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Some Integral Inequalities for s-convex Functions in the Second Sense with Applications

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Abstract – In this paper, author establish some new interesting inequalities for product of convex and s-convex functions in the second sense. Also several applications to special means for positive number are given.

Keywords – *Convexity, s-convexity, Product of two convex functions, Hadamard's inequality.*

1 Introduction

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality (see [1] or [2]) which has generated a wide range of directions for extension and a rich mathematical literature. The following definitions are well known in the mathematical literature: A function $f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR.

In the paper [3], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined in the following way: A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f\left(tx + (1-t)\,y\right) \leq t^s f\left(x\right) + (1-t)^s\, f\left(y\right) \tag{2}$$

holds for all $x, y \in [0, \infty)$, $t \in [0, 1]$, and for some fixed $s \in (0, 1]$. For $s \in (0, 1]$, it is obvious that

$$t^s f\left(x\right) + (1-t)^s\, f\left(y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right). \tag{3}$$

The class of s-convex functions in the second sense is usually denoted with K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In the same paper [3], Hudzik and Maligranda proved that if $s \in (0, 1)$, $f \in K_s^2$ implies $f([0, \infty)) \subseteq [0, \infty)$, i.e., they proved that all functions from K_s^2, $s \in (0, 1)$, are nonnegative.

Example 1.1. *[3] Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define function $f : [0, \infty) \to \mathbb{R}$ as*

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases} \tag{4}$$

It can be easily checked that
(1) If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$
(2) If $b > 0$ and $c < 0$, then $f \notin K_s^2$.

Many important inequalities are established for the class of convex functions, but one of the most famous is so called Hermite-Hadamard inequality (or Hadamard's inequality). This double inequalities are stated as follows [5]: Let f be a convex function on $[a, b] \subset \mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{5}$$

For some recent results connected with integral inequalities for different type convex functions see [1]-[5] and [7]-[11]. The main purpose of this paper is to establish new inequalities for the class of s-convex functions in the second sense by using the elementary inequalities.

2 Main Results

In the next our theorem, we will also make use of Beta function of Euler type, which is for $u, v > 0$ defined as

$$\beta(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

and

$$\beta(u, v) = \beta(v, u).$$

Theorem 2.1. *Let $f : I \to \mathbb{R}$, $I \subset [0, \infty)$, $a, b \in I$, with $a < b$ be an increasing and s-convex function in the second sense for some $s \in (0, 1]$. Then the following inequality hold;*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{f(a)+f(b)}{s+1} + \frac{f(a)+f(b)}{2}\right] + \Psi(a, b) \\ \leq\; & \frac{1}{b-a}\int_a^b f^2(x)\,dx + \Phi(a, b), \end{aligned} \tag{6}$$

where

$$\Psi(a,b)=\frac{f^2(a)+f^2(b)}{s+2}+\frac{2f(a)f(b)}{(s+1)(s+2)}$$

and

$$\Phi(a,b)=\frac{2(s+2)}{3(2s+1)}\left(f^2(a)+f^2(b)\right)+2f(a)f(b)\left[\frac{\Gamma^2(s+1)}{\Gamma(2s+2)}+\frac{1}{6}\right]$$

Proof. Since f is an s-convex function on I, we have that

$$f(ta+(1-t)b)\le t^s f(a)+(1-t)^s f(b)\le tf(a)+(1-t)f(b) \tag{7}$$

for all $a,b\in I$, and $t\in[0,1]$. Using the elementary inequality ([6], p.8) $xy+yz+zx\le x^2+y^2+z^2$ $(x,y,z\in\mathbb{R})$, we have that

$$\begin{aligned} & f^2(ta+(1-t)b) \\ & +t^{2s}f^2(a)+2t^s(1-t)^s f(a)f(b)+(1-t)^{2s}f^2(b) \\ & +t^2f^2(a)+2t(1-t)f(a)f(b)+(1-t)^2f^2(b) \\ \ge\ & f(ta+(1-t)b)\left(t^sf(a)+(1-t)^sf(b)\right) \\ & +t^{s+1}f^2(a)+(1-t)^{s+1}f^2(b) \\ & +\left(t^s(1-t)+t(1-t)^s\right)f(a)f(b) \\ & +f(ta+(1-t)b)\left(tf(a)+(1-t)f(b)\right). \end{aligned}$$

Rewriting this inequality, we have

$$\begin{aligned} & f^2(ta+(1-t)b)+f^2(a)\left[t^{2s}+t^2\right] \\ & +2f(a)f(b)\left[t^s(1-t)^s+t(1-t)\right]+f^2(b)\left[(1-t)^{2s}+(1-t)^2\right] \\ \ge\ & f(ta+(1-t)b)\left(t^sf(a)+(1-t)^sf(b)\right) \\ & +t^{s+1}f^2(a)+(1-t)^{s+1}f^2(b) \\ & +\left(t^s(1-t)+t(1-t)^s\right)f(a)f(b) \\ & +f(ta+(1-t)b)\left(tf(a)+(1-t)f(b)\right). \end{aligned}$$

Integrating this inequality over t on $[0,1]$, we deduce

$$\begin{aligned} & (A:=)\int_0^1 f^2(ta+(1-t)b)\,dt+f^2(a)\int_0^1\left(t^{2s}+t^2\right)dt \\ & +2f(a)f(b)\int_0^1\left(t^s(1-t)^s+t(1-t)\right)dt \\ & +f^2(b)\int_0^1\left((1-t)^{2s}+(1-t)^2\right)dt \\ \ge\ & (B:=)\int_0^1 f(ta+(1-t)b)\left(t^sf(a)+(1-t)^sf(b)\right)dt \\ & +f^2(a)\int_0^1 t^{s+1}dt+f^2(b)\int_0^1(1-t)^{s+1}\,dt \\ & +f(a)f(b)\int_0^1\left(t^s(1-t)+t(1-t)^s\right)dt \\ & +\int_0^1 f(ta+(1-t)b)\left(tf(a)+(1-t)f(b)\right)dt. \end{aligned} \tag{8}$$

A and B expressions to analyze respectively and using increasing of f, and by substituting $ta+(1-t)b=x$, it is easy to observe that

$$\begin{aligned}
\int_0^1 f^2(ta+(1-t)b)\,dt &= \frac{1}{b-a}\int_a^b f^2(x)\,dx, \\
f^2(a)\int_0^1 \left(t^{2s}+t^2\right)dt &= \frac{2(s+2)}{3(2s+1)}f^2(a), \\
2f(a)f(b)\int_0^1 \left(t^s(1-t)^s+t(1-t)\right)dt &= 2f(a)f(b)\left\{\frac{\Gamma^2(s+1)}{\Gamma(2s+2)}+\frac{1}{6}\right\}, \\
f^2(b)\int_0^1 \left((1-t)^{2s}+(1-t)^2\right)dt &= \frac{2(s+2)}{3(2s+1)}f^2(b),
\end{aligned}$$

then, we get

$$\begin{aligned}
& (A:=)\int_0^1 f^2(ta+(1-t)b)\,dt+f^2(a)\int_0^1 \left(t^{2s}+t^2\right)dt \\
& +2f(a)f(b)\int_0^1 \left(t^s(1-t)^s+t(1-t)\right)dt \\
& +f^2(b)\int_0^1 \left((1-t)^{2s}+(1-t)^2\right)dt \\
= \; & \frac{1}{b-a}\int_a^b f^2(x)\,dx+2f(a)f(b)\left\{\frac{\Gamma^2(s+1)}{\Gamma(2s+2)}+\frac{1}{6}\right\} \\
& +\frac{2(s+2)}{3(2s+1)}\left(f^2(a)+f^2(b)\right).
\end{aligned}$$

For proof of the right of (8), by using increasing of f and by substituting $ta+(1-t)b=x$, it is easy to observe that:

$$\begin{aligned}
& \int_0^1 f(ta+(1-t)b)\left(t^s f(a)+(1-t)^s f(b)\right)dt \\
\geq \; & \int_0^1 f(ta+(1-t)b)\,dt\int_0^1 \left(t^s f(a)+(1-t)^s f(b)\right)dt \\
= \; & \frac{f(a)+f(b)}{s+1}\frac{1}{b-a}\int_a^b f(x)\,dx,
\end{aligned}$$

and

$$\begin{aligned}
f^2(a)\int_0^1 t^{s+1}dt+f^2(b)\int_0^1 (1-t)^{s+1}\,dt &= \frac{f^2(a)+f^2(b)}{s+2} \\
f(a)f(b)\int_0^1 \left(t^s(1-t)+t(1-t)^s\right)dt &= \frac{2f(a)f(b)}{(s+1)(s+2)},
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 f(ta+(1-t)b)\left(tf(a)+(1-t)f(b)\right)dt \\
\geq \; & \int_0^1 f(ta+(1-t)b)\,dt\int_0^1 \left(tf(a)+(1-t)f(b)\right)dt \\
= \; & \frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b f(x)\,dx,
\end{aligned}$$

then, we get

$$\begin{aligned}
&(B:=)\int_0^1 f(ta+(1-t)b)\left(t^s f(a)+(1-t)^s f(b)\right)dt \\
&+f^2(a)\int_0^1 t^{s+1}dt+f^2(b)\int_0^1 (1-t)^{s+1}dt \\
&+f(a)f(b)\int_0^1 \left(t^s(1-t)+t(1-t)^s\right)dt \\
&+\int_0^1 f(ta+(1-t)b)\left(tf(a)+(1-t)f(b)\right)dt \\
\geq\ & \frac{f(a)+f(b)}{s+1}\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{f^2(a)+f^2(b)}{s+2} \\
&+\frac{2f(a)f(b)}{(s+1)(s+2)}+\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b f(x)\,dx \\
=\ & \frac{1}{b-a}\int_a^b f(x)\,dx\left[\frac{f(a)+f(b)}{s+1}+\frac{f(a)+f(b)}{2}\right] \\
&+\frac{f^2(a)+f^2(b)}{s+2}+\frac{2f(a)f(b)}{(s+1)(s+2)}.
\end{aligned}$$

When above equalities and inequalities are taken into account, $(B\leq A)$, and by using the left half of the Hadamard's inequality given in (5) on the left side of the inequality $(B\leq A)$, then the inequality (6) is proved. ∎

Corollary 2.2. *With the above assumptions, and under the condition that $s=1$, one has the inequality:*

$$\begin{aligned}
&2f\left(\frac{a+b}{2}\right)\left[\frac{f(a)+f(b)}{2}\right] \\
\leq\ & \frac{1}{b-a}\int_a^b f^2(x)\,dx+\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.
\end{aligned} \tag{9}$$

Theorem 2.3. *Let $f:I\to\mathbb{R}$, $I\subset[0,\infty)$, $a,b\in I$, with $a<b$ be an increasing and s-convex function in the second sense for some $s\in(0,1]$. Then the following inequality hold;*

$$f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{s+1}\leq\frac{1}{8(b-a)}\int_0^1 f^4(x)\,dx+\alpha(a,b) \tag{10}$$

where $\alpha(a,b)=\frac{f^4(a)+f^4(b)}{32s+8}+\frac{3f^2(a)f^2(b)\Gamma^2(2s+1)}{4\Gamma(4s+2)}+\frac{f(a)f(b)\left[f^2(a)+f^2(b)\right]\Gamma(3s+1)\Gamma(s+1)+2\Gamma(4s+2)}{2\Gamma(4s+2)}$.

Proof. Since f is an s-convex function on I, we have

$$f(ta+(1-t)b)\leq t^s f(a)+(1-t)^s f(b)$$

for all $a,b\in I$, and $t\in[0,1]$. Using the elementary inequality ([6], p.9) $8xy\leq x^4+y^4+8$ $(x,y\in\mathbb{R})$, we have

$$\begin{aligned}
&8f(ta+(1-t)b)\left(t^s f(a)+(1-t)^s f(b)\right) \\
\leq\ & f^4(ta+(1-t)b)+\left(t^s f(a)+(1-t)^s f(b)\right)^4+8.
\end{aligned} \tag{11}$$

Integrating this inequality over t on $[0,1]$, we deduce

$$\begin{aligned} & 8\int_0^1 f(ta+(1-t)b)\left(t^s f(a)+(1-t)^s f(b)\right)dt \\ \leq & \int_0^1 f^4(ta+(1-t)b)\,dt+\int_0^1 \left(t^s f(a)+(1-t)^s f(b)\right)^4 dt+8. \end{aligned}$$

Since f is an increasing function, we have

$$\begin{aligned} & \int_0^1 f(ta+(1-t)b)\left(t^s f(a)+(1-t)^s f(b)\right)dt \\ \geq & \int_0^1 f(ta+(1-t)b)\,dt\int_0^1 \left(t^s f(a)+(1-t)^s f(b)\right)dt, \end{aligned}$$

then

$$\begin{aligned} & 8\int_0^1 f(ta+(1-t)b)\,dt\int_0^1 \left(t^s f(a)+(1-t)^s f(b)\right)dt \\ \leq & \int_0^1 f^4(ta+(1-t)b)\,dt+\int_0^1 \left(t^s f(a)+(1-t)^s f(b)\right)^4 dt+8. \end{aligned}$$

As it is easy to see that

$$\int_0^1 f(ta+(1-t)b)\,dt=\frac{1}{b-a}\int_0^1 f(x)\,dx,$$

$$\int_0^1 \left(t^s f(a)+(1-t)^s f(b)\right)dt=\frac{f(a)+f(b)}{s+1},$$

$$\begin{aligned} & \int_0^1 \left(t^s f(a)+(1-t)^s f(b)\right)^4 dt \\ = & f^4(a)\int_0^1 t^{4s}dt+4f^3(a)f(b)\int_0^1 t^{3s}(1-t)^s\,dt \\ & +6f^2(a)f^2(b)\int_0^1 t^{2s}(1-t)^{2s}\,dt \\ & +4f(a)f^3(b)\int_0^1 t^s(1-t)^{3s}\,dt+f^4(b)\int_0^1 (1-t)^{4s}\,dt \\ = & \frac{f^4(a)}{4s+1}+4f^3(a)f(b)\beta(3s+1,s+1) \\ & +6f^2(a)f^2(b)\beta(2s+1,2s+1) \\ & +4f(a)f^3(b)\beta(3s+1,s+1)+\frac{f^4(b)}{4s+1} \\ = & \frac{f^4(a)+f^4(b)}{4s+1}+6f^2(a)f^2(b)\beta(2s+1,2s+1) \\ & +4f(a)f(b)\beta(3s+1,s+1)\left[f^2(a)+f^2(b)\right] \\ = & \frac{f^4(a)+f^4(b)}{4s+1}+6f^2(a)f^2(b)\frac{\Gamma(2s+1)\Gamma(2s+1)}{\Gamma(4s+2)} \\ & +4f(a)f(b)\left[f^2(a)+f^2(b)\right]\frac{\Gamma(3s+1)\Gamma(s+1)}{\Gamma(4s+2)}, \end{aligned}$$

respectively, then the following inequality is obtain

$$\begin{aligned} & \frac{8}{b-a}\int_0^1 f(x)\,dx\frac{f(a)+f(b)}{s+1} \\ \leq\; & \frac{1}{b-a}\int_0^1 f^4(x)\,dx+\frac{f^4(a)+f^4(b)}{4s+1} \\ & +6f^2(a)\,f^2(b)\,\frac{\Gamma(2s+1)\,\Gamma(2s+1)}{\Gamma(4s+2)} \\ & +4f(a)\,f(b)\left[f^2(a)+f^2(b)\right]\frac{\Gamma(3s+1)\,\Gamma(s+1)}{\Gamma(4s+2)}+8, \end{aligned} \tag{12}$$

and by using the left half of the Hadamard's inequality given in (5) on the left side the above inequality (12), then the inequality (10) is proved. ∎

Corollary 2.4. *With the above assumptions, and under the condition that* $s=1$*, one has the inequality:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2} \\ \leq\; & \frac{1}{8(b-a)}\int_a^b f^4(x)\,dx \\ & +\frac{f^4(a)+f(a)^3 f(b)+f(a)^2 f(b)^2+f(a)\,f(b)^3+f^4(b)}{40}+1. \end{aligned} \tag{13}$$

Theorem 2.5. *Let* $f,g: I\rightarrow\mathbb{R}$, $I\subset[0,\infty)$, $a,b\in I$, *with* $a<b$ *be increasing and* s*-convex functions in the second sense. If* f *is* s_1*-convex in the second sense and* g *is* s_2*-convex in the second sense for some* $s_1,s_2\in(0,1]$*, then*

$$\begin{aligned} & \frac{f(a)+f(b)}{(s_1+1)}g\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{(s_2+1)}f\left(\frac{a+b}{2}\right) \\ \leq\; & \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{M(a,b)}{s_1+s_2+1}+\frac{\Gamma(s_1+1)\,\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}N(a,b), \end{aligned} \tag{14}$$

where $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ *and* $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$*.*

Proof. Since f is an s_1-convex and g is an s_2-convex on $[a,b]$, we have

$$\begin{aligned} f(ta+(1-t)\,b) &\leq t^{s_1}f(a)+(1-t)^{s_1}f(b) \\ g(ta+(1-t)\,b) &\leq t^{s_2}g(a)+(1-t)^{s_2}g(b) \end{aligned}$$

for all $a,b\in I$, and $t\in[0,1]$. Now, using the elementary inequality ([6], p.4) $(a-b)(c-d)\geq 0$ ($a,b,c,d\in\mathbb{R}$ and $a<b,c<d$), we get inequality

$$\begin{aligned} & t^{s_1}f(a)\,g(ta+(1-t)\,b)+(1-t)^{s_1}f(b)\,g(ta+(1-t)\,b) \\ & +t^{s_2}g(a)\,f(ta+(1-t)\,b)+(1-t)^{s_2}g(b)\,f(ta+(1-t)\,b) \\ \leq\; & f(ta+(1-t)\,b)\,g(ta+(1-t)\,b)+t^{s_1+s_2}f(a)\,g(a) \\ & +t^{s_1}(1-t)^{s_2}f(a)\,g(b)+t^{s_2}(1-t)^{s_1}f(b)\,g(a) \\ & +(1-t)^{s_1+s_2}f(b)\,g(b). \end{aligned}$$

Integrating this inequality over t on $[0,1]$, we deduce

$$\begin{aligned}
& (A:=)\, f(a)\int_0^1 t^{s_1} g(ta+(1-t)b)\,dt \\
& +f(b)\int_0^1 (1-t)^{s_1} g(ta+(1-t)b)\,dt \\
& +g(a)\int_0^1 t^{s_2} f(ta+(1-t)b)\,dt \\
& +g(b)\int_0^1 (1-t)^{s_2} f(ta+(1-t)b)\,dt \\
\leq\ & (B:=)\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt + f(a)g(a)\int_0^1 t^{s_1+s_2}dt \\
& +f(a)g(b)\int_0^1 t^{s_1}(1-t)^{s_2}\,dt + f(b)g(a)\int_0^1 t^{s_2}(1-t)^{s_1}\,dt \\
& +f(b)g(b)\int_0^1 (1-t)^{s_1+s_2}\,dt.
\end{aligned}$$

A and B expressions to analyze respectively and using increasing of f, g and using the left half of the Hadamard's inequality given in (5) on the left side of the above inequalities, we get

$$\begin{aligned}
(A\ :\ & =) f(a)\int_0^1 t^{s_1} g(ta+(1-t)b)\,dt + f(b)\int_0^1 (1-t)^{s_1} g(ta+(1-t)b)\,dt \\
& +g(a)\int_0^1 t^{s_2} f(ta+(1-t)b)\,dt + g(b)\int_0^1 (1-t)^{s_2} f(ta+(1-t)b)\,dt \\
& \geq f(a)\int_0^1 t^{s_1}dt\int_0^1 g(ta+(1-t)b)\,dt \\
& +f(b)\int_0^1 (1-t)^{s_1}\,dt\int_0^1 g(ta+(1-t)b)\,dt \\
& +g(a)\int_0^1 t^{s_2}dt\int_0^1 f(ta+(1-t)b)\,dt \\
& +g(b)\int_0^1 (1-t)^{s_2}\,dt\int_0^1 f(ta+(1-t)b)\,dt \\
& = \frac{f(a)}{(s_1+1)(b-a)}\int_a^b g(x)\,dx + \frac{f(b)}{(s_1+1)(b-a)}\int_a^b g(x)\,dx \\
& +\frac{g(a)}{(s_2+1)(b-a)}\int_a^b f(x)\,dx + \frac{g(b)}{(s_2+1)(b-a)}\int_a^b f(x)\,dx \\
& = \frac{f(a)+f(b)}{(s_1+1)(b-a)}\int_a^b g(x)\,dx + \frac{g(a)+g(b)}{(s_2+1)(b-a)}\int_a^b f(x)\,dx \\
& \geq \frac{f(a)+f(b)}{(s_1+1)}g\left(\frac{a+b}{2}\right) + \frac{g(a)+g(b)}{(s_2+1)}f\left(\frac{a+b}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}(B \quad : \quad =) & \int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)dt + f\left(a\right)g\left(a\right)\int_0^1 t^{s_1+s_2}dt \\ & + f\left(a\right)g\left(b\right)\int_0^1 t^{s_1}\left(1-t\right)^{s_2}dt + f\left(b\right)g\left(a\right)\int_0^1 t^{s_2}\left(1-t\right)^{s_1}dt \\ & + f\left(b\right)g\left(b\right)\int_0^1 \left(1-t\right)^{s_1+s_2}dt \\ = & \frac{1}{b-a}\int_a^b f\left(x\right)g\left(x\right)dx + \frac{f\left(a\right)g\left(a\right)+f\left(b\right)g\left(b\right)}{s_1+s_2+1} \\ & + f\left(a\right)g\left(b\right)\beta\left(s_1+1,s_2+1\right) + f\left(b\right)g\left(a\right)\beta\left(s_2+1,s_1+1\right) \\ = & \frac{1}{b-a}\int_a^b f\left(x\right)g\left(x\right)dx + \frac{f\left(a\right)g\left(a\right)+f\left(b\right)g\left(b\right)}{s_1+s_2+1} \\ & + \left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]\beta\left(s_1+1,s_2+1\right) \\ = & \frac{1}{b-a}\int_a^b f\left(x\right)g\left(x\right)dx + \frac{f\left(a\right)g\left(a\right)+f\left(b\right)g\left(b\right)}{s_1+s_2+1} \\ & + \frac{\Gamma\left(s_1+1\right)\Gamma\left(s_2+1\right)}{\Gamma\left(s_1+s_2+2\right)}\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]\end{aligned}$$

respectively, $(A \leq B)$ then the inequality (14) is proved. ∎

Corollary 2.6. *With the above assumptions, and under the condition that $s_1 = s_2 = 1$, one has the inequality*

$$\begin{aligned} & \frac{f\left(a\right)+f\left(b\right)}{2}g\left(\frac{a+b}{2}\right)+\frac{g\left(a\right)+g\left(b\right)}{2}f\left(\frac{a+b}{2}\right) \\ \leq & \frac{1}{b-a}\int_a^b f\left(x\right)g\left(x\right)dx + \frac{M\left(a,b\right)}{3}+\frac{N\left(a,b\right)}{6}. \end{aligned} \tag{15}$$

3 Applications to some special means

We now consider the applications of our results to the following special means
The arithmetic mean: $A = A\left(a,b\right) := \frac{a+b}{2}, \;\; a,b \geq 0,$
The geometric mean: $G = G\left(a,b\right) := \sqrt{ab}, \;\; a,b \geq 0,$
The quadratic mean: $K = K\left(a,b\right) := \sqrt{\frac{a^2+b^2}{2}} \;\; a,b \geq 0.$
The following inequality is well known in the resources:

$$G \leq A \leq K$$

In [3], the above Example 1.1 is given: Let $s \in (0,1)$ and $a,b,c \in \mathbb{R}$. We define function $f : [0,\infty) \to \mathbb{R}$ as

$$f\left(t\right) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$. Consequently, for $a = c = 0$, $b = 1$, $s = 1/2$, we have $f : [0,1] \to [0,1]$, $f\left(t\right) = t^{\frac{1}{2}}$, $f \in K_s^2$.

Proposition 3.1. *Let $a, b \in [0, \infty)$, $a < b$. Then one has the inequality*

$$\frac{7}{3}A^{1/2}(a,b)\,A\left(a^{1/2},b^{1/2}\right) \leq \frac{43}{15}A(a,b) + \frac{20\pi - 17}{60}G(a,b). \tag{16}$$

Proof. The assertion follows from Theorem 2.1 applied to s-convex mapping $f : I \to \mathbb{R}$, $f(x) = x^s$, $x \in [a,b]$ and $f(x) = x^{1/2}$ for $s = 1/2$. ∎

Proposition 3.2. *Let $a, b \in [0, \infty)$, $a < b$. Then one has the inequality*

$$\begin{aligned} & \frac{4}{3}A^{1/2}(a,b)\,A\left(a^{1/2},b^{1/2}\right) \\ \leq\; & \frac{K^2(a,b) + G^2(a,b)}{6} + \frac{\pi}{16}G(a,b)\,A(a,b) + 1. \end{aligned} \tag{17}$$

Proof. The assertion follows from Theorem 2.3 applied to s-convex mapping $f : I \to \mathbb{R}$, $f(x) = x^s$, $x \in [a,b]$ and $f(x) = x^{1/2}$ for $s = 1/2$. ∎

Proposition 3.3. *Let $a, b \in [0, \infty), a < b$. Then one has the inequality:*

$$\frac{8}{3}A^{1/2}(a,b)\,A\left(a^{1/2},b^{1/2}\right) \leq 2A(a,b) + \frac{\pi}{4}G(a,b) \tag{18}$$

Proof. The assertion follows from Theorem 2.5 applied to s-convex mapping $f, g : I \to \mathbb{R}$, $f(x) = g(x) = x^s$, $x \in [a,b]$ and $f(x) = g(x) = x^{1/2}$ for $s = 1/2$. ∎

Similar inequalities may be stated for s-convex functions $f(x) = x^s$, or $f(x) = bx^s + c$, $s \in (0,1]$. We omit the details.

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