SOFT SEPARATION AXIOMS AND SOFT PRODUCT OF SOFT TOPOLOGICAL SPACES

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Abstract. In this article, we deal with the soft separation axioms using soft points on soft topological space and discuss the characterizations and properties of them. We extend these separation axioms to the soft product of soft topological spaces. Also we provide correct examples for the wrong examples example:1, example:2 and example:3 given in article [8].

For the vagueness and uncertainty of real life problems, there are several mathematical tools such as fuzzy sets, intuitionistic fuzzy sets, rough sets, vague sets etc. There is one more mathematical tool named soft sets which was introduced by Molodsov[12] in 1999. After that it was developed and used in decision making problems by Maji et. al in [10] and [11]. Aktas and Cagman [1] introduced the applications of soft set theory in algebraic structures in 2007. Kharral and Ahmad [9] introduced and discussed several properties of soft mappings. Shabir and Naz [16] investigated soft seperation axioms defined for crisp points in 2011. Hussain and Ahmad [7] investigate the properties of soft interior, soft closure and soft boundary in 2011. Aygunoglu and Aygun [2] in 2012 generalize Alexander subbase theorem and Tychonoff theorem to the soft topological spaces by defining and using the product of soft topological spaces. Nazmul and Samanta [13] studied the neighbourhood properties of soft topological spaces in 2013. There are several articles related to the properties of soft topological spaces and soft mappings on soft topological spaces. Some of them are [4], [6], [14], [17], [19], [20], [21]. Four different types of seperation axioms were defined and discussed in [5], [8], [16] and [18]. Singh and Noorie [17] derives the relation among these four types of $T_i$, $i = 1, 2, 3, 4$ spaces in 2017.

In the second section of this article, we give some basic definitions and preliminaries of soft topological spaces.

In the third section of this article, we deal with the soft separation axioms using soft points and discuss about the characterizations and properties of them. In fact

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these separation axioms are stronger than other separation axioms. We extend these separation axioms to the product of soft topological spaces. Also we provide correct examples for the wrong examples Example:1, Example:2 and Example:3 given in article [8].

Throughout this paper, $X$ is the universe set, $E$ is a set of parameters and $\mathcal{P}(X)$ is the set of all subsets of $X$.

1. Preliminaries

**Definition 1.1.** [12] A mapping $F : E \to \mathcal{P}(X)$ is called a soft set and is denoted by $(F, E)$. The family of all soft sets over $X$ is denoted as $SS(X, E)$.

**Definition 1.2.** [12] Let $(F, E)$ and $(G, E)$ be two soft sets over $X$. Then $(F, E)$ is a soft subset of $(G, E)$ written as $(F, E) \subseteq (G, E)$, if $F(e) \subseteq G(e)$, for all $e \in E$. Also the soft sets $(F, E)$ and $(G, E)$ are equal written as $(F, E) = (G, E)$, if $(F, E) \subseteq (G, E)$ and $(G, E) \subseteq (F, E)$.

**Definition 1.3.** [12] Let $\{(F_i, E) : i \in I\} \subseteq SS(X, E)$, where $I$ is an arbitrary index set. Then

1. the soft union of $\{(F_i, E) : i \in I\}$ is the soft set $(F, E)$, where $F$ is the mapping defined as $F(e) = \bigcup\{F_i(e) : i \in I\}$, for every $e \in E$ and is denoted as $(F, E) = \bigcup\{(F_i, E) : i \in I\}$.
2. the soft intersection of $\{(F_i, E) : i \in I\}$ is the soft set $(F, E)$, where $F$ is the mapping defined as $F(e) = \bigcap\{F_i(e) : i \in I\}$, for every $e \in E$ and is denoted as $(F, E) = \bigcap\{(F_i, E) : i \in I\}$.

**Definition 1.4.** [21] Let $(F, E)$ be a soft set over $X$. Then the soft relative complement $F^c$ of $(F, E)$ is the mapping from $E$ to $\mathcal{P}(X)$ defined by $F^c(e) = X - F(e)$ for every $e \in E$ and is denoted as $(F, E)^c$ or $(F^c, E)$.

**Definition 1.5.** [12] Let $(F, E)$ be a soft set over $X$. Then

1. $(F, E)$ is called as null soft set, if $F(e) = \phi$, for every $e \in E$. We simply write it as $\hat{\phi}$.
2. $(F, E)$ is called as absolute soft set, if $F(e) = X$, for every $e \in E$. We simply write it as $\hat{X}$.

**Definition 1.6.** [16, 21] Let $\tau \subseteq SS(X, E)$. Then $\tau$ is a soft topology on $X$ if it satisfies the following three conditions:

1. $\hat{\phi}, \hat{X} \in \tau$.
2. The soft union of any number of soft sets in $\tau$ is in $\tau$.
3. The soft intersection of finite number of soft sets in $\tau$ is in $\tau$.

This soft topological space over $X$ is written as $(X, \tau, E)$ and the members of $\tau$ are called soft open sets in $X$. Also the soft complement of soft open sets are called soft closed sets.

**Definition 1.7.** [21] The soft set $(F, E)$ over $X$ is called as a soft point in $X$, denoted by $x_e$, if \( F(e') = \begin{cases} \{x\} & \text{if } e' = e \\ \phi & \text{if } e' \in E - \{e\} \end{cases} \)
Definition 1.8. [2] Let \((X, \tau, E)\) be a soft topological space. A subcollection \(\mathcal{B}\) of \(\tau\) is said to be a base for \(\tau\) if every member of \(\tau\) can be expressed as a union of members of \(\tau\).

Definition 1.9. [2] Let \((X, \tau, E)\) be a soft topological space. A subcollection \(\mathcal{S}\) of \(\tau\) is said to be a subbase for \(\tau\) if the family of all finite intertions of members of \(\mathcal{S}\) forms a base for \(\tau\).

Definition 1.10. [21] A soft set \((G, E)\) in a soft topological space \((X, \tau, E)\) is known as a soft neighbourhood of a soft set \((F, E)\) if there exists a soft open set \((H, E)\) such that \((F, E) \subseteq (H, E) \subseteq (G, E)\).

Definition 1.11. [16] Let \((F, E)\) be a soft set in a soft topological space \((X, \tau, E)\). Then the soft closure of \((F, E)\) is denoted as \(\text{Cl}(F, E)\) and defined as \(\text{Cl}(F, E) = \bigcap \{ (G, E) : (G, E) \subseteq \tau \text{ and } (G, E) \supseteq (F, E) \}\).

Definition 1.12. [16] Let \(Y\) be a nonempty soft subset of a soft topological space \((X, \tau, E)\) and \((F, A)\) be a soft set over \(Y\). Then \((F, A)\) is a soft open set in \(Y\) if and only if \((F, E) = (G, E) \cap \mathcal{E}_Y\), for some \((G, E) \subseteq \tau\).

Proposition 1.1. [16] Let \((Y, \tau_Y, E)\) be a soft subspace of a soft topological space \((X, \tau, E)\) and \((F, A)\) be a soft set over \(Y\). Then \((F, A)\) is a soft subspace of \((X, \tau, E)\), where \(E_Y : E \rightarrow \mathcal{P}(Y)\) such that \(E_Y(e) = Y\), for every \(e \in E\).

Theorem 1.2. [21] A soft set \((F, E)\) is soft open set if and only if \((G, E)\) is a soft neighbourhood of a soft set \((F, E)\), for each soft set \((F, E)\) contained in \((G, E)\).

Proposition 1.3. [16] Let \((X, \tau, E)\) be a soft topological space over \(X\). Then the collection \(\tau_e = \{ F(e) : (F, E) \subseteq \tau \}\) defines a topology on \(X\).

Proposition 1.4. [16] Let \((X, \tau, E)\) be a soft topological space over \(X\) and \(Y \subseteq X\. Then \((Y, \tau_Y)\) is a subspace of \((X, \tau_e)\).

Definition 1.13. [3] Let \((F, E_1) \subseteq \text{SS}(X_1, E_1)\) and \((G, E_2) \subseteq \text{SS}(X_2, E_2)\). Then the cartesian product \((F \times G)_{E_1 \times E_2}\) is defined by \((F \times G)_{E_1 \times E_2}(e_{11}, e_{22}) = F(e_{11}) \times G(e_{22}), \forall (e_{11}, e_{22}) \in E_1 \times E_2\).

Definition 1.14. [2] The soft mappings \((p_i)_{i \in \Delta}\) is called soft projection mappings from \(X_1 \times X_2\) to \(X_i\) defined by \((p_i)_i((F, E_1) \times (F, E_2)) = (p_i)_i(F_1 \times F_2)_{E_1 \times E_2}(e_{11}, e_{22}) = p_i(F_1 \times F_2)_{E_1 \times E_2} = (F, E)_i\), where \((F, E)_1 \in \text{SS}(X_1, E_1), (F, E)_2 \in \text{SS}(X_2, E_2)\) and \(p_i : X_1 \times X_2 \rightarrow X_i\), \(q_i : E_1 \times E_2 \rightarrow E_i\) are projection mappings in classical meaning.

Definition 1.15. [2] Let \{\((\phi_i)_i : S(X, E) \rightarrow (Y_i, \tau_i)\)\}_{i \in \Delta}\) be a family of soft mappings where \{\((Y_i, \tau_i)\)\}_{i \in \Delta}\) is a family of soft topological spaces. Then the topology \(\tau\) generated from the subbase \{\((\phi_i)_i^{-1}((F, E)) : (F, E) \in \tau_i, i \in \Delta\)\) is called the initial soft topology induced by the family of soft mappings \{\((\phi_i)_i\)\}_{i \in \Delta}.

Definition 1.16. [2] Let \{\((X_i, \tau_i)\)\}_{i \in \Delta}\) be a family of soft topological spaces. Then the initial soft product topology on \(X(= \coprod_{i \in \Delta} X_i)\) generated by the family \{\((p_i)_i\)\}_{i \in \Delta}\) is called soft product topology on \(X\), where \((p_i)_i\) are the soft projection mapping from \(X\) to \(X_i\).
Theorem 1.5. Let $X$ and $Y$ be crisp sets, $F_A$, $(F_A)_i \in SS(X, E)$ and $G_B$, $(G_B)_i \in SS(Y, K)$, where $i \in \Delta$, an index set. Then

1. If $(F_A)_1 \subseteq (F_A)_2$, then $\Phi_\psi((F_A)_1) \subseteq \Phi_\psi((F_A)_2)$.
2. If $(G_B)_1 \subseteq (G_B)_2$, then $\Phi_\psi^{-1}((G_B)_1) \subseteq \Phi_\psi^{-1}((G_B)_2)$.
3. $(F_A) \subseteq \Phi_\psi^{-1}(\Phi_\psi(F_A))$, the equality holds if $\Phi_\psi$ is injective.
4. $\Phi_\psi(\Phi_\psi^{-1}(F_A)) \subseteq (F_A)$, the equality holds if $\Phi_\psi$ is surjective.
5. $\Phi_\psi(\bigcup_{i \in \Delta} (F_A)_i) = \bigcup_{i \in \Delta} \Phi_\psi((F_A)_i)$.
6. $\Phi_\psi(\bigcap_{i \in \Delta} (F_A)_i) \subseteq \bigcap_{i \in \Delta} \Phi_\psi((F_A)_i)$.
7. $\Phi_\psi^{-1}(\bigcap_{i \in \Delta} (G_B)_i) = \bigcap_{i \in \Delta} \Phi_\psi^{-1}((G_B)_i)$.
8. $\Phi_\psi^{-1}(\bigcap_{i \in \Delta} (G_B)_i) = \bigcap_{i \in \Delta} \Phi_\psi^{-1}((G_B)_i)$.
9. $\Phi_\psi^{-1}(E_Y) = E_X$ and $\Phi_\psi^{-1}(\phi_Y) = \phi_X$.
10. $\Phi_\psi(E_X) = E_Y$ if $\Phi_\psi$ is surjective.
11. $\Phi_\psi(\phi_Y) = \phi_Y$.

2. Soft separation axioms and product soft topological spaces

Definition 2.1. A soft topological space $(X, \tau, E)$ is said to be a soft $T_0$-space if for every pair of soft points $x_{e_1}, y_{e_2}$ such that $x_{e_1} \neq y_{e_2}$, there exists $(F, E) \in \tau$ such that $x_{e_1} \notin (F, E), y_{e_2} \notin (F, E)$ or there exists $(G, E) \in \tau$ such that $y_{e_2} \notin (G, E), x_{e_1} \notin (G, E)$.

Definition 2.2. A soft topological space $(X, \tau, E)$ is said to be a soft $T_1$-space if for every pair of soft points $x_{e_1}, y_{e_2}$, such that $x_{e_1} \neq y_{e_2}$ there exist $(F, E), (G, E) \in \tau$ such that $x_{e_1} \notin (F, E), y_{e_2} \notin (G, E)$ and $x_{e_1} \notin (G, E), y_{e_2} \notin (F, E)$.

Example 2.1. Example for $T_0$-space.

Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and $\tau = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ where

$F_1(e) = \begin{cases} \{x\} & \text{if } e = e_1, \\ \{y\} & \text{if } e = e_2, \end{cases}$

$F_2(e) = \begin{cases} \{x\} & \text{if } e = e_1, \\ \{y\} & \text{if } e = e_2, \end{cases}$

$F_3(e) = \begin{cases} \{x\} & \text{if } e = e_1, \\ X & \text{if } e = e_2, \end{cases}$

$F_4(e) = \begin{cases} \{x\} & \text{if } e = e_1, \\ \emptyset & \text{if } e = e_2, \end{cases}$

For the soft points $x_{e_1}, y_{e_1}$, there is a soft open set $(F_1, E) \in \tau$ with $x_{e_1} \notin (F_1, E)$ and $y_{e_1} \notin (F_1, E)$. For the soft points $x_{e_2}, y_{e_2}$, there is a $(F_1, E) \in \tau$ with $x_{e_2} \notin (F_1, E)$ and $y_{e_2} \notin (F_1, E)$. For the soft points $x_{e_1}, y_{e_2}$, there is a $(F_2, E) \in \tau$ with $x_{e_2} \notin (F_2, E)$ and $y_{e_2} \notin (F_2, E)$. For the soft points $x_{e_2}, y_{e_1}$, there is a $(F_2, E) \in \tau$ with $x_{e_2} \notin (F_2, E)$ and $y_{e_1} \notin (F_2, E)$. For the soft points $x_{e_1}, x_{e_2}$, there is a $(F_1, E) \in \tau$ with $x_{e_1} \notin (F_1, E)$ and $x_{e_2} \notin (F_1, E)$. For the soft points $y_{e_1}, y_{e_2}$, there is a $(F_1, E) \in \tau$ with $y_{e_1} \notin (F_1, E)$ and $y_{e_2} \notin (F_1, E)$. Thus $(X, \tau, E)$ is a soft $T_0$-space.

Example 2.2. Let $X = \mathbb{Z}$, the set of all integers and $E = \mathbb{N}$, the set of all natural numbers. Define a soft topology on $X$ as $\tau = \{(F, E) : F(e_i) \text{ is finite for each } e_i \in E\} \cup \{\emptyset\}$.

1. Clearly $\emptyset \in \tau$ and $\hat{X} \in \tau$. 
(2) If \((F_{\alpha}, E) \in \tau\) for some \(\alpha \in \Delta\), where \(\Delta\) is some index set, then \(F_{\alpha}^c(e_i)\) is finite for each \(e_i \in E\). Now \(\cap F_{\alpha}^c(e_i) = (\cup F_{\alpha}(e_i))^c\) is finite for each \(e_i \in E\). So that \(\cup (F_{\alpha}, E) \in \tau\).

(3) If \((F_1, E), (F_2, E) \in \tau\), then \(F_1^c(e_i) \cap F_2^c(e_i)\) are finite for each \(e_i \in E\). Now \(F_1^c(e_i) \cup F_2^c(e_i) = (F_1(e_i) \cap F_2(e_i))^c = (F_1(\cap F_2)(e_i))^c = (F_1 \cap F_2)^c(e_i)\) is finite for each \(e_i \in E\). So that \((F_1, E) \cap (F_2, E) \in \tau\).

Thus \((X, \tau, E)\) is a soft topological space. For any two distinct soft points \(x_{e_i}\) and \(y_{e_j}\), \(x_{e_i}^c\) and \(y_{e_j}^c\) are soft open sets such that \(x_{e_i} \in y_{e_j}^c\), \(y_{e_j} \notin y_{e_j}^c\) and \(x_{e_i} \notin x_{e_i}^c\), \(y_{e_j} \in x_{e_i}^c\). Thus \((X, \tau, E)\) is a soft \(T_1\) space.

**Theorem 2.1.** Every soft \(T_1\)-space is a soft \(T_0\)-space.

**Proof.** Proof is straight forward \(\square\)

**Theorem 2.2.** Let \((X, \tau, E)\) be a soft topological space. Then \((X, \tau, E)\) is a soft \(T_0\) space if and only if for any two distinct soft points \(x_{e_i}\) and \(y_{e_j}\), there is soft closed set \((H, E)\) such that \(x_{e_i} \notin (H, E)\), \(y_{e_j} \notin (H, E)\) or there is soft closed set \((K, E)\) such that \(x_{e_i} \notin (K, E)\), \(y_{e_j} \notin (K, E)\).

**Proof.** Let us consider two distinct soft points \(x_{e_i}\) and \(y_{e_j}\). Since \((X, \tau, E)\) is a soft \(T_0\) space, there is soft open set \((F, E)\) such that \(x_{e_i} \notin (F, E)\), \(y_{e_j} \notin (F, E)\) or there is soft open set \((G, E)\) such that \(x_{e_i} \notin (G, E)\), \(y_{e_j} \notin (G, E)\). Let \((H, E) = (G^c, E)\) and \((K, E) = (F^c, E)\). Then \((H, E)\) is a soft closed set such that \(x_{e_i} \notin (H, E)\), \(y_{e_j} \notin (H, E)\) or \((K, E)\) is a soft closed set such that \(x_{e_i} \notin (K, E)\), \(y_{e_j} \notin (K, E)\).

Conversely, for any two distinct soft points \(x_{e_i}\) and \(y_{e_j}\), there is a soft closed set \((H, E)\) such that \(x_{e_i} \notin (H, E)\), \(y_{e_j} \notin (H, E)\) or there is soft closed set \((K, E)\) such that \(x_{e_i} \notin (K, E)\), \(y_{e_j} \notin (K, E)\). Then \((H^c, E)\) is a soft open set such that \(x_{e_i} \notin (H^c, E)\), \(y_{e_j} \notin (H^c, E)\) or \((K^c, E)\) is a soft open set such that \(x_{e_i} \notin (K^c, E)\), \(y_{e_j} \notin (K^c, E)\). This proves that \((X, \tau, E)\) is a soft \(T_0\) space. \(\square\)

Example:1 given in the article \([8]\) for soft \(T_1\) space which is not a soft \(T_0\) space is wrong. Because it is not a soft \(T_0\) space too.

**Example 2.3.** \([8]\) \(X = \{x_1, x_2\}\), \(A = \{e_1, e_2\}\) and \(\tau = \{\emptyset, X, (F, A)\}\) where \(F(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases}\) This \((X, \tau, A)\) is verified as soft \(T_0\) space in \([8]\).

Consider two soft points \(e_F = \begin{cases} \{x_2\} & \text{if } e = e_1 \\ \emptyset & \text{if } e = e_2 \end{cases}\) and \(e_G = \begin{cases} \emptyset & \text{if } e = e_1 \\ \{x_1\} & \text{if } e = e_2 \end{cases}\), then there is no soft open set \((F, A)\) in \((X, \tau, A)\) such that \(e_F \notin (F, A)\) and \(e_G \notin (F, A)\). Thus \((X, \tau, A)\) is not a soft \(T_0\) space.

The following example will be a correct example for example:1 of \([8]\). It also shows that the converse of above theorem \([2.1]\) is not true in general.

**Example 2.4. Example for a soft \(T_0\)-space which is not a soft \(T_1\)-space.**

Let \(X = \{x, y\}\), \(E = \{e_1, e_2\}\) and \(\tau = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}\) where

\[
\begin{align*}
F_1(e) &= \begin{cases} \{x\} & \text{if } e = e_1 \\ \{y\} & \text{if } e = e_2 \end{cases}, \quad F_2(e) &= \begin{cases} \{x\} & \text{if } e = e_1 \\ \emptyset & \text{if } e = e_2 \end{cases}, \\
F_3(e) &= \begin{cases} \emptyset & \text{if } e = e_1 \\ \{x\} & \text{if } e = e_2 \end{cases}
\end{align*}
\]
For the soft points \( x_{e_1}, y_{e_1} \), there is a \((F_2, E) \in \tau\) with \( x_{e_1} \notin (F_2, E) \) and \( y_{e_1} \notin (F_2, E) \). For the soft points \( x_{e_2}, y_{e_2} \), there is a \((F_3, E) \in \tau\) with \( x_{e_2} \notin (F_3, E) \) and \( y_{e_2} \notin (F_3, E) \). For the soft points \( x_{e_1}, y_{e_2} \), there is a \((F_2, E) \in \tau\) with \( x_{e_1} \notin (F_2, E) \) and \( x_{e_2} \notin (F_2, E) \). For the soft points \( x_{e_2}, y_{e_1} \), there is a \((F_3, E) \in \tau\) with \( x_{e_2} \notin (F_3, E) \) and \( y_{e_1} \notin (F_3, E) \). For the soft points \( x_{e_1}, x_{e_2} \), there is a \((F_2, E) \in \tau\) with \( x_{e_1} \notin (F_2, E) \) and \( x_{e_2} \notin (F_2, E) \). For the soft points \( y_{e_1}, y_{e_2} \), there is a \((F_1, E) \in \tau\) with \( y_{e_2} \notin (F_1, E) \) and \( y_{e_1} \notin (F_1, E) \). Thus \((X, \tau, E)\) is a soft \( T_0\)-space. But for the pair of soft points \( y_{e_1}, y_{e_2} \), we don’t have \((K, E) \in \tau\) such that \( y_{e_1} \notin (K, E) \) and \( y_{e_2} \notin (K, E) \). Thus \((X, \tau, E)\) is not a soft \( T_1\)-space.

**Theorem 2.3.**

1. A subspace of a soft \( T_0\)-space is a soft \( T_0\)-space.
2. A subspace of a soft \( T_1\)-space is a soft \( T_1\)-space.

**Proof.**

(1) Let \((X, \tau, E)\) be a soft \( T_0\)-space and \((Y, \tau_Y, E)\) be a soft subspace. Let \( x_{e_1}, y_{e_1} \) be two soft points in \( SS(Y, E) \). Then \( x_{e_1}, y_{e_1} \notin SS(X, E) \).

Since \((X, \tau, E)\) is a soft \( T_0\)-space, there is a soft open set \((F, E) \in (X, \tau, E)\) such that \( x_{e_1} \notin (F, E) \), \( y_{e_1} \notin (F, E) \) or there is a soft open set \((G, E) \in (X, \tau, E)\) such that \( y_{e_1} \notin (G, E) \), \( x_{e_1} \notin (G, E) \). Then \((F, E) \cap E_Y\) is a soft open set in \((Y, \tau_Y, E)\) such that \( x_{e_1} \notin (F, E) \cap E_Y \), \( y_{e_1} \notin (F, E) \cap E_Y \) or \((G, E) \cap E_Y\) is a soft open set in \((Y, \tau_Y, E)\) such that \( y_{e_1} \notin (G, E) \cap E_Y \).

(2) Proof is similar to (1)

**Theorem 2.4.** Let \((X, \tau, E)\) be a soft topological space. Then \((X, \tau, E)\) is a soft \( T_1\)-space if and only if for any soft points \( x_{e_1} \) and \( y_{e_1} \), there exist two soft closed sets \((H, E)\) and \((K, E)\) such that \( x_{e_1} \notin (H, E) \), \( y_{e_1} \notin (H, E) \), \( y_{e_1} \notin (K, E) \) and \( x_{e_1} \notin (K, E) \).

**Proof.**

The following example shows that the product of soft \( T_0\)-spaces need not be a soft \( T_0\)-space.

**Example 2.5.** Let \( X_1 = \{x_1, y_1\}, E_1 = \{e_{11}, e_{12}\}\) and \( \tau_1 = \{\phi, X_1, (F_1, E_1), (F_2, E_1), (F_3, E_1), (F_4, E_1), (F_5, E_1), (F_6, E_1), (F_7, E_1)\} \), \( X_2 = \{x_2, y_2\}, E_2 = \{e_{21}, e_{22}\}\) and \( \tau_2 = \{\phi, X_2, (G_1, E_2), (G_2, E_2), (G_3, E_2), (G_4, E_2), (G_5, E_2), (G_6, E_2), (G_7, E_2)\} \).

\[
F_1(e) = \begin{cases} \{x_1\} & \text{if } e = e_{11} \\ \phi & \text{if } e = e_{12} \end{cases},
G_1(e) = \begin{cases} \{x_2\} & \text{if } e = e_{21} \\ \phi & \text{if } e = e_{22} \end{cases},
F_2(e) = \begin{cases} \phi & \text{if } e = e_{11} \\ \{x_1\} & \text{if } e = e_{12} \end{cases},
F_3(e) = \begin{cases} \{x_2\} & \text{if } e = e_{21} \\ \{x_1\} & \text{if } e = e_{22} \end{cases},
G_3(e) = \begin{cases} \{x_2\} & \text{if } e = e_{21} \\ \{x_1\} & \text{if } e = e_{22} \end{cases}.
\]
SOFT SEPARATION AXIOMS AND SOFT PRODUCT OF SOFT TOPOLOGICAL SPACES

\[ F_4(e) = \begin{cases} 
\{ y_1 \} & \text{if } e = e_{11}, \\
\{ x_1 \} & \text{if } e = e_{12}
\end{cases}, \quad G_4(e) = \begin{cases} 
\{ x_2 \} & \text{if } e = e_{21}, \\
\{ y_2 \} & \text{if } e = e_{22}
\end{cases}, \quad F_5(e) = \begin{cases} 
\{ y_1 \} & \text{if } e = e_{11}, \\
\phi & \text{if } e = e_{12}
\end{cases}, \quad G_5(e) = \begin{cases} 
\phi & \text{if } e = e_{21}, \\
\{ y_2 \} & \text{if } e = e_{22}
\end{cases}, \quad F_7(e) = \begin{cases} 
\{ x_1 \} & \text{if } e = e_{11}, \\
\{ x_2 \} & \text{if } e = e_{21}, \\
\{ x_1 \} & \text{if } e = e_{12}, \\
\{ x_2 \} & \text{if } e = e_{22}
\end{cases}. \]

For the soft points \( x_{11}, y_{11}, \) there is a soft open set \((F_1, E_1) \in \tau_1 \) with \( x_{11} \in (F_1, E_1) \) and \( y_{11} \notin (F_1, E_1) \). For the soft points \( x_{12}, y_{12}, \) there is a soft open set \((F_2, E_1) \in \tau_1 \) with \( x_{12} \in (F_2, E_1) \) and \( y_{12} \notin (F_2, E_1) \). For the soft points \( x_{11}, y_{12}, \) there is \((F_1, E_1) \in \tau_1 \) with \( x_{11} \in (F_1, E_1) \) and \( y_{12} \notin (F_1, E_1) \). Thus \((X_1, \tau_1, E_1) \) is a soft T_0-space.

For the soft points \( x_{21}, y_{21}, \) there is a soft open set \((G_1, E_2) \in \tau_2 \) with \( x_{21} \in (G_1, E_2) \) and \( y_{21} \notin (G_1, E_2) \). For the soft points \( x_{22}, y_{22}, \) there is a soft open set \((G_2, E_2) \in \tau_2 \) with \( x_{22} \in (G_2, E_2) \) and \( y_{22} \notin (G_2, E_2) \). For the soft points \( x_{21}, y_{21}, \) there is a soft open set \((G_2, E_2) \in \tau_2 \) with \( x_{21} \in (G_2, E_2) \) and \( y_{21} \notin (G_2, E_2) \). Thus \((X_2, \tau_2, E_2) \) is a soft T_0-space.

Now \( E_1 \times E_2 = \{(e_{11}, \tilde{e}_{21}), (e_{12}, \tilde{e}_{22}), (e_{21}, \tilde{e}_{12})\} \) and \( \tau_1 \times \tau_2 = \{\tilde{\phi}, \tilde{F}_1 \times \tilde{X}_2, (\tilde{F}_1 \times \tilde{G}_1, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_2, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_3, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_4, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_5, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_6, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_7, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_8, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_9, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_10, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_11, E_1 \times E_2), (\tilde{F}_1 \times \tilde{G}_12, E_1 \times E_2)\}. \)
Suppose if the soft product of \((X_1, \tau_1, E_1)\) and \((X_2, \tau_2, E_2)\) is a soft \(T_0\) space, then for any two distinct soft points \((x_1, y_2)_{(e_{11}, e_{21})} = \begin{cases} (x_1, y_2) & \text{if } e = (e_{11}, e_{21}) \\ \phi & \text{if } e = (e_{11}, e_{22}) \\ \phi & \text{if } e = (e_{12}, e_{21}) \\ \phi & \text{if } e = (e_{12}, e_{22}) \end{cases}\)

and \((y_1, y_2)_{(e_{11}, e_{21})} = \begin{cases} (y_1, y_2) & \text{if } e = (e_{11}, e_{21}) \\ \phi & \text{if } e = (e_{11}, e_{22}) \\ \phi & \text{if } e = (e_{12}, e_{21}) \\ \phi & \text{if } e = (e_{12}, e_{22}) \end{cases}\)

\((F_m \times G_n, E_1 \times E_2)\) in \(\tau_1 \times \tau_2\) such that \((x_1, y_2)_{(e_{11}, e_{21})} \notin (F_m \times G_n, E_1 \times E_2)\) and \((y_1, y_2)_{(e_{11}, e_{21})} \notin (F_m \times G_n, E_1 \times E_2)\), for some \(m, n \in \{1, 2, 3, \ldots, 7\}\). Now \((p_q)_2((x_1, y_2)_{(e_{11}, e_{21})}) \notin (p_q)_2((F_m \times G_n, E_1 \times E_2))\). That is \(p_2(x_1, y_2)_{(e_{q2(e_{11}, e_{21})})} \notin p_2(F_m \times G_n, E_1 \times E_2)\). This implies \(y_{22} \in (G_n, E_2)\), for some \(m, n \in \{1, 2, 3, \ldots, 7\}\). Since \((p_q)_2\) is a soft projection mapping and \((F_m \times G_n, E_1 \times E_2)\) is a soft open set in \(X_1 \times X_2, \tau_1 \times \tau_2\) \((G_n, E_2)\) is a soft open set in \((X_2, \tau_2, E_2)\) containing \(y_{22}\). But there is no soft open set \((G_n, E_2)\) in \((X_2, \tau_2, E_2)\) containing \(y_{22}\), for any \(n \in \{1, 2, 3, \ldots, 7\}\) and hence \((X_1 \times X_2, \tau_1 \times \tau_2, E_1 \times E_2)\) is not a soft \(T_0\) space.

**Definition 2.4.** Let \((X, \tau, E)\) be a soft topological space and \(A = \{x_{e_i} : x_{e_i}\) is a soft points of \((X, \tau, E)\}\).

1. If the number of elements of the set \(A\) is finite, then \((X, \tau, E)\) is called a finite soft topological space.
2. If the number of elements of the set \(A\) is countable, then \((X, \tau, E)\) is called a countable soft topological space.

**Theorem 2.5.** If \((X, \tau, E)\) is a finite soft \(T_1\) space, then \((X, \tau, E)\) is a soft discrete space.

**Proof.** Let \(x_{e_i}\) be a soft point, \(x \in X\) and \(e_i \in E\). \((X, \tau, E)\) is a soft \(T_1\) space, for any soft point \(y_{e_j} \neq x_{e_i}\), there is a soft open set \((F_{x_{e_i}}, E)\) such that \(x_{e_i} \in (F_{x_{e_i}}, E)\) and \(y_{e_j} \notin (F_{x_{e_i}}, E)\). Since \((X, \tau, E)\) is a finite soft topological space, \(\bigcap_{y_{e_j} \neq x_{e_i}} (F_{x_{e_i}}, E)\) is a soft open set such that \(\bigcap_{y_{e_j} \neq x_{e_i}} (F_{x_{e_i}}, E) = \begin{cases} \{x\} & \text{if } e = e_i \\ \phi & \text{if } e \neq e_i \end{cases}\).

Thus \(x_{e_i}\) is soft open and hence \((X, \tau, E)\) is a soft discrete space.

**Definition 2.5.** Let \((X, \tau, E)\) be a soft topological space. Then the soft set \((F, E)\) is called a soft \(G_δ\) set if it is a countable intersection of soft open sets.

**Theorem 2.6.** If \((X, \tau, E)\) is a countable soft \(T_1\) space and if every soft \(G_δ\) set is soft open in \((X, \tau, E)\), then \((X, \tau, E)\) is a soft discrete space.

**Proof.** Let \(x_{e_i}\) be a soft point. Since \((X, \tau, E)\) is a soft \(T_1\) space, for any soft point \(y_{e_j} \neq x_{e_i}\), there is a soft open set \((F_{x_{e_i}}, E)\) such that \(x_{e_i} \in (F_{x_{e_i}}, E)\) and \(y_{e_j} \notin (F_{x_{e_i}}, E)\). Since every soft \(G_δ\) set is soft open and \((X, \tau, E)\) is a countable soft topological space, \(\bigcap_{y_{e_j} \neq x_{e_i}} (F_{x_{e_i}}, E)\) is a soft open set such that \(\bigcap_{y_{e_j} \neq x_{e_i}} (F_{x_{e_i}}, E) = \begin{cases} \{x\} & \text{if } e = e_i \\ \phi & \text{if } e \neq e_i \end{cases}\). Thus \(x_{e_i}\) is soft open and hence \((X, \tau, E)\) is a soft discrete space.
Theorem 2.7. Product of soft $T_1$-spaces is a soft $T_1$-space

Proof. Let $\{(X_i, \tau_i, E_i) : i \in I\}$ be a family of soft topological spaces and $(\prod X_i, \prod \tau_i, \prod E_i)$ be their product soft topological space. Suppose $x_e$ and $y_f$ be two distinct soft points, where $x_e = \langle x_i \rangle_{i \in I}$, $y_f = \langle y_i \rangle_{i \in I}$, $x_i, y_i \in X_i$ and $e = \langle e_i \rangle_{i \in I}$, $f = \langle f_i \rangle_{i \in I}, e_i, f_i \in E_i$. Then there exists at least one $\beta \in I$ such that $x_\beta \neq y_\beta$ or there exist $e_{i_k}, e_{i_m} \in E_i$ such that $e_{i_k} \neq e_{i_m}$.

Case 1: If $x_\beta \neq y_\beta$, $(p_\beta)(x_e) = (p_\beta)(y_f) = p_\beta(x_e) = x_{\beta, x_\beta}$ and $(p_\beta)(y_f) = (p_\beta)(y_f) = p_\beta(y_f) = y_{\beta, y_\beta}$. Since $X_\beta$ is a soft $T_1$ space, there exist soft open sets $(F_\beta, E_\beta)$ and $(G_\beta, E_\beta)$ such that $x_{\beta, x_\beta} \notin (F_\beta, E_\beta)$, $y_{\beta, y_\beta} \notin (F_\beta, E_\beta)$ and $y_{\beta, y_\beta} \notin (G_\beta, E_\beta)$. Then the soft subbasic members $(p_\beta)^{-1}(F_\beta, E_\beta)$ and $(p_\beta)^{-1}(G_\beta, E_\beta)$ are the soft open sets containing $x_e$ and $y_f$ respectively. Suppose if $y_f \notin (p_\beta)^{-1}(F_\beta, E_\beta)$, then $p_\beta(y_f) \notin (p_\beta)^{-1}(F_\beta, E_\beta)$ which is a contradiction. Similarly we can prove $x_e \notin (p_\beta)^{-1}(G_\beta, E_\beta), y_f \notin (p_\beta)^{-1}(G_\beta, E_\beta), x_e \notin (p_\beta)^{-1}(F_\beta, E_\beta)$ and $y_f \notin (p_\beta)^{-1}(F_\beta, E_\beta)$ are the soft open sets such that $x_e \notin (p_\beta)^{-1}(F_\beta, E_\beta)$, $y_f \notin (p_\beta)^{-1}(F_\beta, E_\beta)$.

Case 2: If $e_{i_k} \neq e_{i_m}$, there are soft open sets $(F_{i_k}, E_{i_k})$ and $(F_{i_m}, E_{i_m})$ in $(X_i, \tau_i, E_i)$ such that $x_{e_{i_k}} \notin (F_{i_k}, E_{i_k})$, $x_{e_{i_m}} \notin (F_{i_m}, E_{i_m})$, $y_{e_{i_k}} \notin (F_{i_k}, E_{i_k})$, $y_{e_{i_m}} \notin (F_{i_m}, E_{i_m})$. Then $(p_{i_k})^{-1}(F_{i_k}, E_{i_k})$ and $(p_{i_m})^{-1}(F_{i_m}, E_{i_m})$ are soft open sets such that $x_e \notin (p_{i_k})^{-1}(F_{i_k}, E_{i_k})$, $y_f \notin (p_{i_m})^{-1}(F_{i_m}, E_{i_m})$. We can prove $y_f \notin (p_{i_k})^{-1}(F_{i_k}, E_{i_k})$ and $x_e \notin (p_{i_m})^{-1}(F_{i_m}, E_{i_m})$ as we proved in case 1. This completes the proof.

Theorem 2.8. Let $(X, \tau, E)$ be a soft topological space. Then the following are equivalent.

1. $(X, \tau, E)$ is a soft $\tau_1$-space
2. $x_e = \hat{\cap}\{(G, E) : (G, E) \in \tau \land x_e \notin (G, E)\}$
3. $x_e = \hat{\cap}\{(F, E) : (F, E) \in \tau^c \land x_e \notin (F, E)\}$

Proof. (i) $\Rightarrow$ (ii). Clearly $x_e \notin \hat{\cap}\{(G, E) : (G, E) \in \tau \land x_e \notin (G, E)\}$. Suppose if $y_{e_j} \notin \hat{\cap}\{(G, E) : (G, E) \in \tau \land x_e \notin (G, E)\}$ such that $x_{e_i} \neq y_{e_j}$. Then $x \neq y$ or $e_i = e_j$. In either cases, by our assumption, there is a soft open set $(G, E)$ such that $x_{e_i} \notin (G, E)$ and $y_{e_j} \notin (G, E)$. So $y_{e_j} \notin \hat{\cap}\{(G, E) : (G, E) \in \tau \land x_{e_i} \notin (G, E)\}$. Thus $x_{e_i} \notin \hat{\cap}\{(G, E) : (G, E) \in \tau \land x_{e_i} \notin (G, E)\}$.

(ii) $\Rightarrow$ (iii). Clearly $x_{e_i} \notin \hat{\cap}\{(F, E) : (F, E) \in \tau^c \land x_{e_i} \notin (F, E)\}$. Let $y_{e_j} \notin \hat{\cap}\{(F, E) : (F, E) \in \tau^c \land x_{e_i} \notin (F, E)\}$ such that $x_{e_i} \neq y_{e_j}$. By (ii), there exists $(G, E) \in \tau$ such that $y_{e_j} \notin (G, E)$ and $x_{e_i} \notin (G, E)$. Now $(G, E)^c \in \tau^c$ and $y_{e_j} \notin (G, E)^c$ and $x_{e_i} \notin (G, E)^c$ and hence $y_{e_j} \notin \hat{\cap}\{(F, E) : (F, E) \in \tau^c \land x_{e_i} \notin (F, E)\}$. Thus $x_{e_i} = \hat{\cap}\{(F, E) : (F, E) \in \tau^c \land x_{e_i} \notin (F, E)\}$.

(iii) $\Rightarrow$ (i). Let $x_{e_i}$ and $y_{e_j}$ be two distinct soft points. Then by (iii), $x_{e_i} \neq y_{e_j} = \hat{\cap}\{(F, E) : (F, E) \in \tau^c \land y_{e_j} \notin (F, E)\}$. There is some soft closed set $(F_1, E)$ such that $y_{e_j} \notin (F_1, E)$ and $x_{e_i} \notin (F_1, E)$. Then $(F_1, E)^c$ is a soft open set such that $x_{e_i} \notin (F_1, E)^c$ and $y_{e_j} \notin (F_1, E)^c$. Similarly, from $y_{e_j} \neq x_{e_i} = \hat{\cap}\{(F, E) : (F, E) \in \tau^c \land y_{e_j} \notin (F, E)\}$, we can find another soft open set $(F_2, E)^c$ such that $x_{e_i} \notin (F_2, E)^c$ and $y_{e_j} \notin (F_2, E)^c$. This proves that $(X, \tau, E)$ is a soft $\tau_1$-space.
Remark.  (1) From (iii) of theorem 2.8, it is clear that each soft point $x_e$ is a soft closed set in a soft $T_1$ space.

(2) Let $T_i = \text{Number of elements in } F(e_i), i \in I$ an indexed set of $E$. If $T = \sum_{i \in I} T_i$ is finite, then the soft set $(F, E)$ can be written as a finite union of soft points. Each soft point is a soft closed set, we have $(F, E)$ is a soft closed set.

(3) If $T = \sum_{i \in I} T_i$ is infinite, $(F, E)$ need not be a closed set. Following example shows this.

Example 2.6. Let $X$ be an infinite set and $E = \mathbb{N}$. Let $\tau = \{(F, E)^c : \{e_i : F(e_i) \neq \phi\} \text{ is finite}\} \cup \{\phi\}$.

(1) Clearly $\phi \in \tau$ and $X \in \tau$.

(2) If $(F_{a_1}, E) \in \tau, a_i \in I, \text{ for some index set } I$, then $\{e_j : F_{a_1}(e_j) \neq \phi\}$ is a finite set. Now $\{e_j : (\cup F_{a_1})^c(e_j) \neq \phi\} = \{e_j : F_{a_1}^c(e_j) \neq \phi\}$, for all $a_k \in I$. Since $\{e_j : F_{a_1}(e_j) \neq \phi\}$ is a finite set, $\{e_j : (\cup F_{a_1})^c(e_j) \neq \phi\}$ is a finite set and hence $\cup (F_{a_1}, E) \in \tau$.

(3) If $(F_{a_1}, E)$ and $(F_{a_2}, E) \in \tau$, then $\{e_j : F_{a_1}(e_j) \neq \phi\}$ and $\{e_j : F_{a_2}(e_j) \neq \phi\}$ are finite sets. Now $\{e_j : (F_{a_1} \cup F_{a_2})^c(e_j) \neq \phi\} = \{e_j : (F_{a_1} \cap F_{a_2})^c(e_j) \neq \phi\}$ is a finite set. Thus $(F_{a_1}, E) \cap (F_{a_2}, E) \in \tau$.

Thus $(X, \tau, E)$ is a soft topological space. Let us take two distinct soft points $x_e, y_e$. Then either $x \neq y$ or $e_i \neq e_j$. In either cases $x_e, y_e$ are two soft open sets such that $x_e \subset y_e, y_e \not\subset x_e$. This proves that $(X, \tau, E)$ is a soft $T_1$ space. Let us consider a soft set $(G, E)$ such that $G(e_i) = \begin{cases} \{x\} & \text{if } e_i \text{ is even} \\ \phi & \text{if } e_i \text{ is odd} \end{cases}$. Define $T(e_i) = \begin{cases} 1 & \text{if } e_i \text{ is even} \\ 0 & \text{if } e_i \text{ is odd} \end{cases}$, $T = \sum T(e_i) = \infty$, because $2\mathbb{N}$ is an infinite set. Since $\{e_j : G(e_j) \neq \phi\}$ is not a finite set, $(G, E)$ is not a soft closed set.

Definition 2.6. A soft topological space $(X, \tau, E)$ is said to be a soft $T_2$-space if for every pair of soft points $x_e, y_e$, such that $x_e \neq y_e$, there exist soft open sets $(F, E)$ and $(G, E)$ such that $x_e \in (F, E), y_e \in (G, E)$ and $(F, E) \cap (G, E) = \phi$.

Example 2.7. $X = \{x_1, x_2\}, A = \{e_1, e_2\}$ and $\tau = \{\phi, \hat{X}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}$ where $F_1(e) = \begin{cases} \{x_2\} & \text{if } e = e_1 \\ \{x_1\} & \text{if } e = e_2 \end{cases}$, $F_2(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases}$, $F_3(e) = \begin{cases} \{x_1\} & \text{if } e = e_1 \\ \phi & \text{if } e = e_2 \end{cases}$, $F_4(e) = \begin{cases} X & \text{if } e = e_1 \\ \{x_1\} & \text{if } e = e_2 \end{cases}$.

This $(X, \tau, A)$ is verified as soft $T_1$ and soft $T_2$ spaces.

Consider two soft points $e_F = \begin{cases} \{x_1\} & \text{if } e = e_1 \text{ and } e_G = \begin{cases} \phi & \text{if } e = e_1 \\ \{x_2\} & \text{if } e = e_2 \end{cases} \end{cases}$, then there is no soft open set $(F_i, A), i \in \{1, 2, 3, 4\}$ in $(X, \tau, A)$ such that $e_G \notin (F_i, A)$ and $e_F \notin (F_i, A)$. Thus $(X, \tau, A)$ is not a soft $T_1$ space.
Similarly, there is no two soft open sets \((F_i, A) (F_j, A)\), \(i, j \in \{1, 2, 3, 4\}\), \(i \neq j\) in \((X, \tau, A)\) such that \(e_F \in \tilde{(F_i, A)}\), \(e_G \in \tilde{(F_j, A)}\) and \((F_i, A) \cap (F_j, A) = \emptyset\). Thus \((X, \tau, A)\) is not a soft \(T_2\) space too.

Next the example 3 given in article [8] is wrong.

**Example 2.8.** \(X = \{x_1, x_2\}, A = \{e_1, e_2\}\) and \(\tau = \{\emptyset, \tilde{X}, (F_1, A), (F_2, A), (F_3, A)\}\) where

\[
F_1(e) = \begin{cases} \{x_1\} & \text{if } e = e_1, \\ \emptyset & \text{if } e = e_2, \end{cases} \quad F_2(e) = \begin{cases} \emptyset & \text{if } e = e_1, \\ \{x_2\} & \text{if } e = e_2, \end{cases} \quad F_3(e) = \begin{cases} \{x_1\} & \text{if } e = e_1, \\ \{x_2\} & \text{if } e = e_2. \end{cases}
\]

This \((X, \tau, A)\) is verified as soft \(T_1\) and soft \(T_0\) spaces in [8].

Consider two soft points \(e_F = \{x_2\}\) if \(e = e_1\) and \(e_G = \{x_1\}\) if \(e = e_2\),

then there is no soft open set \((F_i, A)\), \(i \in \{1, 2, 3\}\) in \((X, \tau, A)\) such that \(e_F \in \tilde{(F_i, A)}\) and \(e_G \in \tilde{(F_i, A)}\). Thus \((X, \tau, A)\) is not a soft \(T_1\) space. Also there is no soft open set \((F_i, A)\), \(i \in \{1, 2, 3\}\) in \((X, \tau, A)\) such that \(e_F \notin \tilde{(F_i, A)}\) and \(e_G \notin \tilde{(F_i, A)}\). Hence \((X, \tau, A)\) is not a soft \(T_0\) space too.

Correct example for soft \(T_1\) space which is not a soft \(T_2\) space is given below.

**Example 2.9.** Consider a soft topological space \((X, \tau, E)\) discussed in Example: 2.6. It is a soft \(T_1\) space.

Let \(x_{e_1}\) and \(y_{e_2}\) be two distinct soft points. Then either \(x \neq y\) or \(e_1 \neq e_2\).

Assume that there exists two soft open sets \((F, E)\) and \((G, E)\) such that \(x_{e_1} \in \tilde{(F, E)}\) and \(y_{e_2} \in \tilde{(G, E)}\). Since \((F, E)\) and \((G, E)\) are soft open sets, \(\{e_j : F^c(e_j) \neq \emptyset\}\) and \(\{e_j : G^c(e_j) \neq \emptyset\}\) are finite sets. Now \(E - \{e_j : (F(e_j) \cap G(e_j))^c \neq \emptyset\} \neq \emptyset\). For any \(e_k \in E - \{e_j : (F(e_j) \cap G(e_j))^c \neq \emptyset\}\), \(F^c(e_k) = \emptyset\) and \(G^c(e_k) = \emptyset\). That is \(F(e_k) \cap G(e_k) = \emptyset\) and hence \((F, E) \cap (G, E) = \emptyset\). This proves that \((X, \tau, E)\) is not a soft \(T_2\) space.

**Theorem 2.9.** Every soft \(T_2\) space is a soft \(T_1\) space.

**Proof.** Proof is straightforward.

**Theorem 2.10.** Soft subspace of soft \(T_2\)-space is a soft \(T_2\)-space.

**Proof.** Let \((X, \tau, E)\) be a soft \(T_2\)-space and \((Y, \tau_Y, E)\) be a soft subspace. Let \(x_{e_1}, y_{e_2}\) be two soft points in \((Y, \tau, E)\). Then \(x_{e_1}, y_{e_2} \in SS(X, E)\). Since \((X, \tau, E)\) is a soft \(T_2\) space, there exist two soft open sets \((F, E)\) and \((G, E)\) in \((X, \tau, E)\) such that \(x_{e_1} \in (F, E), y_{e_2} \in (G, E)\) and \((F, E) \cap (G, E) = \emptyset\). Now \((F, E) \cap (G, E) = \emptyset\) and \((G, E) \cap (F, E) = \emptyset\). Thus \((Y, \tau_Y, E)\) is a soft \(T_2\) space.

**Lemma 2.11.** Let \((X, \tau, E)\) be a finite soft \(T_2\) space. Then \((X, \tau, E)\) is a soft discrete space.

**Proof.** Proof follows from theorem 2.9 and theorem 2.5.

**Lemma 2.12.** If \((X, \tau, E)\) is a countable soft \(T_2\) space and if every soft \(G_\delta\) set is soft open in \((X, \tau, E)\), then \((X, \tau, E)\) is a soft discrete space.

**Proof.** Proof follows from theorem 2.9 and theorem 2.6.
Theorem 2.13. Let \((X, \tau, E)\) be a soft topological space. Then \((X, \tau, E)\) is a soft \(T_2\) space if and only if for any two distinct soft points \(x_{e_i}\) and \(y_{e_j}\), there exist two soft closed neighbourhoods \((H, E)\) and \((K, E)\) containing disjoint soft open sets containing \(x_{e_i}\) and \(y_{e_j}\) respectively such that \((H, E) \tilde{\cup} (K, E) = X\).

Proof. Since \((X, \tau, E)\) is a soft \(T_2\) space, for any two distinct soft points \(x_{e_i}\) and \(y_{e_j}\), there exist two soft open sets \((F, E)\) and \((G, E)\) such that \(x_{e_i} \in (F, E)\) and \(y_{e_j} \in (G, E)\). Now \(x_{e_i} \in (F^c, E)\) and \(y_{e_j} \in (G^c, E)\). Note that \((F, E) \subseteq (G^c, E)\) and \((G, E) \subseteq (F^c, E)\). Let \((F^c, E) = (K, E)\) and \((G^c, E) = (H, E)\). Then we have two soft closed neighbourhoods \((H, E)\) and \((K, E)\) containing disjoint soft open sets \((F, E)\) and \((G, E)\) respectively, such that \(x_{e_i} \in (F, E)\) and \(y_{e_j} \in (G, E)\). This proves that \((X, \tau, E)\) is a soft \(T_2\) space.

Theorem 2.14. Product of soft \(T_2\)-spaces is a soft \(T_2\)-space.

Proof. Let \(\{\{X_i, \tau_i, E_i\} : i \in I\}\) be the collection of soft topological spaces and \((\prod X_i, \prod \tau_i, \prod E_i)\) be the product soft topological space. Suppose \(x_e\) and \(y_I\) be two distinct soft points. Then there exist two soft closed neighbourhoods \((H, E)\) and \((K, E)\) and two soft open sets \((L, E)\) containing \(x_{e_i}\) and \((M, E)\) containing \(y_{e_i}\), such that \((L, E) \subseteq (H, E)\) and \((M, E) \subseteq (K, E)\), \((L, E) \cap (M, E) = \emptyset\) and \((H, E) \tilde{\cup} (K, E) = X\). This proves that \((X, \tau, E)\) is a soft \(T_2\) space.

Definition 2.7. Let \((X, \tau, E)\) be a soft topological space. Then \((X, \tau, E)\) is a soft Urysohn space or soft \(T_{2\frac{1}{2}}\) space if for any two soft points \(x_{e_i}\) and \(y_{e_j}\), there exist two soft open sets \((F, E)\) and \((G, E)\) such that \(x_{e_i} \in (F, E)\), \(y_{e_j} \in (G, E)\) and \(Cl(F, E) \cap Cl(G, E) = \emptyset\).
**Theorem 2.15.** Every soft $T_{2\frac{1}{2}}$-space is a soft $T_2$-space.

*Proof.* Proof is similar to theorem 2.14.

**Theorem 2.16.** Soft subspace of soft $T_{2\frac{1}{2}}$-space is a soft $T_{2\frac{1}{2}}$-space.

*Proof.* Proof is similar to theorem 2.10.

**Theorem 2.17.** Let $(X, \tau, E)$ be a soft topological space. Then $(X, \tau, E)$ is a soft $T_{2\frac{1}{2}}$ space if and only if for any two soft points $x_{e_i}$ and $y_{e_j}$, there exist two soft open sets $(H, E)$ and $(K, E)$ such that $x_{e_i} \in (H, E)$, $y_{e_j} \in (K, E)$ and $(H, E)$ containing the disjoint closed soft neighbourhoods of $x_{e_i}$ and $y_{e_j}$ respectively with $(H, E) \cap (K, E) = \emptyset$.

*Proof.* Since $(X, \tau, E)$ is a soft $T_{2\frac{1}{2}}$ space, for any soft points $x_{e_i}$ and $y_{e_j}$, there exist two soft open sets $(F, E)$ and $(G, E)$ such that $x_{e_i} \in (F, E)$ and $y_{e_j} \in (G, E)$ such that $\text{Cl}(F, E) \cap \text{Cl}(G, E) = \emptyset$. Now $x_{e_i} \in \text{Cl}(F, E)$ and $y_{e_j} \in \text{Cl}(G, E)$ such that $\text{Cl}(F, E) \cap \text{Cl}(G, E) = \emptyset$. Note that $\text{Cl}(F, E) \cap \text{Cl}(G, E) = \emptyset$ and $\text{Cl}(G, E) \subseteq \text{Cl}(F, E)$. Let $\text{Cl}(F, E) = (K, E)$ and $\text{Cl}(G, E) = (H, E)$. Then we have two soft open sets $(H, E)$ and $(K, E)$ containing $x_{e_i}$ and $y_{e_j}$ respectively, such that $x_{e_i} \in (F, E)$ and $y_{e_j} \in (G, E)$ and $\text{Cl}(G, E) \subseteq \text{Cl}(F, E) \cap (K, E) = \emptyset$. Thus $(H, E)$ and $(K, E)$ are soft open sets containing the disjoint closed soft neighbourhoods $\text{Cl}(F, E)$ and $\text{Cl}(G, E)$, respectively such that $x_{e_i} \in \text{Cl}(F, E)$, $y_{e_j} \in \text{Cl}(G, E)$ and $(H, E) \cap (K, E) = \emptyset$.

Conversely, let $x_{e_i}$ and $y_{e_j}$ be two distinct soft points. By our assumption, there exist two soft open sets $(H, E)$ and $(K, E)$ containing disjoint closed neighbourhoods $(L, E)$ and $(M, E)$ of $x_{e_i}$ and $y_{e_j}$ respectively such that $(H, E) \cap (K, E) = \emptyset$. Note that there are soft open sets $(F, E)$ and $(G, E)$ such that $(F, E) \cap (G, E) = \emptyset$ and $(H, E), (G, E) \subseteq \text{Cl}(F, E) \subseteq (L, E)$ and $(M, E) \subseteq \text{Cl}(G, E) \subseteq (K, E)$. Thus $(F, E)$ and $(G, E)$ are soft open sets containing $x_{e_i}$ and $y_{e_j}$ such that $\text{Cl}(F, E) \cap \text{Cl}(G, E) = \emptyset$. Thus $(X, \tau, E)$ is a soft $T_{2\frac{1}{2}}$ space.

Soft single point space discussed in [5] is not a soft $T_0$ or $T_1$ or $T_2$ or $T_{2\frac{1}{2}}$ space. Because for the soft points $x_{e_i}$ and $x_{e_j}$, there is no soft open set containing $x_{e_i}$ not containing $x_{e_j}$.

**Theorem 2.18.** Product of soft $T_{2\frac{1}{2}}$-spaces is a soft $T_{2\frac{1}{2}}$-space.

*Proof.* Proof is similar to theorem 2.14.

**Lemma 2.19.** Let $(X, \tau, E)$ be a finite soft $T_{2\frac{1}{2}}$ space. Then $(X, \tau, E)$ is a soft discrete space.

*Proof.* Proof follows from theorem 2.15, theorem 2.9 and theorem 2.5.

**Lemma 2.20.** If $(X, \tau, E)$ is a countable soft $T_{2\frac{1}{2}}$ space and if every soft $G_{\delta}$ set is soft open in $(X, \tau, E)$, then $(X, \tau, E)$ is a soft discrete space.

*Proof.* Proof follows from theorem 2.15, theorem 2.9 and theorem 2.6.
3. Conclusion

For the soft separation axioms of soft points defined on soft topological space, we discuss the characterizations and properties of soft\( T_0 \), \( T_1 \), \( T_2 \) and soft \( T_{2,1} \) spaces. Also it is verified that the product of soft \( T_i \) spaces, \( i = 1, 2, T_2 \) is a soft \( T_i \) space. But there is an example given here for the product of soft \( T_0 \) spaces need not be a soft \( T_0 \) space. Also we provide correct examples for the wrong examples example:1, example:2 and example:3 given in article [8].

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References

SOFT SEPARATION AXIOMS AND SOFT PRODUCT OF SOFT TOPOLOGICAL SPACES

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