On orthomorphism elements in ordered algebra

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Abstract: Let \( C \) be an ordered algebra with a unit \( e \). The class of orthomorphism elements \( \text{Orthe}(C) \) of \( C \) was introduced and studied by Alekhn in "The order continuity in ordered algebras". If \( C = L(G) \), where \( G \) is a Dedekind complete Riesz space, this class coincides with the band \( \text{Orth}(G) \) of all orthomorphism operators on \( G \). In this study, the properties of orthomorphism elements similar to properties of orthomorphism operators are obtained. Firstly, it is shown that if \( C \) is an ordered algebra such that \( C_\gamma \), the set of all regular elements of \( C \), is a Riesz space with the principal projection property and \( \text{Orthe}(C) \) is topologically full with respect to \( I_e \), then \( B_e = \text{Orthe}(C) \) holds, where \( B_e \) is the band generated by \( e \) in \( C_\gamma \). Then, under the same hypotheses, it is obtained that \( \text{Orthe}(C) \) is an \( f \)-algebra with a unit \( e \).

Key words: Ordered algebra, orthomorphism elements, orthomorphism, \( f \)-algebra

1. Introduction

All vector spaces are considered over the reals only. An ordered vector space (Riesz space) \( C \) under an associative multiplication is said to be an ordered algebra (Riesz algebra) whenever the multiplication makes \( C \) an algebra, and in addition it satisfies the following property: \( a, b \in C^+ \) implies \( ab \in C^+ \). A Riesz algebra \( C \) is called an \( f \)-algebra if \( C \) has the additional property that \( a \wedge b = 0 \) implies \( ac \wedge b = ca \wedge b = 0 \) for each \( c \in C^+ \).

Throughout the study, we will assume \( C \neq \{0\} \) and \( C \) has a unit element \( e > 0 \). An element \( a \in C \) is called a regular element if \( a = b - c \) with \( b \) and \( c \) positive, the space of all regular elements of \( C \) will be denoted by \( C_\gamma \). Obviously, \( C_\gamma \) is a real ordered algebra. Let \( C \) be an ordered vector space and an element \( a \in C^+ \), the order ideal \( I_a \) generated by \( a \) is the set \( I_a = \{ b \in C : -\lambda a \leq b \leq \lambda a \text{ for some } \lambda \in \mathbb{R}^+ \} \). Under the algebraic operations and the ordering induced by \( C \), \( I_a \) is an ordered vector subspace of \( C \). Moreover, \( I_e \) is an ordered algebra [1].

An element \( q \in C \) is said to be an order idempotent whenever \( 0 \leq q \leq e \) and \( q^2 = q \). Under the partial ordering induced by \( C \), the set of all order idempotents \( \text{OI}(C) \) of \( C \) is a Boolean algebra and its lattice operations satisfy the identities \( p \wedge q = pq \) and \( p \lor q = p + q - pq \) for all \( p, q \in \text{OI}(C) \). If \( c \in C \) and the modulus \( |c| \) of \( c \) exists, then \( q|c| = |qc| \) and \( |c| q = |cq| \) for all \( q \in \text{OI}(C) \) [2].

Definition 1.1 [1] Let \( C \) be an ordered algebra, an element \( a \in C \) is said to be an order idempotent preserving element whenever \( (e - q)a + q = 0 \) for all \( q \in \text{OI}(C) \). An element \( a \) is said to be an orthomorphism element of

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an ordered algebra $C$ whenever $a$ is an order idempotent preserving element that is also regular.

The collection of all orthomorphism elements of an ordered algebra $C$ will be denoted by $\text{Orthe}(C)$. An operator $\pi : G \to G$ on a Riesz space $G$ is said to be band preserving whenever $\pi(B) \subseteq B$ holds for each band $B$ of $G$. $\pi$ is a band preserving operator if and only if $\pi(x) \perp y$ whenever $x \perp y$ in $G$. A band preserving and order bounded operator $\pi$ is called orthomorphism of $G$ and the set of all orthomorphisms of $G$ is denoted by $\text{Orth}(G)$. If $G$ has the principal projection property, then an operator $\pi : G \to G$ is band preserving if and only if $\pi p = p \pi$ (or $(I - p)\pi p = 0$) for every order projection $p$ on $G$ [3, Theorem 8.3]. If $C = L(G)$ is taken, where $G$ is a Dedekind complete Riesz space, then the set of all order idempotents $\text{OI}(C)$ of $C$ is the set of all order projections on $G$ [3, Theorem 3.10] and the band $B_e$ generated by $e$ in $C_r$ is equal to $\text{Orth}(G) = \text{Orthe}(C)$ [3, Theorem 8.11]. In general, the equality $B_e = \text{Orthe}(C)$ does not hold in the case of an arbitrary ordered algebra $C$. Therefore, the following question might come into mind. Under what condition $\text{Orthe}(C)$ could be identified to $B_e$? In this work, we try to provide an answer to this question. Moreover, we will show that, under the same hypothesis, $\text{Orthe}(C)$ has the similar properties of orthomorphisms.

We refer to [3, 5, 7, 9] for definitions and notations which are not explained here. All Riesz spaces in this paper are assumed to be Archimedean.

2. Orthomorphism elements

**Proposition 2.1** Let $C$ be an ordered algebra such that $C_r$ is a Riesz space. Then, $\text{Orthe}(C)$ is a band in $C_r$ so that $B_e \subseteq \text{Orthe}(C)$ where $B_e$ is the band generated by $e$ in $C_r$.

**Proof** Since $q|a| = |qa|$ and $|a|q = |aq|$ for all $q \in \text{OI}(C)$ and $a \in C_r$, it is easy to show that $\text{Orthe}(C)$ is an order ideal. To see that $\text{Orthe}(C)$ is a band in $C_r$, let $0 \leq (b_\alpha) \uparrow b$ in $C_r$ with $(b_\alpha) \subseteq \text{Orthe}(C)$. Then, for all $\alpha$ we have

$$0 \leq (e - q) bq = (e - q)(b - b_\alpha)q + (e - q)b_\alpha q = (e - q)(b - b_\alpha)q \leq (b - b_\alpha).$$

Thus, $b - b_\alpha \downarrow 0$ implies $(e - q) bq = 0$ and $b \in \text{Orthe}(C)$. $B_e \subseteq \text{Orthe}(C)$ is obtained from the definition of $B_e$. \hfill \Box

**Lemma 2.2** Let $C$ be an ordered algebra such that $C_r$ is a Riesz space with the principal projection property and $b \in C_r$. Then, $b \in \text{Orthe}(C)$ if and only if $ba = ab$ for all $a \in I_e$.

**Proof** Let $b \in C_r$. If $ba = ab$ for all $a \in I_e$ then $b \in \text{Orthe}(C)$ as $\text{OI}(C) \subseteq I_e$. Now, let $b \in \text{Orthe}(C)$. From Freudenthal’s Spectral Theorem [3, Theorem 6.8], there exists a sequence $(u_n)$ of $e$-step function satisfying

$$0 \leq a - u_n \leq n^{-1}e$$

for each $n$ and $u_n \uparrow a$ for every $a \in I_e$. As $u_n$ $e$-step function, there exist $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{R}$ and $p_1, p_2, ..., p_k \in \text{OI}(C)$ such that $u_n = \sum_{i=1}^{k} \lambda_i p_i$. Thus, we have $bu_n = u_n b$ for each $n$. This yields

$$0 \leq |ab - ba| = |ab - u_n b + u_n b - ba| \leq |ab - u_n b| + |bu_n - ba| \leq n^{-1}b + n^{-1}b$$

for each $n$. Since $C$ is Archimedean, we have $ab = ba$ for every $a \in I_e$. \hfill \Box

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If $C = L(G)$, where $G$ is a Dedekind complete Riesz space, then $\text{Orth}(G) = \text{Orth}(C) = B_I$ where $B_I$ is the generated by the identity operator $I$ in $C_r$. In general, the equality $B_c = \text{Orth}(C)$ does not hold in the case of an ordered algebra $C$.

**Example 2.3** Let $G$ be the Riesz space of all continuous piecewise linear functions on $[0,1]$, then $\text{Orth}(G) = \{\lambda I : \lambda \in \mathbb{R}\}$ by the Problem 7 in [3, p. 124]. If we take $C = L(G)$, then we have $OI(C) = \{0, I\}$ as $OI(C) \subseteq \text{Orth}(G)$ holds. As a result of these simple observations we obtain that $\text{Orth}(C) = L_r(G) \neq B_I$.

Now, we will investigate when $B_c = \text{Orth}(C)$ holds.

**Definition 2.4** Let $C$ be an ordered algebra such that $C_r$ is a Riesz space and $\text{Orth}(C)$ has separating order dual. Let $b, c \in \text{Orth}(C)$ be arbitrary and $0 \leq b \leq c$. $\text{Orth}(C)$ is said to be topologically full with respect to $I_e$ if there exists a net $0 \leq a_\alpha \leq e$ with $a_\alpha c \to b$ in $\sigma(\text{Orth}(C), \text{Orth}(C)\^\sim)$.

**Example 2.5** Let $G$ be a Dedekind complete Riesz space with separating order dual. If we take $C = L(G)$, then $\text{Orth}(C) = \text{Orth}(G)$ is topologically full with respect to $I_e = Z(G)$ from the Theorem 4.3 in [6].

Let $C$ be a Riesz algebra such that $C_r$ is a Riesz space. It is easy to see that $(bc)g = q(bc)$ for each $b, c \in \text{Orth}(C)$ and $q \in OI(C)$. Thus, $\text{Orth}(C)$ is a Riesz algebra. For $b \in \text{Orth}(C)$, let us define $L_b : \text{Orth}(C) \to \text{Orth}(C) : L_b(c) = bc$ and $R_b : \text{Orth}(C) \to \text{Orth}(C) : R_b(c) = cb$ for each $c \in \text{Orth}(C)$.

$L_b$, $R_b$ are regular operators and so that the adjoint operators $L_b^\sim$, $R_b^\sim$ are regular operators on $\text{Orth}(C)\^\sim$. Let us consider positive linear maps

$$S_h : \text{Orth}(C) \to I_e^\sim, \ b \to S_{b,h} : S_{b,h}(a) = h(ab)$$

$$V_h : \text{Orth}(C) \to I_e^\sim, \ b \to V_{b,h} : V_{b,h}(a) = h(ba)$$

for each $b \in \text{Orth}(C)$, $a \in I_e$ and $h \in \text{Orth}(C)\^\sim_+$. If $\text{Orth}(C)$ is topologically full with respect to $I_e$, then we can say more about the positivity of the maps $S_h$ and $V_h$. The proof of the following Lemma is the adaptation of the Lemma in [8, p.65].

**Lemma 2.6** If $C$ is an ordered algebra such that $C_r$ is a Riesz space with the principal projection property and $\text{Orth}(C)$ is topologically full with respect to $I_e$, then $S_h, V_h : \text{Orth}(C) \to I_e^\sim$ are lattice homomorphisms for each $h \in \text{Orth}(C)\^\sim_+$.

**Proof** Let $0 \leq h \in \text{Orth}(C)\^\sim_+$. To see that $S_h$ is a lattice homomorphism, it is enough to show that $S_{b,h} \land S_{c,h} = 0$ for each $b, c \in \text{Orth}(C)$ satisfying $b \land c = 0$. Let $d = b + c$ and $I_b$, $I_c$, $I_d$ be respectively, the order ideals generated by $b$, $c$, and $d$. Then $I_d$ is actually the order direct sum of $I_b$ and $I_c$ by the Theorem 17.6 [5]. We denote by $p$ the order projection of $I_d$ onto $I_b$. Let $R$ be the restriction to $I_d$ of order bounded functionals on $\text{Orth}(C)$. Then $R$ is an order ideal in $I_d^\sim$ by the Theorem 2.3 in [3]. The adjoint $p^\sim : I_d^\sim \to I_d^\sim$ of $p$ satisfies $0 \leq p^\sim \leq I$ and as a consequence we obtain $p^\sim(R) \subseteq R$. As a result of these simple observations we obtain that the pair $< I_d, R >$ constitutes a Riesz pair and $p : (I_d, \sigma(I_d, R)) \to (I_d, \sigma(I_d, R))$ is continuous. Since $0 \leq p(d) \leq d$ there exists $(a_\alpha)$ in $I_e$ such that $0 \leq a_\alpha \leq e$ with $a_\alpha d \to p(d) = b$ in $\sigma(\text{Orth}(C), \text{Orth}(C)\^\sim)$. As $L_{a_\alpha} \in Z(I_d)$ for each $\alpha$ it is easy to see that $a_\alpha d \to b$ in $\sigma(I_d, R)$ and
\(a_{\alpha} p(d) = p(a_{\alpha} d)\). By the continuity of \(p\) now yields \(a_{\alpha} p(d) = a_{\alpha} b \rightarrow b\) in \(\sigma(I_{d}, R)\). Since \(a_{\alpha} d = a_{\alpha} b + a_{\alpha} c\) for each \(\alpha\), we have \(a_{\alpha} c \rightarrow 0\) in \(\sigma(I_{d}, R)\). As \((S_{b,h} \wedge S_{c,h})(a) \leq h((a - aa_{\alpha})b + (aa_{\alpha})c)\) for each \(\alpha\), we obtain

\[
0 \leq (S_{b,h} \wedge S_{c,h})(a) \leq \lim_{\alpha} h((a - aa_{\alpha})b + (aa_{\alpha})c) \\
= \lim_{\alpha} h(L_{a}(b - a_{\alpha}b + a_{\alpha}c)) \\
= \lim_{\alpha} L_{a}^{\sim}(h)(b - a_{\alpha}b + a_{\alpha}c) \\
= 0
\]

as \(L_{a}^{\sim}(\text{Orthe}(C)^{\sim}) \subseteq \text{Orthe}(C)^{\sim}\), which implies that \(S_{h}\) is lattice homomorphism. On the other hand, by the Lemma 2.2 \(ba_{\alpha} \rightarrow b\) and \(ca_{\alpha} \rightarrow 0\) in \(\sigma(I_{d}, R)\) holds. Similarly, taking \(V_{h}\) instead of \(S_{h}\) and \(R_{a}\) instead of \(L_{a}\), we get \(V_{h}\) is lattice homomorphism. \(\square\)

**Corollary 2.7** Let the hypotheses in the Lemma 2.6 hold. If \(b, c \in \text{Orthe}(C)\) and \(b \wedge c = 0\) then \(|S_{b,h}| \wedge |S_{c,t}| = 0\) for each \(h, t \in \text{Orthe}(C)^{\sim}\).

**Proof** Let \(b, c \in \text{Orthe}(C)\) and \(b \wedge c = 0\). From the Lemma 2.6 we have

\[
0 \leq |S_{b,h}| \wedge |S_{c,t}| \leq |S_{b|h}| \wedge |S_{c|t}| \leq |S_{b|h\vee|t|} \wedge S_{c|h\vee|t|}| = |S_{b\wedge c|h\vee|t|}| = 0.
\]

\(\square\)

**Proposition 2.8** Let \(C\) be an ordered algebra such that \(C_{r}\) is a Riesz space with the principal projection property and \(\text{Orthe}(C)\) is topologically full with respect to \(I_{c}\). Then, \(B_{e} = \text{Orthe}(C)\) holds (where \(B_{e}\) is the band generated by \(e\) in \(\text{Orthe}(C)\)).

**Proof** Let \(b \in \text{Orthe}(C)\) with \(|b| \wedge e = 0\). Clearly,

\[S_{b,h}(a) = h(ab) = h(L_{b}(a)) = L_{b}^{\sim}(h)(ae) = S_{e,L_{b}^{\sim}(h)}(a)\]

holds for each \(h \in \text{Orthe}(C)^{\sim}\). Then, it follows that

\[
0 \leq |S_{b,h}| = |S_{b,h}| \wedge |S_{b,h}| \leq |S_{b,h}| \wedge S_{e,L_{b}^{\sim}(h)} = 0
\]

and so \(S_{b,h} = 0\) for each \(h \in \text{Orthe}(C)^{\sim}\). Thus, we have \(b = 0\) which implies that \(B_{e} = \{e\}^{dd} = \text{Orthe}(C)\). \(\square\)

**Corollary 2.9** Let the hypotheses be as in the Proposition 2.8. Then, the band \(B_{e}\) generated by \(e\) in \(C_{r}\) is equal to \(\text{Orthe}(C)\).

**Proof** It is clear that the band generated by \(e\) in \(\text{Orthe}(C)\) is equal to the band generated by \(e\) in \(C_{r}\) as \(\text{Orthe}(C)\) is a band in \(C_{r}\). \(\square\)

By the Example 2.5, we have known that if \(G\) is a Dedekind complete Riesz space with separating order dual and \(C = L(G)\), then \(\text{Orthe}(C)\) has separating order dual and \(\text{Orthe}(C) = \text{Orth}(G)\) is topologically full with respect to \(I_{c} = Z(G)\). By using this observation and the above result, we can obtain the following Corollary being previously proved as a theorem in a different manner.
**Corollary 2.10** Let $G$ be a Dedekind complete Riesz space and $G$ has separating order dual. Then the band $B_I$ generated by the identity operator in $L_r(G)$ is equal to $\text{Orth}(G)$.

**Theorem 2.11** If $C$ is an ordered algebra such that $C_r$ is a Riesz space with the principal projection property and $\text{Orth}(C)$ is topologically full with respect to $I_e$, then $\text{Orth}(C)$ is an $f$-algebra. Moreover, it is a full subalgebra of $C$.

**Proof** Let $b, c, d \in \text{Orth}(C)^+$ and $b \wedge c = 0$. For each $0 \leq h \in \text{Orth}(C)^\sim$ and $a \in I_e$

$$0 \leq S_{db \wedge c,h}(a) = (S_{db,h} \wedge S_{c,h})(a)$$

$$\leq S_{db,h}(a) \wedge S_{c,h}(a)$$

$$= h(a(db)) \wedge S_{c,h}(a)$$

$$= h(d(ab)) \wedge S_{c,h}(a)$$

$$= h(L_d(ab)) \wedge S_{c,h}(a)$$

$$= L_d^\sim(h)(ab) \wedge S_{c,h}(a)$$

$$= S_{b,L_d^\sim(h)}(a) \wedge S_{c,h}(a)$$

$$= 0$$

holds, which proves that $db \wedge c = 0$. Similarly, taking $V$ instead of $S$ and $R_d$ instead of $L_d$, we have $bd \wedge c = 0$.

Let $b \in \text{Orth}(C)$ be invertible in $C$. We will show that $b^{-1} \in \text{Orth}(C)$. As $b \in \text{Orth}(C)$ \( bq = qb \) for each $q \in OI(C)$. It is easy to see that $b^{-1}q = qb^{-1}$ for each $q \in OI(C)$. Thus, $\text{Orth}(C)$ is a full subalgebra of $C$.

**Corollary 2.12** Let $G$ be a Dedekind complete Riesz space and $G$ has separating order dual. Then, $\text{Orth}(G)$ is an $f$-algebra. Moreover, it is a full subalgebra of $L_r(G)$.

As each unital $f$-algebra $C$ with separating order dual is topologically full with respect to $I_e$ [8], we can give the following corollary.

**Corollary 2.13** Let $C$ be an ordered algebra such that $C_r$ is a Riesz space with the principal projection property and $\text{Orth}(C)$ has separating order dual. Then, $\text{Orth}(C)$ is an $f$-algebra if and only if $\text{Orth}(C)$ is topologically full with respect to $I_e$.

As we said before, if $G$ is a Dedekind complete Riesz space with separating order dual and $C = L(G)$ then $\text{Orth}(C) = \text{Orth}(G)$ is topologically full with respect to $I_e = Z(G)$. However, even if $C$ is a Dedekind complete ordered algebra, $\text{Orth}(C)$ may not be topologically full with respect to $I_e$. We now give an example of a Dedekind complete ordered algebra which is not topologically full with respect to $I_e$.

**Example 2.14** Let $f$ be a multiplicative functional on $l_\infty$ satisfying $f(e_0) = 0$ and $C$ be the linear space $l_\infty \oplus \mathbb{R}$. $C$ is a Dedekind complete ordered Banach algebra with unit $(e, 0)$ under the multiplication

$$(u_1, \lambda_1) \ast (u_2, \lambda_2) = (u_1 u_2, \lambda_1 f(u_2) + \lambda_2 f(u_1) + \lambda_1 \lambda_2),$$
the norm
\[ \|(u, \lambda)\| = \|u\| + |\lambda| \]
and the order induced by the cone
\[ C^+ = \{(u, \lambda) : u \in l_1^+ \text{ and } \lambda \in \mathbb{R}\}. \]

Furthermore,
\[ OI(C) = \{(p, 0) : p \in OI(l_\infty)\} \text{ and } \text{Orthe}(C) = \{(u, \lambda) : u \in \text{Orthe}(l_\infty) \text{ and } \lambda \in \mathbb{R}\} \text{ [1].} \]

Since \( C \) is Dedekind complete, \( C_r \) is a Riesz space with the principal projection property. As \( \text{Orthe}(C) \) is order closed, \( \text{Orthe}(C) \) is norm closed [9, Theorem 100.7]. This implies \( \text{Orthe}(C) \) Banach lattices, hence \( \text{Orthe}(C)^\sim = \text{Orthe}(C)' \) and so \( \text{Orthe}(C) \) has separating order dual. It is easy that, \((0, 1), (e, 0) \in \text{Orthe}(C)\) and \((0, 1), (e, 0) \). On the other hand, we have
\[ (0, 1) \ast (e, 0) = (0e, 1f(e) + 0f(0) + 01) = (0, 1) \neq 0 \]
so that \( \text{Orthe}(C) \) is not an \( f \)-algebra. By the Corollary 2.13, \( \text{Orthe}(C) \) is not topologically full with respect to \( I_e \).

Since each \( f \)-algebra is commutative, we can give the following corollary.

**Corollary 2.15** Let \( C \) be an ordered algebra such that \( C_r \) is a Riesz space with the principal projection property and \( \text{Orthe}(C) \) is topologically full with respect to \( I_e \). Then, \( \text{Orthe}(C) \) is a commutative algebra.

**References**