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A New Marshall-Olkin Extended Family of Distributions with Bounded Support

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Highlights

- This paper presents a new Marshall-Olkin extended family of distributions with bounded support.
- The mathematical properties of the proposed model were derived.
- Two data sets were used to show the flexibility of the proposed model over some existing models

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Abstract

This paper presents a new Marshall-Olkin extended family of distributions with bounded support. Some of the Mathematical properties of the proposed distribution were studied and the method of maximum likelihood estimation was employed to estimate the unknown parameters of the proposed distribution. A Monte Carlo simulation study was carried out to examine the asymptotic behaviour of the parameter estimates of the distribution. Finally, two real data sets defined on a unit interval were used to show the applicability of the proposed distribution in analyzing real data sets.

1. INTRODUCTION

Lifetime distributions are statistical models used in describing the length of life of a system or a device. Most well-known lifetime distributions which include; the exponential, Weibull, Lindley and gamma distributions are widely used in both reliability theory and survival analysis. However, the classical lifetime distributions have a limited range of behavior and cannot represent all situations in applications. For example, the one parameter exponential distribution has been widely applied to model survival data set in the last decades, but one of the major disadvantages of the exponential distribution is that it has a constant hazard rate property. Moreover, the probability density function (pdf) of the exponential distribution is a decreasing function. Due to this reason, several generalizations of the exponential distribution have been suggested in the literature. The Weibull, gamma, generalized exponential distributions are different extensions of the exponential distribution, which contain exponential distribution as a special case. All the three distributions can have increasing, decreasing or unimodal density functions and monotone (increasing or decreasing) hazard rate properties. Unfortunately, none of the distributions exhibits a non-monotone (bathtub or inverted bathtub) hazard rate properties. The limitations of these classical distributions often arouse the interest of researchers in finding new distributions by extending existing ones.

In recent years, several ways of generating new distributions from classic ones have been introduced in literature. [1] introduced the family of Weibull distributions with exponential distribution as sub-model. The model was constructed by taking power of exponentially distributed random variables. [2] introduced a new method of adding a parameter into a family of distributions which they called the Marshall-Olkin extended family of distributions. [3] introduced the beta-generated family of distributions and most recently, [4] introduced the *T-R{Y}* framework for generalizing existing classical distributions. Many

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generalized distributions arising from the Marshall-Olkin extended family of distributions are found in [5-11].

Suppose the survival function of a known probability distribution is defined by $\bar{F}(x)$, [2] defined the survival function of the Marshall-Olkin extended family of distributions as

$$\bar{G}(x,\alpha) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)} = \frac{\alpha\bar{F}(x)}{F(x)+\alpha\bar{F}(x)}, \quad -\infty < x < \infty, \;\; 0 < \alpha < \infty. \tag{1}$$

If $F(x)$ is a cumulative distribution function with a density function $f(x)$, then $G(x)$ has a density function given as

$$g(x,\alpha) = \frac{\alpha f(x)}{\{1-\bar{\alpha}\bar{F}(x)\}^2}, \tag{2}$$

where $\bar{\alpha} = 1-\alpha$ is called a "tilt parameter", since the hazard $h(x)$ of the transformed distribution is shifted below $(\alpha \geq 1)$ or above $(0 < \alpha \leq 1)$ from the hazard $r(x)$ of the baseline distribution. In fact, for all $x \geq 0$, $h(x) \leq r(x)$ when $\alpha \geq 1$, and $h(x) \geq r(x)$ when $0 < \alpha \leq 1$. A unique characteristic of the Marshall-Olkin extended family of distributions is the property of allowing the random variable of the transformed distribution to follow the same support with the baseline distribution.

In this paper, we introduce a new Marshall-Olkin Extended Topp Leone (MOETL) distribution by taking the baseline distribution defined in Equation (1) as the survival function of the one parameter Topp Leone distribution reported in [12].

[12] defined a one-parameter Topp-Leone distribution with bounded support, which is a sub-model of the two-parameter Topp-Leone distribution introduced by [13]. The survival function and the probability density function (pdf) of the one-parameter Topp Leone distribution are defined as

$$\bar{F}(x) = 1-(1-(1-x)^2)^{\lambda} \tag{3}$$

and

$$f(x) = 2\lambda\,(1-x)(1-(1-x)^2)^{\lambda-1}. \tag{4}$$

The rest Sections of this paper are organized as follows: Section 2 presents some mathematical properties of the proposed distribution which include; the survival function, cumulative distribution function, probability density function, Hazard rate function, Quantile function, Median, Moments and Renyi entropy. The parameter estimation and simulation study on the maximum likelihood estimates of the proposed distribution are given in Section 3. Finally, in Section 4, we applied the proposed distribution to two real data sets and compared its fit alongside with some existing distributions defined on a unit interval.

2. MATHEMATICAL PROPERTIES OF THE MOETL DISTRIBUTION

2.1. Survival, Cumulative Distribution, Density and Hazard Functions of the MOETL Distribution

Let *X* be a random variable. Then, the survival function of the MOETL distribution can be obtained as

$$\bar{G}(x) = \frac{\alpha\left\{1-\left(1-(1-x)^2\right)^{\lambda}\right\}}{1-\bar{\alpha}\{1-(1-(1-x)^2)^{\lambda}\}} = \frac{\alpha\left\{1-\left(1-(1-x)^2\right)^{\lambda}\right\}}{\alpha+\bar{\alpha}(1-(1-x)^2)^{\lambda}}. \tag{5}$$

The cumulative distribution function of the MOETL distribution can be obtained as

$$G(x) = 1-\bar{G}(x) = \frac{\left\{1-\left(1-(1-x)^2\right)^{\lambda}\right\}}{1-\bar{\alpha}\{1-(1-(1-x)^2)^{\lambda}\}} = \frac{\left(1-(1-x)^2\right)^{\lambda}}{\alpha+\bar{\alpha}(1-(1-x)^2)^{\lambda}}. \tag{6}$$

The corresponding density function of the MOETL distribution is obtained as

$$g(x) = \frac{2\alpha\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}}{\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^\lambda\}\right]^2}$$
$$= 2\alpha\lambda(1-x)[1-(1-x)^2]^{\lambda-1}\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^\lambda\}\right]^{-2}. \quad (7)$$

[14] reported that for any positive real numbers and $|z| < 1$, a generalized binomial expansion is given by

$$(1+z)^{-s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} (-1)^k z^k. \quad (8)$$

Now, using Equation (8) in (7), the density function of the MOETL distribution can be expressed in series representation as follows;

$$\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^\lambda\}\right]^{-2} = \sum_{i=0}^{\infty} \binom{i+1}{i} \bar{\alpha}^i \{1-(1-(1-x)^2)^\lambda\}^i$$

$$\left[1-(1-(1-x)^2)^\lambda\right]^i = \sum_{j=0}^{i} \binom{i}{j} (-1)^j (1-(1-x)^2)^{\lambda j}$$

$$[1-(1-x)^2]^{\lambda(j+1)-1} = \sum_{k=0}^{\lambda(j+1)-1} \binom{\lambda(j+1)-1}{k} (-1)^k (1-x)^{2k}$$

$$(1-x)^{2k+1} = \sum_{m=0}^{2k+1} \binom{2k+1}{m} (-1)^m x^m$$

so that Equation (7) becomes

$$g(x) = 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1} \binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m} \times (-1)^{j+k+m}(1-\alpha)^i x^m. \quad (9)$$

The graphical plots of the density function of the MOETL distribution is displayed in Figure 1.

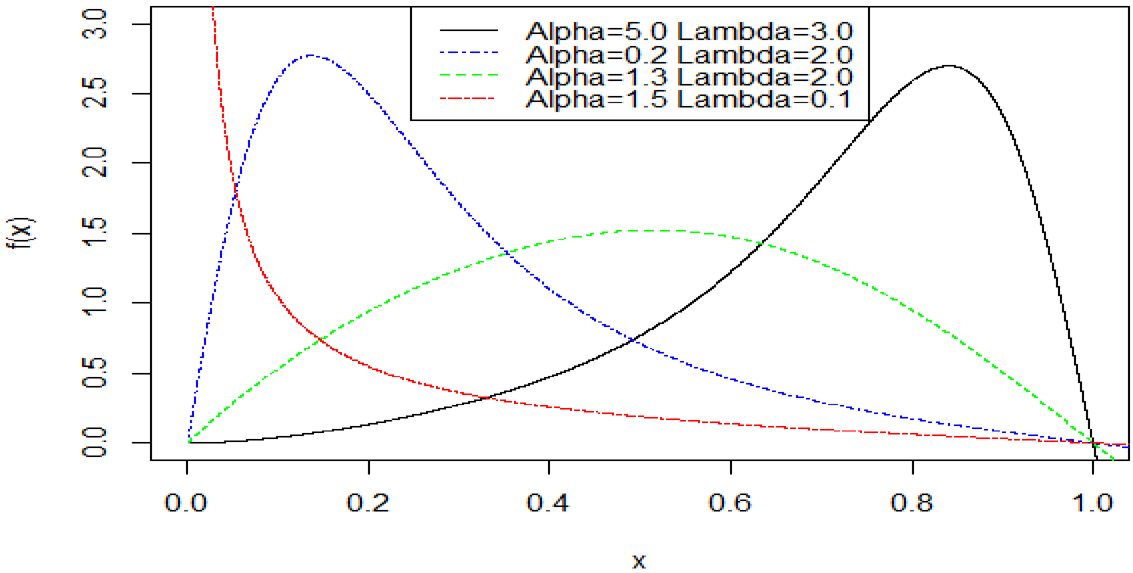

Figure 1. *Probability Density Function of the MOETL distribution*

The graphical plots in Figure 1 indicates that the density function of the MOETL distribution exhibits a decreasing (reversed-J), left-skewed, right-skewed and symmetric shapes.
The hazard rate function of a continuous random variable X is defined by

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{g(x)}{\bar{G}(x)}. \quad (10)$$

Thus, hazard rate function of the MOETL distribution is obtained as

$$h(x) = \frac{2\alpha\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}}{\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^\lambda\}\right]^2} \times \frac{1-\bar{\alpha}\left\{1-\left(1-(1-x)^2\right)^\lambda\right\}}{\alpha\{1-(1-(1-x)^2)^\lambda\}}$$
$$= \frac{2\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}}{\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^\lambda\}\right]\{1-(1-(1-x)^2)^\lambda\}}. \quad (11)$$

The graphical plots of the hazard rate function of the MOETL distribution is shown in Figure 2.

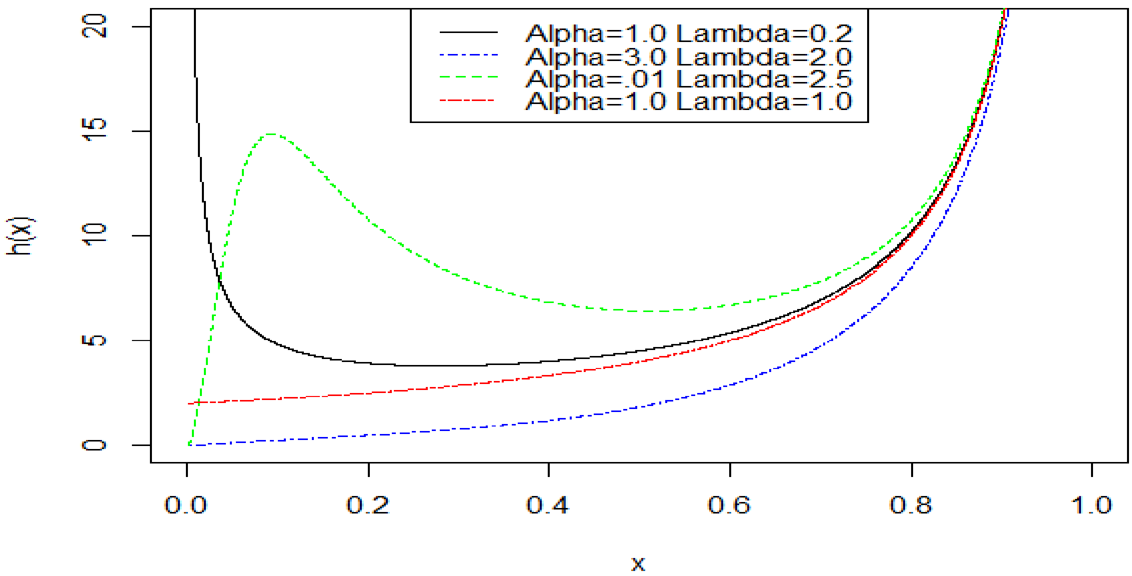

Figure 2. *Hazard Rate Function of the MOETL distribution*

The graphical plots in Figure 2 reveals that the hazard rate function of the MOETL distribution exhibits an increasing, bathtub and inverted bathtub shaped property.

2.2. The Quantile Function of the MOETL Distribution

Given the cumulative distribution function $G(x)$ defined by Equation (6), the quantile function of the MOETL distribution can be obtain as $Q_X(p) = G^{-1}(p)$.

The p^{th} quantile function is obtained by solving $G(x) = p$ i.e.,

$$\frac{\left\{1-\left(1-(1-x)^2\right)^\lambda\right\}}{1-\bar{\alpha}\{1-(1-(1-x)^2)^\lambda\}} = p$$
$$(1-(1-x)^2)^\lambda = p\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^\lambda\}\right]$$
$$(1-(1-x)^2)^\lambda - p\bar{\alpha}(1-(1-x)^2)^\lambda = p\alpha$$
$$(1-p\bar{\alpha})\left[(1-(1-x)^2)^\lambda\right] = p\alpha$$
$$(1-(1-x)^2)^\lambda = \frac{p\alpha}{(1-p+p\alpha)}$$
$$(1-x)^2 = 1 - \left[\frac{p\alpha}{(1-p+p\alpha)}\right]^{1/\lambda}$$

$$x = 1 - \left[1 - \left[\frac{p\alpha}{(1-p+p\alpha)}\right]^{1/\lambda}\right]^{1/2}. \tag{12}$$

The median of the MOETL distribution is obtained by substituting for $p = 1/2$ in Equation (12) which yields,

$$median = 1 - \left[1 - \left[\frac{0.5\alpha}{(0.5+0.5\alpha)}\right]^{1/\lambda}\right]^{1/2}. \tag{13}$$

2.3. The r^{th} Moments of the MOETL Distribution

Let X be a continuous random variable with probability density function $g(x)$, then the r^{th} moment about the origin of X is defined by,

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r g(x) dx. \tag{14}$$

The moments of the MOETL distribution cannot be expressed in a closed form, hence, we consider the series representation of the density function of the MOETL distribution defined in Equation (9) to obtain the r^{th} moment of the distribution in terms of infinite series. By substituting Equation (9) into (14), we define the r^{th} moment of the MOETL distribution as

$$\mu_r' = E(X^r) = \int_0^1 x^r 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1} \binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k} \times \binom{2k+1}{m}(-1)^{j+k+m}(1-\alpha)^i x^m \, dx \tag{15}$$

$$= 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1} \binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k} \times \binom{2k+1}{m}(-1)^{j+k+m}(1-\alpha)^i \int_0^1 x^{r+m} \, dx. \tag{16}$$

Evaluating the integral part of the expression in Equation (16), we have

$$\int_0^1 x^{r+m} \, dx = \left.\frac{x^{r+m+1}}{r+m+1}\right|_0^1 = \frac{1}{r+m+1}.$$

Equation (16) can further be simplified as

$$\mu_r' = 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1} \binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}(-1)^{j+k+m}\frac{(1-\alpha)^i}{r+m+1}. \tag{17}$$

The first four r^{th} moments of the MOETL distribution in terms of infinite series are obtained from Equation (17) as;

$$\mu_1' = \mu = 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1} \binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}(-1)^{j+k+m}\frac{(1-\alpha)^i}{m+2}$$

$$\mu_2' = 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1} \binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}(-1)^{j+k+m}\frac{(1-\alpha)^i}{m+3}$$

$$\mu_3' = 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1}\binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}(-1)^{j+k+m}\frac{(1-\alpha)^i}{m+4}$$

$$\mu_4' = 2\alpha\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\lambda(j+1)-1}\sum_{m=0}^{2k+1}\binom{i+1}{i}\binom{i}{j}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}(-1)^{j+k+m}\frac{(1-\alpha)^i}{m+5}.$$

Furthermore, the variance, measures of skewness and kurtosis of the MOETL distribution can be derived by substituting the values of the r^{th} moments into the expressions reported in [15];

Variance $(\mu_2) = (\mu_2' - \mu^2)$, Skewness $(S_k) = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}$,

Kurtosis $(K_s) = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{(\mu_2' - \mu^2)^2}$.

Numerical computations of the theoretical moments of the MOETL distribution for selected values of the parameters are shown in Tables 1 and 2.

Table 1. *Theoretical Moments of MOETL distribution* $(\alpha = 2)$

μ_r'	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 3$
μ_1'	0.3048	0.4292	0.6222
μ_2'	0.1548	0.2447	0.4259
μ_3'	0.0944	0.1589	0.3099
μ_4'	0.0637	0.1115	0.2354
σ^2	0.0619	0.0605	0.0388
S_k	0.6159	0.1312	-0.4366
K_s	2.3496	2.0104	2.5140

Table 2. *Theoretical Moments of MOETL distribution* $(\alpha = 4)$

μ_r'	$\lambda = 0.5$	$\lambda = 3$	$\lambda = 6$
μ_1'	0.4061	0.6965	0.7784
μ_2'	0.2331	0.5188	0.6259
μ_3'	0.1526	0.4032	0.5155
μ_4'	0.1079	0.3230	0.4324
σ^2	0.0682	0.0337	0.0200
S_k	0.1437	-0.8220	-0.9993
K_s	1.9515	3.2936	3.4810

From Tables 1 and 2 we observed that the MOETL distribution exhibits a right-skewed ($S_k > 0$), left-skewed $(S_k < 0)$ and approximately symmetric $(S_k \approx 0)$ shapes. On the other hand, the MOETL distribution can be leptokurtic $(K_s > 3)$, platykurtic $(K_s < 3)$ and mesokurtic $(K_s \approx 3)$. This claim supports the graphical illustration of the density function of the MOETL distribution in Figure 1.

2.4. The Renyi Entropy of the MOETL Distribution

[16] defined an entropy of a random variable X as a measure of randomness associated with the random variable X. The Renyi entropy of X with density function $f(x)$, is defined by,

$$\tau_R(\xi) = \frac{1}{1-\xi} log_e \int f^{\xi}(x)dx, \qquad \xi > 0, \quad \xi \neq 1. \tag{18}$$

It is observed from the density function of the MOETL distribution that the mathematical expression of the Renyi entropy cannot be expressed in a closed form. Hence, we also consider the Renyi entropy of the distribution in terms of infinite series by substituting Equation (9) into (18), to obtain the Renyi entropy of the random variable X following the MOETL distribution as

$$\tau_R(\xi) = \frac{1}{1-\xi} log_e \int \left[2\alpha\lambda(1-x)[1-(1-x)^2]^{\lambda-1}\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^{\lambda}\}\right]^{-2}\right]^{\xi} dx. \tag{19}$$

Again, using the binomial expansion in Equation (8), we have

$$\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^{\lambda}\}\right]^{-2\xi} = \sum_{i=0}^{\infty} \binom{i+\xi}{i} \bar{\alpha}^{i\xi} \{1-(1-(1-x)^2)^{\lambda}\}^{i\xi}$$

$$\left[1-(1-(1-x)^2)^{\lambda}\right]^{i\xi} = \sum_{j=0}^{i\xi} \binom{i\xi}{j} (-1)^j (1-(1-x)^2)^{\xi\lambda j}$$

$$[1-(1-x)^2]^{\xi\lambda(j+1)-\xi} = \sum_{k=0}^{\xi\lambda(j+1)-\xi} \binom{\xi\lambda(j+1)-\xi}{k} (-1)^k (1-x)^{2\xi k}$$

$$(1-x)^{2\xi k+1} = \sum_{m=0}^{2\xi k+1} \binom{2\xi k+1}{m} (-1)^m x^{\xi m}.$$

Substituting these expressions into Equation (19), we have

$$\tau_R(\xi) = \frac{1}{(1-\xi)} log_e \left[\sum_{i=0}^{\infty} \sum_{j=0}^{i\xi} \sum_{k=0}^{\xi\lambda(j+1)-\xi} \sum_{m=0}^{2\xi k+1} \binom{i+\xi}{i} \binom{i\xi}{j} \binom{\xi\lambda(j+1)-\xi}{k} \binom{2\xi k+1}{m} (-1)^{j+k+m} \frac{(1-\alpha)^{i\xi}(2\alpha\lambda)^{\xi}}{(\xi m+1)}\right]. \tag{20}$$

[17] provided some important properties of the measure given by Equation (18):

(i) The Renyi entropy can be negative;
(ii) For any $\xi_1 < \xi_2$, $R_{\xi_2} \leq R_{\xi_1}$ and equality holds if and only if X is a uniform random variable.

According to [18], the Renyi entropy is more or less sensitive to the shape of the probability distribution due to the parameter ξ. For large values of ξ, the measure given in Equation (18) is more sensitive to events that occur often, while for small values of ξ, it is more sensitive to event which rarely occur. For instance, [19] reported that an application of Equation (18) in speech recognition, different values of ξ determines different concepts of noisiness. Basically, small values of ξ tend to emphasize the noise content of the signal, while large values of ξ tend to emphasize the harmonic content of the signal. Numerical computations of the Renyi entropy of the TLPLD for varying values of parameter ξ is shown in Table 3.

Table 3. *Numerical Computation of the Renyi Entropy of the MOETLD*

ξ	$\lambda = 1, \alpha = 3$	$\lambda = 1, \alpha = 5$	$\lambda = 3, \alpha = 3$	$\lambda = 3, \alpha = 5$
0.01	-0.0011	-0.0013	-0.0066	-0.0086
0.03	-0.0032	-0.0039	-0.0193	-0.0251
0.5	-0.0404	-0.0564	-0.2158	-0.2887
0.7	-0.0519	-0.0755	-0.2672	-0.3591
2	-0.1018	-0.1704	-0.4466	-0.5994
4	-0.1464	-0.2558	-0.5543	-0.7331
7	-0.1861	-0.3202	-0.6231	-0.8133
9	-0.2034	-0.3449	-0.6484	-0.8418

From Table 3, we clearly observe that for any two consecutive values of parameter ξ_i , Say ($\xi_1$and ξ_2), the Renyi entropy R_i, Say (R_1 and R_2), satisfies the condition $\xi_1 < \xi_2$, $R_{\xi_2} \leq R_{\xi_1}$ as suggested by [17].

3. PARAMETER ESTIMATION OF THE MOETL DISTRIBUTION

3.1. Maximum Likelihood Estimation

Suppose $(x_1, x_2, \ldots, x_n)$ are random samples from a known probability density function $f(x)$, then the likelihood function associated with the random variable X is defined by

$$L(x) = \prod_{i=0}^{n} f(x_i). \tag{21}$$

The corresponding log-likelihood function of Equation (21) is given by

$$\ell(x, \varphi) = \sum_{i=0}^{n} log[f(x_i, \varphi)]. \tag{22}$$

From Equation (22), we define the log-likelihood function of the MOETL distribution as

$$\ell(x, \varphi) = \sum_{i=0}^{n} log\left[\frac{2\alpha\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}}{\left[1-\bar{\alpha}\{1-(1-(1-x)^2)^{\lambda}\}\right]^2}\right], \qquad \varphi = (\alpha, \lambda)$$

$$= nlog(2\alpha\lambda) + \sum_{i=0}^{n} log(1 - x_i) + (\lambda - 1)\sum_{i=0}^{n}[1 - (1 - x_i)^2]$$

$$-2\sum_{i=0}^{n} log\left[1 - \bar{\alpha}\{1 - (1 - (1 - x)^2)^{\lambda}\}\right]. \tag{23}$$

Taking partial derivative of Equation (23) with respect to the parameters, we obtain the score function as;

$$\frac{\ell(x, \varphi)}{d\alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n} \frac{2\{1 - (1 - (1 - x)^2)^{\lambda}\}}{[1 - \bar{\alpha}\{1 - (1 - (1 - x)^2)^{\lambda}\}]}$$

$$\frac{\ell(x, \varphi)}{d\lambda} = \frac{n}{\lambda} + \sum_{i=0}^{n} log(1 - (1 - x)^2) + \sum_{i=0}^{n} \frac{2\bar{\alpha}(1 - (1 - x)^2)log(1 - (1 - x)^2)}{[1 - \bar{\alpha}\{1 - (1 - (1 - x)^2)^{\lambda}\}]}.$$

The maximum likelihood estimator $\hat{\varphi}$ of φ can be obtained by solving the system of equation $\frac{\ell(x,\varphi)}{d\varphi} = 0$. This equation cannot be solved analytically, hence, we use the "bbmle" in *R* statistical package to evaluate the maximum likelihood estimates of the parameters of the MOETL distribution.

3.2. Interval Estimate

The asymptotic confidence intervals (CIs) for the parameters of MOETLD (α, λ) is obtained according to the asymptotic distribution of the maximum likelihood estimates of the parameters. Suppose $\hat{\varphi} = (\hat{\alpha},, \hat{\lambda},)$ be MLE of φ, then the estimators are approximately bi-variate normal with mean (α, λ) and the Fisher information matrix is given by:

$$I(\varphi_k) = -E\big(H(\varphi_k)\big). \tag{24}$$

The approximate (1-δ)100 CIs for the parameters α and λ respectively, are

$$\hat{\alpha} \pm Z_{\delta/2}\sqrt{var(\hat{\alpha})} \quad \text{and} \quad \hat{\lambda} \pm Z_{\delta/2}\sqrt{var(\hat{\lambda})}$$

where $var(\hat{\alpha})$ and $var(\hat{\lambda})$ are the variance of α and λ which are given by the diagonal elements of the variance-covariance matrix $I^{-1}(\varphi_k)$ and $Z_{\delta/2}$ is the upper $(\delta/2)$ percentile of the standard normal distribution.

3.3. Simulation Study

In this section, we investigate the asymptotic behaviour of the maximum likelihood estimate of the parameters of the Marshall Olkin Extended Topp-Leone distribution (MOETLD) through a simulation study. A Monte Carlo simulation study is repeated 1000 times for different sample sizes *n* = 30, 50, 75, 100 and parameter values $(\alpha = 1.5,\ \lambda = 1)$, $(\alpha = 1.5,\ \lambda = 2)$ and $(\alpha = 0.3, \lambda = 2)$. An algorithm for the simulation study is given by the following steps;

1. Choose the value *N* (i.e. number of Monte Carlo simulation);
2. Choose the values $\varphi_0 = (\alpha_0,\ \lambda_0)$ corresponding to the parameters of the MOETLD (α, λ);
3. Generate a sample of size *n* from MOETLD;
4. Compute the maximum likelihood estimates $\hat{\varphi}_k$ of φ_k;
5. Repeat steps (3-4), *N*-times;
6. Compute the:

(i) Average Bias = $\frac{1}{N}\sum_{k=0}^{N}(\hat{\varphi}_k - \varphi_k)$;

(ii) Mean Square Error (MSE) = $\frac{1}{N}\sum_{k=0}^{N}(\hat{\varphi}_k - \varphi_k)^2$ and

(iii) Coverage Probability of the 95% confidence interval of the estimates $\hat{\varphi}_k$ given by

$$CP(\varphi_0) = \frac{1}{N}\sum_{k=1}^{N} I\left(\hat{\varphi}_k - Z_{\delta/2}\sqrt{var(\hat{\varphi})} < \varphi_0 < \hat{\varphi}_k + Z_{\delta/2}\sqrt{var(\hat{\varphi})}\right)$$

where $I(\cdot)$ is an indicator function and $\sqrt{var(\hat{\varphi})}$ is the standard error of the estimate φ_k. The coverage probability computes the proportion of times the confidence interval contains the true value of the parameter φ_0.

Table 4. *Simulation Results for Average Bias, MSE and CP of Parameter α*

Parameter	*N*	Average Bias *(α)*	*MSE(α)*	*CP(α)*
$\alpha = 1.5$ $\lambda = 1.0$	30	0.1229	1.4792	0.8380
	50	0.0895	0.7595	0.8840
	75	0.0723	0.4478	0.8860
	100	0.0469	0.3258	0.9200
$\alpha = 1.5$ $\lambda = 2.0$				
	30	0.1759	1.4817	0.8340
	50	0.0911	0.8222	0.8600
	75	0.0175	0.4866	0.8840
	100	0.0115	0.2943	0.9000
$\alpha = 0.3$ $\lambda = 2.0$				
	30	0.0343	0.0508	0.8580
	50	0.0135	0.0195	0.9180
	75	0.0088	0.0153	0.9000
	100	0.0060	0.0113	0.9200

Table 5. *Simulation Results for Average Bias, MSE and CP of Parameter λ*

Parameter	*N*	Average Bias *(α)*	*MSE(α)*	*CP(α)*
$\alpha = 1.5$ $\lambda = 1.0$	30	0.1768	0.2642	0.9500
	50	0.0882	0.1229	0.9560
	75	0.0516	0.0813	0.9540
	100	0.0417	0.0553	0.9640
$\alpha = 1.5$ $\lambda = 2.0$				
	30	0.3003	1.0841	0.9340
	50	0.2298	0.6365	0.9300
	75	0.1610	0.3376	0.9520
	100	0.1248	0.2374	0.9540
$\alpha = 0.3$ $\lambda = 2.0$				
	30	0.1283	0.3170	0.9640
	50	0.0646	0.1491	0.9660
	75	0.0552	0.1143	0.9500
	100	0.0447	0.0886	0.9460

Tables 4 and 5 present the simulation results for the average bias, mean square error and coverage probability of the 95% confidence interval of the parameter estimates of the MOETLD at different choice of parameter values. Clearly from the Tables, we observed that both parameters α and λ are positively biased, and the value of the biasness and mean square error of the parameter estimates decreases (tends to zero) are the value of the sample size n increases. Also, we observed that the coverage probabilities of the CIs of the parameter estimates are close to the nominal level of 95%.

4. APPLICATION OF THE PROPOSED MOETL DISTRIBUTION

In this section, the MOETL distribution is applied to two real data sets defined on a unit interval. Some well-known lifetime distributions whose support lies on a unit interval will also be used to fit the data sets. These distributions with their density function include;

1. Marshall-Olkin Extended Kumaraswamy Distribution (MOEKD);

$$f(x) = \frac{\alpha a b x^{a-1}(1-x^{a})^{b-1}}{\left[1-\bar{\alpha}(1-x^{a})^{b}\right]^{2}},$$

2. Kumaraswamy Distribution;

$$f(x) = abx^{a-1}(1-x^a)^{b-1},$$

3. Beta Distribution;

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Dataset 1: The first data set consist of 48 rock samples from a petroleum reservoir reported in [20]. The data set is defined on a unit interval which is positively (right) skewed with skewness value (S_k = 1.1330) and leptokurtic with kurtosis value ($K_s = 3.9404$). The data set include;
0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

Dataset 2: The second data set represents 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania and is reported in [21]. The data set is positively skewed with skewness value (S_k = 0.9940) and leptokurtic with kurtosis value ($K_s = 3.3054$).The data set include; 0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74.

The parameter estimates of the distributions, Log-likelihood, Akaike Information Criterion (AIC), Kolmogorov-Smirnov Statistic $(K-S)$ and Crammer-von Mises test statistic (W^*) with their respective *p*-values will be employed to compare the fitness of the distributions to the two data sets under study.

Table 6. *Summary Statistics for the Rock Samples from a Petroleum Reservoir Data Set*

Distributions	*Parameter Estimates*	*Log-Lik*	*AIC*	*K-S* ***(p-value)***	W^* ***(p-value)***
MOETLD	$\alpha = 0.0034$ $\lambda = 5.5834$	57.6628	−111.3256	0.0832 (**0.8938**)	0.0388 (**0.9412**)
MOEKD	$\alpha = 0.0214$ $a = 4.8120$ $b = 46.3554$	57.7042	−109.4082	0.0911 (**0.8201**)	0.0462 (**0.9013**)
Kumaraswamy	$a = 2.7189$ $b = 44.6737$	52.4915	−100.9831	0.1533 (**0.2094**)	0.2059 (**0.2568**)
Beta	$a = 5.9416$ $b = 21.2079$	55.6002	−107.2004	0.1427 (**0.2828**)	0.1298 (**0.4588**)

Table 7. *Summary Statistics for the Maximum Flood Level Data Set*

Distributions	*Parameter Estimates*	*Log-Lik*	*AIC*	*K-S* ***(p-value)***	W^* ***(p-value)***
MOETLD	$\alpha = 0.0216$ $\lambda = 8.6519$	16.1644	−28.3289	0.1214 (**0.9298**)	0.0387 (**0.9439**)
MOEKD	$\alpha = 0.0153$ $a = 6.4543$ $b = 5.4128$	15.9235	−25.8471	0.1297 (**0.8899**)	0.0411 (**0.9320**)
Kumaraswamy	$a = 3.3773$ $b = 12.0005$	12.9733	−21.9465	0.2176 (**0.3002**)	0.1654 (**0.3479**)
Beta	$a = 5.8307$ $b = 9.2364$	14.1836	−24.3671	0.2062 (**0.3627**)	0.1242 (**0.4823**)

Tables 6 and 7 reveal the summary statistics for the rock samples from a petroleum reservoir and the maximum flood level data sets. The parameter estimates, log-likelihood, Akaike Information Criterion (AIC), Kolmogorov-Smirnov Statistic $(K - S)$ and Crammer-von Mises test statistic (W^*) with their respective *p*-values of the distributions were estimated for the two data sets. The Tables indicates that the proposed Marshall-Olkin Topp Leone distribution, having the least value in terms of AIC, $(K - S)$ and W^* test statistics with the highest corresponding *p-values*, outperforms the Marshall-Olkin Kumaraswamy distribution, Kumaraswamy distribution and the Beta distribution in analyzing the data set under study. Furthermore, Figures 3-6 respectively showing the graphical illustration of the density and cumulative distribution fit and the Probability-Probability (P-P) plots of the distributions for the two data sets validate the claim of the superiority of the proposed MOETL distribution over the competing distributions under study.

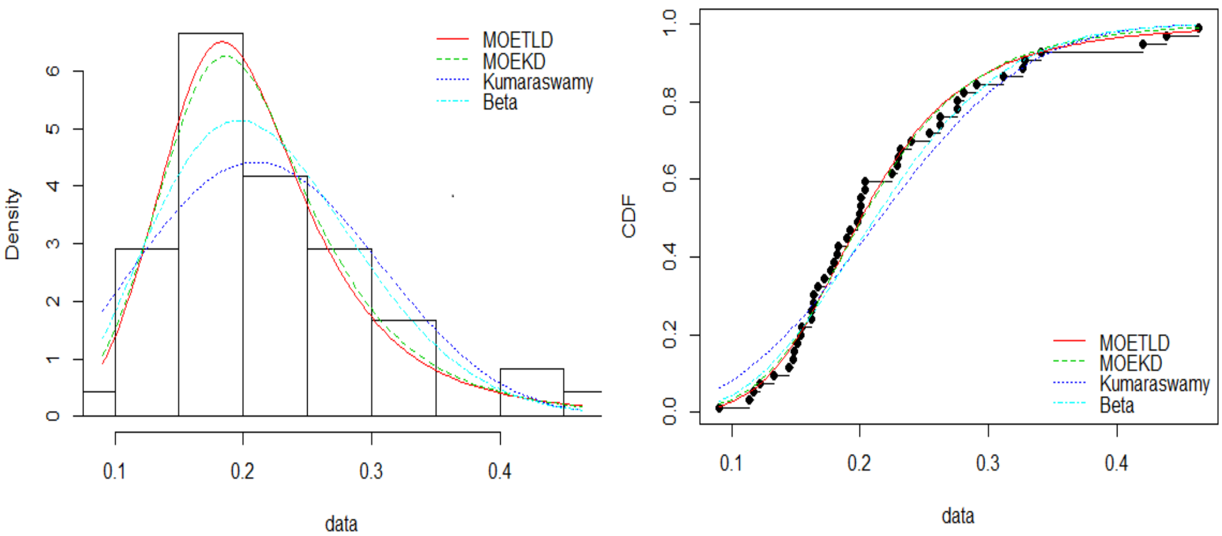

Figure 3. *Density and Cumulative Distribution fit of the Distributions for the Dataset* 1

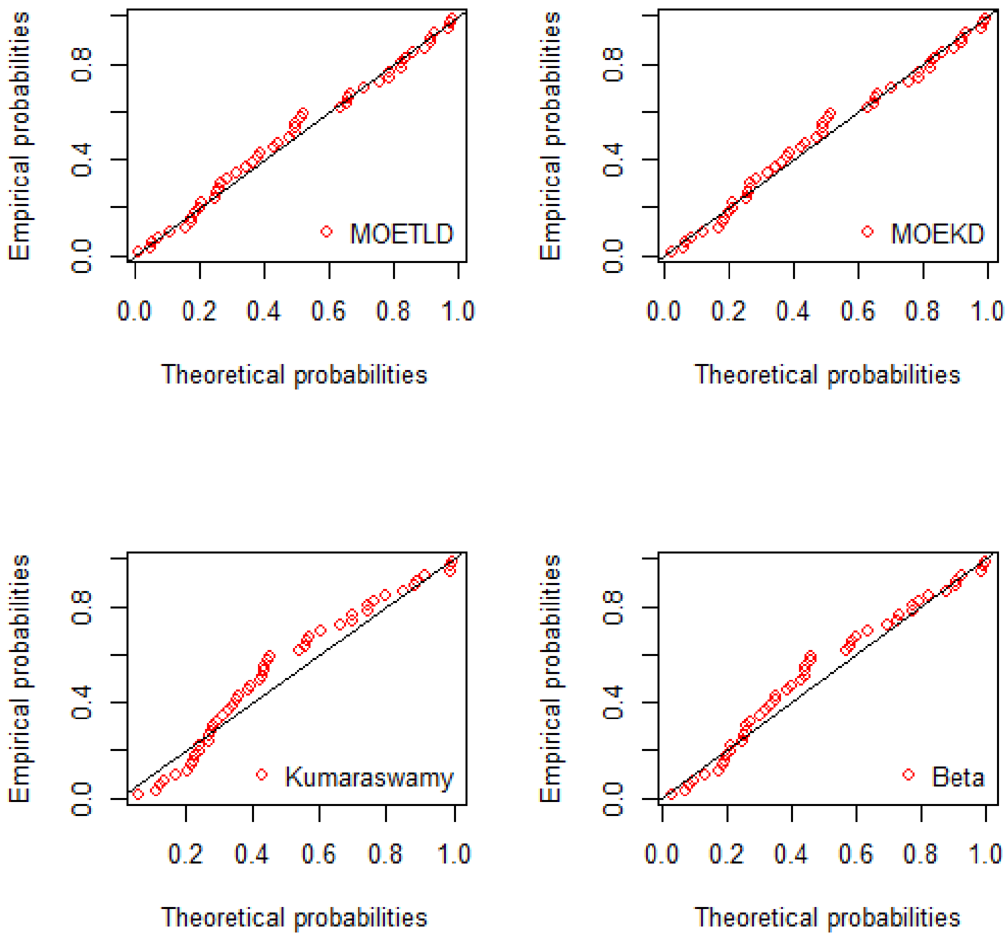

Figure 4. *Probability-Probability (P-P) plots of the Distributions for the Dataset* 1

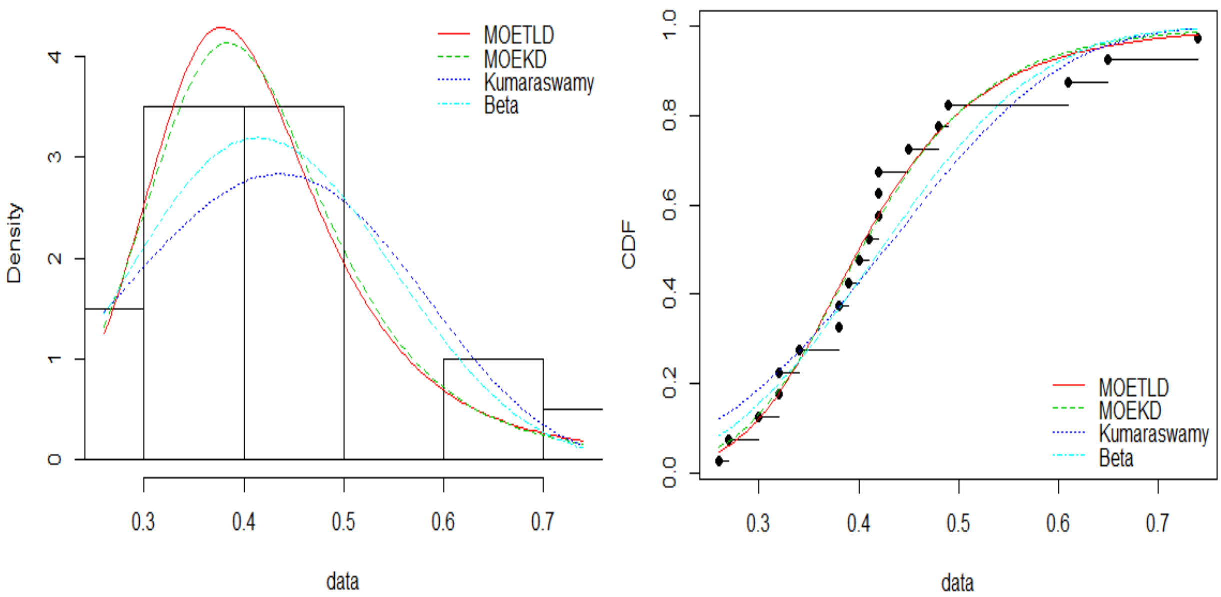

Figure 5. *Density and Cumulative Distribution fit of the Distributions for the Dataset* 2

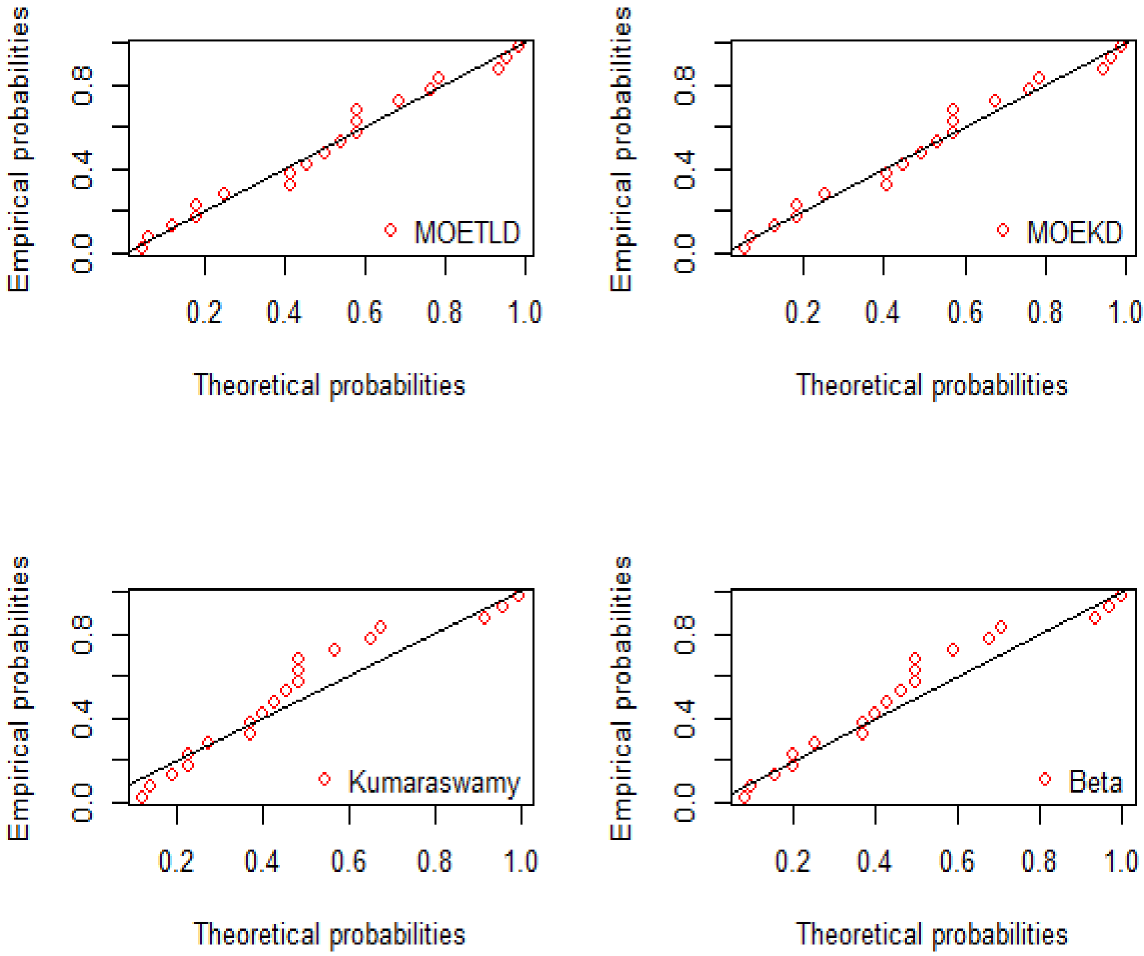

Figure 6. *Probability-Probability (P-P) plots of the Distributions for the Dataset* 2

5. CONCLUSION

This study proposed a new Marshall Olkin extended family of distributions which we called the Marshall Olkin extended Topp-Leone distribution (MOETLD). The mathematical properties of the proposed distribution were derived. Numerical computations of the moments as well as the Renyi entropy were established and the method of maximum likelihood estimation was used in estimating the parameters of the proposed distribution. Finally, an application of the proposed MOETL distribution to two real data sets, revealed its superiority over the existing Marshall-Olkin Kumaraswamy distribution (MOEKD), Kumaraswamy distribution and the Beta distribution, in modeling the two data sets under study.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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