $\subseteq$ X$\setminus$K where X$\setminus$K $\in$ RO(X). By (5) there exists an $\star$-open set U of X such that H $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ X$\setminus$K. If V = X$\setminus$cl*(U), then U and V are disjoint $\star$-open sets of X such that H $\subseteq$ U and K $\subseteq$ V.

Theorem 3.4. *Every $\star$-normal space is mildly $\star$-normal but not conversely.*

Proof: Let (X, τ, $\mathcal{I}$) be a $\star$-normal space and A and B be any two disjoint regular closed sets in X. Since A and B are regular closed in X, they are closed in X. (X, τ, $\mathcal{I}$) is $\star$-normal implies there exist disjoint $\star$-open sets U and W such that A $\subseteq$ U and B $\subseteq$ W. Hence U and W satisfy the conditions of mildly $\star$-normality and (X, τ, $\mathcal{I}$) is mildly $\star$-normal.

Example 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $X = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then $(X, \tau, \mathcal{I})$ is a mildly $\star$-normal space but not a $\star$-normal space.*

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then X is mildly normal if and only if it is mildly $\star$-normal.*

Proof: Suppose that A and B are disjoint regular closed sets in X. Since X is mildly normal, there exist disjoint open sets U and V such that A $\subseteq$ U and B $\subseteq$ V. But every open set is $\star$-open set. Hence X is mildly $\star$-normal.

Conversely, suppose that A and B are disjoint regular closed sets of X. Since X is mildly $\star$-normal, there exist disjoint $\star$-open sets U and V such that A $\subseteq$ U and B $\subseteq$ V. Since $\mathcal{I}$ is completely codense. By Lemma 2.14, $\tau^\star \subseteq \tau^\alpha$ and so U, V $\in \tau^\alpha$. Hence A $\subseteq$ U $\subseteq$ int(cl(int(U))) = G and B $\subseteq$ V $\subseteq$ int(cl(int(V))) = H. Therefore, G and H are disjoint open sets containing A and B respectively. Therefore, X is mildly normal.

Theorem 3.7. *[16] Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_g$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
4. *For a regular closed set A and a regular open set V containing A, there exists an $\mathcal{I}_{rg}$-open set U of X such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*
5. *For a regular closed set A and a regular open set V containing A, there exists an $\star$-open set U of X such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*
6. *For disjoint regular closed sets A and B, there exist disjoint $\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

4 Some New Ideal Topological Functions

Definition 4.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be*

1. *$\mathcal{I}g$-continuous [1] if $f^{-1}(F)$ is $\mathcal{I}g$-closed in X for every closed set F of Y;*
2. *$\mathcal{I}rg$-continuous [1] if $f^{-1}(F)$ is $\mathcal{I}rg$-closed in X for every closed set F of Y;*
3. *completely $\star$-continuous if $f^{-1}(F)$ is regular $\mathcal{I}$-closed in X for every closed set F of Y;*
4. *R-$\star$-continuous if $f^{-1}(F)$ is regular $\mathcal{I}$-closed in X for every $F \in RC(Y)$;*
5. *almost $\star$-continuous if $f^{-1}(F)$ is $\star$-closed in X for every $F \in RC(Y)$;*
6. *almost $\mathcal{I}g$-continuous if $f^{-1}(F)$ is $\mathcal{I}g$-closed in X for every $F \in RC(Y)$;*
7. *almost $\mathcal{I}rg$-continuous if $f^{-1}(F)$ is $\mathcal{I}rg$-closed in X for every $F \in RC(Y)$.*

Example 4.2. *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, c\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Define $f : (X, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then the function f is $\star$-continuous but not completely $\star$-continuous.*

Example 4.3. *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Define $f : (X, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then the function f is $\mathcal{I}g$-continuous but not $\star$-continuous.*

Example 4.4. *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{b, c\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Define $f : (X, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then the function f is $\mathcal{I}rg$-continuous but not $\mathcal{I}g$-continuous.*

Example 4.5. *The function f in the Example 4.2 is R-$\star$-continuous but not completely $\star$-continuous.*

Example 4.6. *The function f in the Example 4.4 is almost $\star$-continuous but not $\star$-continuous.*

Example 4.7. *The function f in the Example 4.4 is almost $\mathcal{I}g$-continuous but not $\mathcal{I}g$-continuous.*

Example 4.8. *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, c\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Define $f : (X, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then the function f is almost $\mathcal{I}rg$-continuous but not $\mathcal{I}rg$-continuous.*

Example 4.9. *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Define $f : (X, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then the function f is almost $\mathcal{I}rg$-continuous but not almost $\mathcal{I}g$-continuous.*

Example 4.10. *The function f in the Example 4.3 is almost $\mathcal{I}g$-continuous but not almost $\star$-continuous.*

Example 4.11. *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Define $f : (X, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then the function f is almost $\star$-continuous but not R-$\star$-continuous.*

Remark 4.12. *From the definitions stated above and the examples given above, we obtain the following diagram.*

$$
\begin{tikzcd}
\text{completely } \star\text{-continuity} \arrow[r, shift left] \arrow[d, shift right] & \text{R-}\star\text{-continuity} \arrow[l, shift left, "/" marking] \arrow[d, shift right] \\
\star\text{-continuity} \arrow[r, shift left] \arrow[u, shift right, "/" marking] \arrow[d, shift right] & \text{almost } \star\text{-continuity} \arrow[l, shift left, "/" marking] \arrow[u, shift right, "/" marking] \arrow[d, shift right] \\
\mathcal{I}g\text{-continuity} \arrow[r, shift left] \arrow[u, shift right, "/" marking] \arrow[d, shift right] & \text{almost } \mathcal{I}g\text{-continuity} \arrow[l, shift left, "/" marking] \arrow[u, shift right, "/" marking] \arrow[d, shift right] \\
\mathcal{I}rg\text{-continuity} \arrow[r, shift left] \arrow[u, shift right, "/" marking] & \text{almost } \mathcal{I}rg\text{-continuity} \arrow[l, shift left, "/" marking] \arrow[u, shift right, "/" marking]
\end{tikzcd}
$$

Definition 4.13. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-regular-$T_{1/2}$ if every $\mathcal{I}rg$-closed set in X is regular $\mathcal{I}$-closed in X.*

Proposition 4.14. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\mathcal{I}rg$-continuous and X is $\mathcal{I}$-regular-$T_{1/2}$, then f is completely $\star$-continuous.*

Proof: Let F be any closed set of Y. Since f is $\mathcal{I}rg$-continuous, $f^{-1}(F)$ is $\mathcal{I}rg$-closed in X and hence $f^{-1}(F)$ is regular $\mathcal{I}$-closed in X. Therefore f is completely $\star$-continuous.

Definition 4.15. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be $\mathcal{I}rg$-irresolute if $f^{-1}(F)$ is $\mathcal{I}rg$-closed in X for every $\mathcal{J}rg$-closed set F of Y.*

Remark 4.16. *Every $\mathcal{I}rg$-irresolute function is $\mathcal{I}rg$-continuous but not conversely.*

Example 4.17. *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, Y\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}\}$. Define $f : (X, \tau_1, \mathcal{I}) \to (Y, \tau_2, \mathcal{J})$ by $f(a) = a$; $f(b) = b$; $f(c) = c$. Then the function f is $\mathcal{I}rg$-continuous but not $\mathcal{I}rg$-irresolute function.*

Proposition 4.18. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is almost $\mathcal{I}rg$-continuous and X is $\mathcal{I}$-regular-$T_{1/2}$, then f is a R-$\star$-continuous.*

Proof: Let $V \in RC(Y)$. Since f is almost $\mathcal{I}rg$-continuous, $f^{-1}(V)$ is $\mathcal{I}rg$-closed in X. But X is $\mathcal{I}$-regular-$T_{1/2}$. Therefore $f^{-1}(V)$ is regular $\mathcal{I}$-closed in X. Hence f is a R-$\star$-continuous.

Definition 4.19. *(a) A function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is said to be*

1. *regular $\mathcal{I}$-closed if f(F) is regular $\mathcal{I}$-closed in Y for every closed set F of X;*
2. *$\star$-closed if f(F) is $\star$-closed in Y for every closed set F of X;*
3. *$\mathcal{I}g$-closed if f(F) is $\mathcal{I}g$-closed in Y for every closed set F of X;*
4. *$\mathcal{I}rg$-closed if f(F) is $\mathcal{I}rg$-closed in Y for every closed set F of X;*
5. *$\mathcal{I}rc$-preserving if f(F) is regular $\mathcal{I}$-closed in Y for every $F \in RC(X)$;*
6. *almost $\star$-closed if f(F) is $\star$-closed in Y for every $F \in RC(X)$;*
7. *almost $\mathcal{I}g$-closed if f(F) is $\mathcal{I}g$-closed in Y for every $F \in RC(X)$;*
8. *almost $\mathcal{I}rg$-closed if f(F) is $\mathcal{I}rg$-closed in Y for every $F \in RC(X)$.*

(b) A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be

1. *$\mathcal{I}g$-$\star$-continuous if $f^{-1}(F)$ is $\mathcal{I}g$-closed in X for every $\star$-closed set F of Y;*
2. *$\mathcal{I}rg$-$\star$-continuous if $f^{-1}(F)$ is $\mathcal{I}rg$-closed in X for every $\star$-closed set F of Y;*
3. *$\mathcal{J}g$-$\star$-closed if f(F) is $\mathcal{J}g$-closed in Y for every $\star$-closed set F of X;*
4. *$\star\star$-closed if f(F) is $\star$-closed in Y for every $\star$-closed set F of X.*

Remark 4.20. *From the definitions stated above, we obtain the following diagram.*

$$\begin{tikzcd}
\text{regular } \mathcal{I}\text{-closed} \arrow[r] \arrow[d] & \mathcal{I}rc\text{-preserving} \arrow[d] \\
\star\text{-closed} \arrow[r] \arrow[d] & \text{almost } \star\text{-closed} \arrow[d] \\
\mathcal{I}g\text{-closed} \arrow[r] \arrow[d] & \text{almost } \mathcal{I}g\text{-closed} \arrow[d] \\
\mathcal{I}rg\text{-closed} \arrow[r] & \text{almost } \mathcal{I}rg\text{-closed}
\end{tikzcd}$$

Remark 4.21. *The following examples enable us to realize that none of the implications in the above diagram is reversible.*

Example 4.22. *Let* $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Define* $f : (X, \tau_1) \to (Y, \tau_2, \mathcal{I})$ *by* $f(a) = a$; $f(b) = b$; $f(c) = c$. *Then the function f is* $\star$*-closed but not regular* $\mathcal{I}$*-closed.*

Example 4.23. *Let* $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, Y\}$ *and* $\mathcal{I} = \{\emptyset\}$. *Define* $f : (X, \tau_1) \to (Y, \tau_2, \mathcal{I})$ *by* $f(a) = a$; $f(b) = b$; $f(c) = c$. *Then the function f is* $\mathcal{I}g$*-closed but not* $\star$*-closed.*

Example 4.24. *Let* $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, Y\}$ *and* $\mathcal{I} = \{\emptyset\}$. *Define* $f : (X, \tau_1) \to (Y, \tau_2, \mathcal{I})$ *by* $f(a) = a$; $f(b) = b$; $f(c) = c$. *Then the function f is* $\mathcal{I}rg$*-closed but not* $\mathcal{I}g$*-closed.*

Example 4.25. *Let* $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{c\}, \{b, c\}, Y\}$ *and* $\mathcal{I} = \{\emptyset\}$. *Define* $f : (X, \tau_1) \to (Y, \tau_2, \mathcal{I})$ *by* $f(a) = a$; $f(b) = b$; $f(c) = c$. *Then the function f is* $\mathcal{I}rc$*-preserving but not regular* $\mathcal{I}$*-closed.*

Example 4.26. *The function f in the Example 4.25 is almost* $\star$*-closed but not* $\star$*-closed.*

Example 4.27. *Let* $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\tau_2 = \{\emptyset, \{c\}, \{b, c\}, Y\}$ *and* $\mathcal{I} = \{\emptyset\}$. *Define* $f : (X, \tau_1) \to (Y, \tau_2, \mathcal{I})$ *by* $f(a) = a$; $f(b) = b$; $f(c) = c$. *Then the function f is almost* $\mathcal{I}g$*-closed but not* $\mathcal{I}g$*-closed.*

Example 4.28. *Let* $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, c\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ *and* $\mathcal{I} = \{\emptyset\}$. *Define* $f : (X, \tau_1) \to (Y, \tau_2, \mathcal{I})$ *by* $f(a) = a$; $f(b) = b$; $f(c) = c$. *Then the function f is almost* $\mathcal{I}rg$*-closed but not* $\mathcal{I}rg$*-closed.*

Example 4.29. *The function f in the Example 4.24 is almost* $\mathcal{I}rg$*-closed but not almost* $\mathcal{I}g$*-closed.*

Example 4.30. *The function f in the Example 4.23 is almost* $\mathcal{I}g$*-closed but not almost* $\star$*-closed.*

Example 4.31. *Let* $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Define* $f : (X, \tau_1) \to (Y, \tau_2, \mathcal{I})$ *by* $f(a) = a$; $f(b) = b$; $f(c) = c$. *Then the function f is almost* $\star$*-closed but not* $\mathcal{I}rc$*-preserving.*

Proposition 4.32. *Let f : (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ, $\mathcal{J}$) be a function. Then*

1. *if f is $\mathcal{I}rg$-continuous, rc-preserving then it is $\mathcal{I}rg$-irresolute;*
2. *if f is R-continuous and $\mathcal{J}rg$-closed then f(A) is $\mathcal{J}rg$-closed in Y for every $\mathcal{I}rg$-closed set A of X.*

Proof: (1) Let B be any $\mathcal{J}rg$-closed set of Y and let U $\in$ RO(X) contain $f^{-1}(B)$. Put V = Y\f(X\U), then we have B $\subseteq$ V, $f^{-1}(V) \subseteq U$ and V $\in$ RO(Y) since f is rc-reserving. Hence we obtain $cl^\star(B) \subseteq V$ and hence $f^{-1}(cl^\star(B)) \subseteq U$. By the $\mathcal{I}rg$-continuity of f we have $cl^\star(f^{-1}(B)) \subseteq cl^\star(f^{-1}(cl^\star(B))) \subseteq U$. This shows that $f^{-1}(B)$ is $\mathcal{I}rg$-closed in X. Therefore f is $\mathcal{I}rg$-irresolute.

(2) Let A be any $\mathcal{I}rg$-closed set of X and let V $\in$ RO(Y) contain f(A). Since f is a R-continuous, $f^{-1}(V) \in RO(X)$ and $A \subseteq f^{-1}(V)$. Therefore, we have $cl^\star(A) \subseteq f^{-1}(V)$ and hence $f(cl^\star(A)) \subseteq V$. Since f is $\mathcal{J}rg$-closed, $f(cl^\star(A))$ is $\mathcal{J}rg$-closed in Y and hence we obtain $cl^\star(f(A)) \subseteq cl^\star(f(cl^\star(A))) \subseteq V$. This shows that f(A) is $\mathcal{J}rg$-closed in Y.

Corollary 4.33. *Let f : (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ, $\mathcal{J}$) be a function. Then*

1. *if f is $\star$-continuous, rc-preserving, then $f^{-1}(B)$ is $\mathcal{I}rg$-closed in X for every $\mathcal{J}rg$-closed set B of Y.*
2. *if f is a R-continuous and $\star$-closed, then f(A) is $\mathcal{J}rg$-closed in Y for every $\mathcal{I}rg$-closed set A of X.*

Proof: This is an immediate consequence of Proposition 4.32

Proposition 4.34. *A surjection f : (X, τ) $\rightarrow$ (Y, σ, $\mathcal{I}$) is almost $\mathcal{I}rg$-closed (or) almost $\mathcal{I}g$-closed if and only if for each subset S of Y and each U $\in$ RO(X) containing $f^{-1}(S)$ there exists respectively a $\mathcal{I}rg$-open (or) $\mathcal{I}g$-open set V of Y such that S $\subseteq$ V and $f^{-1}(V) \subseteq U$.*

Proof: We prove only the first case, the proof of the other being entirely analogus.

Necessity: Suppose that f is almost $\mathcal{I}rg$-closed. Let S be a subset of Y and let U $\in$ RO(X) contain $f^{-1}(S)$. Put V = Y\f (X\U), then V is a $\mathcal{I}rg$-open set of Y such that S $\subseteq$ V and $f^{-1}(V) \subseteq U$.

Sufficiency: Let F be any regular closed set of X. Then $f^{-1}(Y \backslash f(F)) \subseteq (X \backslash F)$ and $(X \backslash F) \in RO(X)$. There exists a $\mathcal{I}rg$-open set V of Y such that $(Y \backslash f(F)) \subseteq V$ and $f^{-1}(V) \subseteq (X \backslash F)$. Therefore, we have $f(F) \supseteq (Y \backslash V)$ and $F \subseteq f^{-1}(Y \backslash V)$. Hence we obtain $f(F) = Y \backslash V$ and f(F) is $\mathcal{I}rg$-closed in Y. This shows that f is almost $\mathcal{I}rg$-closed .

5 Preservation Theorems

In this section we investigate preservation theorems concerning mildly $\star$-normal spaces in ideal topological spaces.

Theorem 5.1. *If f : (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ, $\mathcal{J}$) is an $\mathcal{I}rg$-$\star$-continuous $\mathcal{J}rc$-preserving or almost closed injection and Y is mildly $\star$-normal or $\star$-normal respectively, then X is mildly $\star$-normal.*

Proof: Let A and B be disjoint regular closed sets of X. Since f is an $\mathcal{J}rc$-preserving (almost closed) injection, f(A) and f(B) are disjoint regular $\mathcal{J}$-closed (closed) sets of Y. By the mild $\star$-normality ($\star$-normality) of Y, there exist disjoint $\star$-open sets U and V of Y such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is $\mathcal{I}rg$-$\star$-continuous, f^{-1}(U) and f^{-1}(V) are disjoint $\mathcal{I}rg$-open sets of X containing A and B, respectively. It follows from Theorem 3.3 that X is mildly $\star$-normal.

Theorem 5.2. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a completely $\star$-continuous $\mathcal{J}g$-$\star$-closed surjection and X is mildly $\star$-normal then Y is $\star$-normal.*

Proof: Let A and B be disjoint closed sets of Y. Since f is completely $\star$-continuous, f^{-1}(A) and f^{-1}(B) are disjoint regular $\mathcal{I}$-closed sets of X. Since X is mildly $\star$-normal, there exist disjoint $\star$-open sets U and V of X such that f^{-1}(A) $\subseteq$ U and f^{-1}(B) $\subseteq$ V. Let G = Y\f(X\U) and H = Y\f(X\V), then G and H are disjoint $\mathcal{J}g$-open sets of Y such that A $\subseteq$ G and B $\subseteq$ H. Since G and H are $\mathcal{J}g$-open, by Theorem 2.6(1), we obtain A $\subseteq$ $int^\star$(G), B $\subseteq$ $int^\star$(H) and $int^\star(G)\cap int^\star(H) = \emptyset$. This shows that Y is $\star$-normal.

Corollary 5.3. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a completely $\star$-continuous $\star\star$-closed surjection and X is mildly $\star$-normal, then Y is $\star$-normal.*

Theorem 5.4. *Let f: $(X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a R-$\star$-continuous (resp. almost continuous) and $\mathcal{J}rg$-$\star$-closed surjection. If X is mildly $\star$-normal (resp. $\star$-normal), then Y is mildly $\star$-normal.*

Proof: Let A and B be disjoint regular closed sets of Y. Then f^{-1}(A) and f^{-1}(B) are disjoint regular $\mathcal{I}$-closed sets (or) closed sets of X. Since X is respectively mildly $\star$-normal (or) $\star$-normal, there exist disjoint $\star$-open sets U and V of X such that f^{-1}(A) $\subseteq$ U and f^{-1}(B) $\subseteq$ V. Let G = Y\f(X\U) and H = Y\f(X\V), then G and H are disjoint $\mathcal{J}rg$-open sets of Y such that A $\subseteq$ G and B $\subseteq$ H. Since G and H are $\mathcal{J}rg$-open, by Theorem 2.6(2), we obtain A $\subseteq$ $int^\star$(G), B $\subseteq$ $int^\star$(H) and $int^\star(G)\cap int^\star(H) =\emptyset$. This shows that Y is mildly $\star$-normal.

6 Conclusion

The notions of the sets, functions and spaces in ideal topological spaces are highly developed and used extensively in many practical and engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design research and mathematical sciences. Also, topology plays a significant role in space time geometry and high-energy physics. Thus generalized continuity is one of the most important subjects on topological spaces. Hence we studied new types of generalizations of non-continuous functions, obtained some of their properties in ideal topological spaces. Moreover, the ideal topological version of the concepts and the results introduced in this paper may be applied by using the concepts of fuzzy sets and fuzzy functions.

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