Results in Nonlinear Analysis 3 (2020) No. 4, 167–178
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

ψ–Caputo Fractional Differential Equations with Multi-point Boundary Conditions by Topological Degree Theory

Zidane Baitiche[a], Choukri Derbazi[a], Mouffak Benchohra[b]

[a]Laboratory of Mathematics And Applied Sciences, University of Ghardaia, 47000, Algeria
[b]Laboratory of Mathematics, Djillali Liabes University of Sidi-Bel-Abbes, Algeria.

Abstract

In this article, we discuss the existence and uniqueness of solutions to some nonlinear fractional differential equations involving the ψ–Caputo fractional derivative with multi-point boundary conditions. Our results rely on the technique of topological degree theory for condensing maps and the Banach contraction principle. Also, two illustrative examples are presented to illustrate our main results.

Keywords: Fractional differential equations; ψ–Caputo fractional derivative; multi-point boundary conditions.; existence, uniqueness; topological degree theory; condensing maps.
2010 MSC: 34A08; 26A33

1. Introduction

Fractional calculus and fractional differential equations describe various phenomena in diverse areas of natural science such as physics, aerodynamics, biology, control theory, chemistry, and so on, see ([20, 26, 28, 35]). Recently Almeida [11], introduced the so-called ψ–Caputo fractional derivative that generalizes a wide class of other formulations of fractional derivatives such as Caputo, Caputo–Hadamard. For some results and recent development on initial and boundary value problems involving ψ–Caputo fractional derivative, we refer the reader to a series of papers [7, 12, 13, 14, 15, 29] and the references given therein. Moreover, different kinds of fixed point theorems are widely used as fundamental tools in order to prove the existence and uniqueness of solutions for various classes of fractional differential equations, for details, see [1, 4, 2, 3, 5, 6, 9, 17, 18, 23, 25].

Email addresses: baitichezidane19@gmail.com (Zidane Baitiche), choukriedp@yahoo.com (Choukri Derbazi), benchohra@yahoo.com (Mouffak Benchohra)

Received April 20, 2020, Accepted: September 22, 2020, Online: September 25, 2020.

Isaia [21] proved a new fixed theorem that was obtained via coincidence degree theory for condensing operators. After that, the fixed theorem due to Isaia was used by several researchers with the end goal to study the existence of solutions for certain classes of nonlinear differential equations: we suggest some works [10, 16, 21, 22, 30, 31, 32, 33, 36].

Motivated by the mentioned works, in this paper, we generalize the results obtained in [34] to the fractional setting. More specifically, we are concerned with the following ψ-Caputo fractional differential equation of the form

$$\begin{cases} {}^{c}D_{a^+}^{\alpha;\psi}u(t) + f(t,u(t)) = 0, \ \ t \in J := [a,b], \\ u(a) = u'(a) = 0, \quad u(b) = \sum_{i=1}^{m} \lambda_i u(\eta_i), \quad a < \eta_i < b, \end{cases} \tag{1.1}$$

where ${}^{c}D_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $2 < \alpha \leq 3$, $f\colon [a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function and $a < \eta_i < b, \lambda_i, \ i = 1, 2, \dots, m$ are real constants with satisfying

$$\Delta = \sum_{i=1}^{m} \lambda_i(\psi(\eta_i) - \psi(a))^2 - (\psi(b) - \psi(a))^2 \neq 0. \tag{1.2}$$

To the best of our knowledge, this is the first paper dealing with a class of fractional differential equations containing ψ–Caputo fractional derivative and multi-point boundary conditions via topological degree theory. Our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem. For example, If we take $\alpha = 3, \psi(t) = t$ then the BVP (1.1) corresponds to the usual form of the third-order problem given by

$$\begin{cases} u'''(t) + f(t,u(t)) = 0, \ \ t \in J := [a,b], \\ u(a) = u'(a) = 0, \quad u(b) = \sum_{i=1}^{m} \lambda_i u(\eta_i), \quad a < \eta_i < b. \end{cases}$$

The manuscript is structured as follows. Section 2, in which we describe some basic notations of fractional derivatives and integrals, definitions of differential calculus, and important results that will be used in subsequent parts of the paper. In Section 3, based on the coincidence degree theory for condensing maps, we establish a theorem on the existence of solutions for problem (1.1). Next by using the Banach contraction principle, we give a uniqueness results for problem (1.1). Additionally, Section 4 provides a couple of examples to illustrate the applicability of the results developed. Finally, the paper is concluded in Section 5.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Assume that $(X, \|\cdot\|)$ is a Banach space. By $\mathrm{B}_r(0)$ we denote the closed ball centered at 0 with radius r. If A is non-empty subset of A, then $\overline{A}$ and $\mathrm{conv}A$ denote the closure and the convex hull of A, respectively. When A is a bounded subset, $\mathrm{diam}(A)$ denotes the diameter of A. Also, we denote by $\mathfrak{M}_X$ the class of non-empty and bounded subsets of X.

We state here the results given below from [8, 19].

Definition 1. *The Kuratowski measure of non-compactness is the mapping $\kappa\colon \mathfrak{M}_X \longrightarrow [0,\infty)$ defined as:*

$$\kappa(B) = \inf\Big\{\varepsilon > 0 : B \textit{ can be covered by finitely many sets with diameter} \leq \varepsilon\Big\}.$$

Properties 1. *The Kuratowski measure of noncompactness satisfies the following properties.*

(1) $A \subset B \Rightarrow \kappa(A) \leq \kappa(B)$,

(2) $\kappa(A) = 0$ *if and only if* A *is relatively compact,*

(3) $\kappa(A) = \kappa(\overline{A}) = \kappa(conv(A))$,
(4) $\kappa(A + B) \leq \kappa(A) + \kappa(B)$,
(5) $\kappa(\lambda A) = |\lambda|\kappa(A), \lambda \in \mathbb{R}$.

Definition 2. *Let $\mathcal{T}: A \longrightarrow X$ be a continuous bounded map. The operator $\mathcal{T}$ is said to be κ-Lipschitz if we can find a constant $\ell \geq 0$ satisfying the following condition,*

$$\kappa(\mathcal{T}(B)) \leq \ell\kappa(B), \text{ for every } B \subset A.$$

Moreover, $\mathcal{T}$ is called strict κ-contraction if $\ell < 1$.

Definition 3. *The function $\mathcal{T}$ is called κ-condensing if*

$$\kappa(\mathcal{T}(B)) < \kappa(B),$$

for every bounded and nonprecompact subset B of A.
In other words,

$$\kappa(\mathcal{T}(B)) \geq \kappa(B), \text{ implies } \kappa(B) = 0.$$

Further we have $\mathcal{T}: A \longrightarrow X$ is Lipschitz if we can find $\ell > 0$ such that

$$\|\mathcal{T}(u) - \mathcal{T}(v)\| \leq \ell\|u - v\|, \text{ for all } u, v \in A,$$

if $\ell < 1$, $\mathcal{T}$ is said to be strict contraction.

For the following results, we refer to [21].

Proposition 4. *If $\mathcal{T}, \mathcal{S}: A \longrightarrow X$ are κ-Lipschitz mapping with constants ℓ_1 and ℓ_2 respectively, then $\mathcal{T} + \mathcal{S} : A \longrightarrow X$ are κ-Lipschitz with constant $\ell_1 + \ell_2$.*

Proposition 5. *If $\mathcal{T}: A \longrightarrow X$ is compact, then $\mathcal{T}$ is κ-Lipschitz with constant $\ell = 0$.*

Proposition 6. *If $\mathcal{T}: A \longrightarrow X$ is Lipschitz with constant ℓ, then $\mathcal{T}$ is κ-Lipschitz with the same constant ℓ.*

Isaia [21] present the following results using topological degree theory.

Theorem 7. *Let $\mathcal{K}: A \longrightarrow X$ be κ-condensing and*

$$\Theta = \{u \in X : \text{ there exist } \xi \in [0, 1] \text{ such that } u = \xi\mathcal{K}u\}.$$

If Θ is a bounded set in X, so there exists $r > 0$ such that $\Theta \subset \mathrm{B}_r(0)$, then the degree

$$\deg(I - \xi\mathcal{K}, \mathrm{B}_r(0), 0) = 1, \text{ for all } \xi \in [0, 1].$$

Consequently, $\mathcal{K}$ has at least one fixed point and the set of the fixed points of $\mathcal{K}$ lies in $\mathrm{B}_r(0)$.

Now, we give some results and properties from the theory of of fractional calculus. We begin by defining ψ-Riemann–Liouville fractional integrals and derivatives. In what follows,

Definition 8 ([11, 24])**.** *For $\alpha > 0$, the left-sided ψ–Riemann–Liouville fractional integral of order α for an integrable function $u: \mathrm{J} \longrightarrow \mathbb{R}$ with respect to another function $\psi: \mathrm{J} \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in \mathrm{J}$ is defined as follows*

$$I_{a^+}^{\alpha;\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s)\mathrm{d}s, \tag{2.1}$$

where Γ is the gamma function. Note that Eq. (2.1) is reduced to the Riemann–Liouville and Hadamard fractional integrals when $\psi(t) = t$ and $\psi(t) = \ln t$, respectively.

Definition 9 ([11]). *Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(\mathrm{J}, \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in \mathrm{J}$. The left-sided ψ–Riemann–Liouville fractional derivative of a function u of order α is defined by*

$$\begin{aligned} D_{a^+}^{\alpha;\psi} u(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} u(s)\mathrm{d}s, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 10 ([11]). *Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(\mathrm{J}, \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in \mathrm{J}$. The left-sided ψ-Caputo fractional derivative of u of order α is defined by*

$$ {}^cD_{a^+}^{\alpha;\psi} u(t) = I_{a^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t), $$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.
To simplify notation, we will use the abbreviated symbol

$$u_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t).$$

From the definition, it is clear that

$$ {}^cD_{a^+}^{\alpha;\psi} u(t) = \begin{cases} \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} u_\psi^{[n]}(s)\mathrm{d}s & , \quad \text{if } \alpha \notin \mathbb{N}, \\ u_\psi^{[n]}(t) & , \quad \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{2.2}$$

This generalization (2.2) yields the Caputo fractional derivative operator when $\psi(t) = t$. Moreover, for $\psi(t) = \ln t$, it gives the Caputo–Hadamard fractional derivative.

We note that if $u \in C^n(\mathrm{J}, \mathbb{R})$ the ψ–Caputo fractional derivative of order α of u is determined as

$$ {}^cD_{a^+}^{\alpha;\psi} u(t) = D_{a^+}^{\alpha;\psi}\left[u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]} u(a)}{k!}(\psi(t)-\psi(a))^k\right].$$

(see, for instance, [11, Theorem 3]).

Lemma 11 ([13]). *Let $\alpha, \beta > 0$, and $u \in L^1(\mathrm{J}, \mathbb{R})$. Then*

$$I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} u(t) = I_{a^+}^{\alpha+\beta;\psi} u(t), \ \text{a.e. } t \in \mathrm{J}.$$

In particular,

If $u \in C(\mathrm{J}, \mathbb{R})$. Then $I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} u(t) = I_{a^+}^{\alpha+\beta;\psi} u(t)$, $t \in \mathrm{J}$.

Next, we recall the property describing the composition rules for fractional ψ-integrals and ψ-derivatives.

Lemma 12 ([13]). *Let $\alpha > 0$, The following holds:*

If $u \in C(\mathrm{J}, \mathbb{R})$ then

$$ {}^cD_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} u(t) = u(t), \ t \in \mathrm{J}.$$

If $u \in C^n(\mathrm{J}, \mathbb{R})$, $n - 1 < \alpha < n$. Then

$$I_{a^+}^{\alpha;\psi}\ {}^cD_{a^+}^{\alpha;\psi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(a)}{k!}\left[\psi(t)-\psi(a)\right]^k,$$

for all $t \in \mathrm{J}$.

Lemma 13 ([13, 24]). *Let $t > a$, $\alpha \geq 0$, and $\beta > 0$. Then*

- $I_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}$,
- ${}^{c}D_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1}$,
- ${}^{c}D_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k} = 0$, *for all* $k \in \{0,\ldots,n-1\}, n \in \mathbb{N}$.

3. Main Results

Let us recall the definition and lemma of a solution for problem (1.1). First of all, we define what we mean by a solution for the boundary value problem (1.1).

Definition 14. *A function $u \in C(\mathrm{J},\mathbb{R})$ is said to be a solution of Eq. (1.1) if u satisfies the equation ${}^{c}D_{a^+}^{\alpha;\psi}u(t) + f(t,u(t)) = 0$, a.e. on J and the condition*

$$u(a) = u'(a) = 0, \quad u(b) = \sum_{i=1}^{m} \lambda_i u(\eta_i), \quad a < \eta_i < b.$$

For the existence of solutions for the problem (1.1) we need the following lemma:

Lemma 15. *For a given $h \in C(\mathrm{J},\mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^{c}D_{a^+}^{\alpha;\psi}u(t) + h(t) = 0, \ t \in \mathrm{J} := [a,b], \\ u(a) = u'(a) = 0, \quad u(b) = \sum_{i=1}^{m} \lambda_i u(\eta_i), \quad a < \eta_i < b, \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} u(t) = &-\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s \\ &+ \frac{(\psi(t)-\psi(a))^2}{\Delta}\sum_{i=1}^{m}\lambda_i \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s \\ &- \frac{(\psi(t)-\psi(a))^2}{\Delta}\int_a^b \frac{(\psi(b)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s, \end{aligned} \tag{3.2}$$

in which

$$\Delta = \sum_{i=1}^{m}\lambda_i(\psi(\eta_i)-\psi(a))^2 - (\psi(b)-\psi(a))^2 \neq 0.$$

Proof. Taking the ψ-Riemann–Liouville fractional integral of order α to the first equation of (3.1), and using Lemma 12, we get

$$u(t) = -I_{a^+}^{\alpha;\psi}h(t) + c_0 + c_1(\psi(t)-\psi(a)) + c_2(\psi(t)-\psi(a))^2, \quad c_0, c_1, c_2 \in \mathbb{R}. \tag{3.3}$$

Since $u(a) = 0$ and $u'(a) = 0$, we deduce that $c_0 = c_1 = 0$. Thus

$$u(t) = -I_{a^+}^{\alpha;\psi}h(t) + c_2(\psi(t)-\psi(a))^2. \tag{3.4}$$

Together with the condition $u(b) = \sum_{i=1}^{m}\lambda_i u(\eta_i), \quad a < \eta_i < b$, this yields

$$-I_{a^+}^{\alpha;\psi}h(b) + c_2(\psi(b)-\psi(a))^2 = \sum_{i=1}^{m}\lambda_i\left[-I_{a^+}^{\alpha;\psi}h(\eta_i) + c_2(\psi(\eta_i)-\psi(a))^2\right].$$

Then, we can get that

$$c_2 = \frac{\sum_{i=1}^{m} \lambda_i I_{a^+}^{\alpha;\psi} h(\eta_i) - I_{a^+}^{\alpha;\psi} h(b)}{\sum_{i=1}^{m} \lambda_i (\psi(\eta_i) - \psi(a))^2 - (\psi(b) - \psi(a))^2}.$$

Substituting the values of c_0, c_1 and c_2 into (3.3), we get the integral equation (3.2). The converse follows by direct computation which completes the proof. □

Now, we shall present our main result concerning the existence of solutions of problem (1.1). Let us introduce the following hypotheses

(H1) There exists a constant $L > 0$ such that

$$|f(t,u) - f(t,v)| \leq L|u - v|, \text{ for each } t \in \mathrm{J} \text{ and for each } u, v \in \mathbb{R}. \tag{3.5}$$

(H2) The functions f satisfies the following growth condition for constants $M, N > 0, p \in (0,1)$

$$|f(t,u)| \leq M|u|^p + N \text{ for each } t \in \mathrm{J} \text{ and each } u \in \mathbb{R}. \tag{3.6}$$

In the following, for computational convenience we put

$$\omega = \frac{(\psi(b) - \psi(a))^2}{|\Delta|\Gamma(\alpha + 1)} \left[(\psi(b) - \psi(a))^\alpha + \sum_{i=1}^{m} |\lambda_i| (\psi(\eta_i) - \psi(a))^\alpha \right]. \tag{3.7}$$

In view of Lemma 15, we consider two operators $\mathcal{T}, \mathcal{S} \colon C(\mathrm{J}, \mathbb{R}) \longrightarrow C(\mathrm{J}, \mathbb{R})$ as follows:

$$\mathcal{S}u(t) = -\int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \mathrm{d}s, \ t \in \mathrm{J},$$

and

$$\begin{aligned} \mathcal{T}u(t) = {} & \frac{(\psi(t) - \psi(a))^2}{\Delta} \sum_{i=1}^{m} \lambda_i \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \mathrm{d}s \\ & - \frac{(\psi(t) - \psi(a))^2}{\Delta} \int_a^b \frac{(\psi(b) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \mathrm{d}s, \ t \in \mathrm{J}. \end{aligned}$$

Then the integral equation (3.2) in Lemma 15 can be written as an operator equation

$$\mathcal{K}u(t) = \mathcal{S}u(t) + \mathcal{T}u(t), \quad t \in \mathrm{J}.$$

The continuity of f shows that the operator $\mathcal{K}$ is well defined and fixed points of the operator equation are solutions of the integral equations (3.2) in Lemma 15.

Lemma 16. *$\mathcal{T}$ is Lipschitz with constant $\ell_f = L\omega$. Moreover, $\mathcal{T}$ satisfies the growth condition given below*

$$\|\mathcal{T}u\| \leq \omega(M\|u\|^p + N),$$

where ω is given by (3.7).

Proof. To show that the operator $\mathcal{T}$ is Lipschitz with constant ℓ_f. Let $u, v \in C(\mathrm{J}, \mathbb{R})$, then we have

$$\begin{aligned}
|\mathcal{T}u(t) - \mathcal{T}v(t)| &\leq \frac{(\psi(t)-\psi(a))^2}{|\Delta|} \sum_{i=1}^{m} |\lambda_i| \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) - f(s,v(s))| \mathrm{ds} \\
&+ \frac{(\psi(t)-\psi(a))^2}{|\Delta|} \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) - f(s,v(s))| \mathrm{ds} \\
&\leq \frac{L(\psi(t)-\psi(a))^2}{|\Delta|} \sum_{i=1}^{m} |\lambda_i| \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|u-v\| \mathrm{ds} \\
&+ \frac{L(\psi(t)-\psi(a))^2}{|\Delta|} \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|u-v\| \mathrm{ds} \\
&\leq \frac{L\|u-v\|(\psi(b)-\psi(a))^2}{|\Delta|} \sum_{i=1}^{m} |\lambda_i| \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds} \\
&+ \frac{L\|u-v\|(\psi(b)-\psi(a))^2}{|\Delta|} \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds} \\
&\leq \frac{L(\psi(b)-\psi(a))^2}{|\Delta|\Gamma(\alpha+1)} \left[(\psi(b)-\psi(a))^{\alpha} + \sum_{i=1}^{m} |\lambda_i| (\psi(\eta_i)-\psi(a))^{\alpha} \right] \|u-v\| \\
&= L\omega \|u-v\|.
\end{aligned}$$

for all $t \in \mathrm{J}$. Taking supremum over t, we obtain

$$\|\mathcal{T}u - \mathcal{T}v\| \leq \ell_f \|u-v\|.$$

Hence, $\mathcal{T} \colon C(\mathrm{J}, \mathbb{R}) \longrightarrow C(\mathrm{J}, \mathbb{R})$ is a Lipschitzian on $C(\mathrm{J}, \mathbb{R})$ with Lipschitz constant $\ell_f = L\omega$. By Proposition 6, $\mathcal{T}$ is κ–Lipschitz with constant ℓ_f. Moreover, for growth condition, we have

$$\begin{aligned}
|\mathcal{T}u(t)| &\leq \frac{(M\|u\|^p + N)(\psi(t)-\psi(a))^2}{|\Delta|} \sum_{i=1}^{m} |\lambda_i| \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds} \\
&+ \frac{(M\|u\|^p + N)(\psi(t)-\psi(a))^2}{|\Delta|} \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds} \\
&\leq \frac{(M\|u\|^p + N)(\psi(b)-\psi(a))^2}{|\Delta|\Gamma(\alpha+1)} \left[(\psi(b)-\psi(a))^{\alpha} + \sum_{i=1}^{m} |\lambda_i| (\psi(\eta_i)-\psi(a))^{\alpha} \right] \\
&= \omega(M\|u\|^p + N)
\end{aligned}$$

Hence it follows that

$$\|\mathcal{T}u\| \leq \omega(M\|u\|^p + N).$$

□

Lemma 17. *$\mathcal{S}$ is continuous and satisfies the growth condition given as below,*

$$\|\mathcal{S}u\| \leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} (M\|u\|^p + N).$$

Proof. To prove that $\mathcal{S}$ is continuous. Let $u_n, u \in C(\mathrm{J}, \mathbb{R})$ with $\lim_{n\to+\infty} \|u_n - u\| \to 0$. It is trivial to see that $\{u_n\}$ is a bounded subset of $C(\mathrm{J}, \mathbb{R})$. As a result, there exists a constant $r > 0$ such that $\|u_n\| \leq r$ for all $n \geq 1$. Taking limit, we see $\|u\| \leq r$. It is easy to see that $f(s, u_n(s)) \to f(s, u(s))$, as $n \to +\infty$. due to the continuity of f. On the other hand taking (H2) into consideration we get the following inequality:

$$\frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\|f(s,u_n(s))-f(s,u(s))\| \leq 2\frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}(Mr^p+N).$$

We notice that since the function $s \mapsto 2\frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}(Mr^p+N)$ is Lebesgue integrable over $[a,t]$. This fact together with the Lebesgue dominated convergence theorem implies that

$$\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\|f(s,u_n(s))-f(s,u(s))\|\mathrm{d}s \to 0 \ \text{ as } \ n \to +\infty.$$

It follows that $\|\mathcal{S}u_n - \mathcal{S}u\| \to 0$ as $n \to +\infty$. Which implies the continuity of the operator $\mathcal{S}$.

For the growth condition, using the assumption (H2) we have

$$\begin{aligned}|\mathcal{S}u(t)| &\leq (M\|u\|^p+N)\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s \\ &\leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}(M\|u\|^p+N).\end{aligned}$$

Therefore,

$$\|\mathcal{S}u\| \leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}(M\|u\|^p+N). \tag{3.8}$$

This completes the proof of Lemma 17. □

Lemma 18. *The operator $\mathcal{S}\colon C(\mathrm{J},\mathbb{R}) \longrightarrow C(\mathrm{J},\mathbb{R})$ is compact. Consequently, $\mathcal{S}$ is κ-Lipschitz with zero constant.*

Proof. In order to show that $\mathcal{S}$ is compact. Let us take a bounded set $\Omega \subset B_r$. We are required to show that $\mathcal{S}(\Omega)$ is relatively compact in $C(\mathrm{J},\mathbb{R})$. For arbitrary $u \in \Omega \subset B_r$, then with the help of the estimates (3.8) we can obtain

$$\|\mathcal{S}u\| \leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}(Mr^p+N),$$

which shows that $\mathcal{S}(\Omega)$ is uniformly bounded. Furthermore, for arbitrary $u \in C(\mathrm{J},\mathbb{R})$ and $t \in \mathrm{J}$.

Now, for equi-continuity of $\mathcal{S}$ take $t_1, t_2 \in \mathrm{J}$ with $t_1 < t_2$, and let $u \in \Omega$. Thus, we get

$$|\mathcal{S}u(t_2)-\mathcal{S}u(t_1)| \leq \frac{Mr^p+N}{\Gamma(\alpha+1)}\left[(\psi(t_2)-\psi(a))^{\alpha}-(\psi(t_1)-\psi(a))^{\alpha}\right].$$

From the last estimate, we deduce that $\|(\mathcal{S}x)(t_2)-(\mathcal{S}x)(t_1)\| \to 0$ when $t_2 \to t_1$. Therefore, $\mathcal{S}$ is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator $\mathcal{S}$ is compact and hence by Proposition 5. $\mathcal{S}$ is κ–Lipschitz with zero constant. □

Theorem 19. *Suppose that (H1)–(H2) are satisfied, then the BVP (1.1) has at least one solution $u \in C(\mathrm{J},\mathbb{R})$ provided that $\ell_f < 1$ and the set of the solutions is bounded in $C(\mathrm{J},\mathbb{R})$.*

Proof. Let $\mathcal{T}, \mathcal{S}, \mathcal{K}$ are the operators defined in the start of this section. These operators are continuous and bounded. Moreover, by Lemma 16, $\mathcal{T}$ is κ–Lipschitz with constant ℓ_f and by Lemma 18, $\mathcal{S}$ is κ–Lipschitz with constant 0. Thus, $\mathcal{K}$ is κ–Lipschitz with constant ℓ_f. Hence $\mathcal{K}$ is strict κ–contraction with constant ℓ_f. Since $\ell_f < 1$, so $\mathcal{K}$ is κ-condensing.

Now consider the following set

$$\Theta = \{u \in C(\mathrm{J}, \mathbb{R}) : \text{ there exist } \xi \in [0,1] \text{ such that } x = \xi \mathcal{K} u\}.$$

We will show that the set Θ is bounded. For $u \in \Theta$, we have $u = \xi\mathcal{K}u = \xi(\mathcal{T}(u) + S(u))$, which implies that

$$\begin{aligned} \|u\| &\leq \xi(\|\mathcal{T}u\| + \|\mathcal{S}u\|) \\ &\leq \left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \omega\right](M\|u\|^{p} + N), \end{aligned}$$

where ω is given by (3.7). From the above inequalities, we conclude that Θ is bounded in $C(\mathrm{J}, \mathbb{R})$. If it is not bounded, then dividing the above inequality by $\beta := \|u\|$ and letting $\beta \to \infty$, we arrive at

$$1 \leq \left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] \lim_{\beta\to\infty} \frac{M\beta^{p} + N}{\beta} = 0,$$

which is a contradiction. Thus the set Θ is bounded and the operator $\mathcal{K}$ has at least one fixed point which represent the solution of BVP (1.1). □

Remark 20. *If the growth condition (H2) is formulated for $p = 1$, then the conclusions of Theorem 19 remain valid provided that*

$$\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] M < 1.$$

To end this section, we give an existence and uniqueness result.

Theorem 21. *Under assumption (H1) the BVP (1.1) has a unique solution if*

$$\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] L < 1. \tag{3.9}$$

Proof. Let $u, v \in C(\mathrm{J}, \mathbb{R})$ and $t \in \mathrm{J}$, then we have

$$\begin{aligned} |\mathcal{K}u(t) - \mathcal{K}v(t)| \leq &\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) - f(s,v(s))| \mathrm{ds} \\ &+ \frac{(\psi(t)-\psi(a))^2}{|\Delta|} \sum_{i=1}^{m} |\lambda_i| \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) - f(s,v(s))| \mathrm{ds} \\ &+ \frac{(\psi(t)-\psi(a))^2}{|\Delta|} \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) - f(s,v(s))| \mathrm{ds} \\ &\leq L\|u-v\| \left[\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds}\right. \\ &+ \frac{(\psi(t)-\psi(a))^2}{|\Delta|} \sum_{i=1}^{m} |\lambda_i| \int_a^{\eta_i} \frac{\psi'(s)(\psi(\eta_i)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds} \\ &\left.+ \frac{(\psi(t)-\psi(a))^2}{|\Delta|} \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{ds}\right] \\ &\leq L\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] \|u-v\|. \end{aligned}$$

In view of the given condition $\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] L < 1$, it follows that the mapping $\mathcal{K}$ is a contraction. Hence, by the Banach fixed point theorem, $\mathcal{K}$ has a unique fixed point which is a unique solution of problem (1.1). This completes the proof. □

4. Examples

In this section, in order to illustrate our results, we consider two examples.

Example 22. *Let us consider problem (1.1) with specific data:*

$$\begin{aligned} \alpha &= \frac{5}{2}, a = 0, b = 1, \psi(t) = t, \\ \eta_i &= \frac{1}{2^i}; \lambda_i = i; i = 1, \ldots, 10. \end{aligned} \tag{4.1}$$

Using the given values of the parameters in (1.2) and (3.7), by the Matlab program, we find that

$$\begin{aligned} \Delta &= -0.5556 \\ \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)} &= 0.3009 \\ \omega &= 0.7823 \\ \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)} + \omega &= 1.0832. \end{aligned} \tag{4.2}$$

In order to illustrate Theorem 19, we take

$$f(t, u(t)) = \frac{1}{e^t + 9}\left(1 + \frac{|u(t)|}{1 + |u(t)|}\right), \tag{4.3}$$

in (1.1) and note that

$$\begin{aligned} |f(t,u) - f(t,v)| &= \frac{1}{e^t + 9}\left(\left|\frac{|u|}{1+|u|} - \frac{|v|}{1+|v|}\right|\right) \\ &\leq \frac{1}{e^t+9}\left(\frac{|u-v|}{(1+|u|)(1+|v|)}\right) \\ &\leq \frac{1}{10}|u-v|. \end{aligned}$$

Hence the condition (H1) holds with $L = \frac{1}{10}$. Further from the above given data it is easy to calculate

$$\ell_f = L\omega = 0.0782.$$

On the other hand, for any $t \in J, u \in \mathbb{R}$ we have

$$|f(t,u)| \leq \frac{1}{10}(|u| + 1).$$

Hence condition (H2) holds with $M = N = \frac{1}{10}, p = 1$. In view of Theorem 19,

$$\Theta = \{u \in C(\mathrm{J}, \mathbb{R}) : \text{ there exist } \xi \in [0,1] \text{ such that } x = \xi \mathcal{K} u\},$$

is the solution set; then

$$\|u\| \leq \xi(\|\mathcal{T}u\| + \|\mathcal{S}u\|) \leq \left[\frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)} + \omega\right](M\|u\| + N).$$

From which, we have

$$\|u\| \leq \frac{\left[\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + \omega\right] N}{1 - \left[\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + \omega\right] M} = 0.1215.$$

By Theorem 19 the BVP (1.1) with the data (4.1) and (4.3) has at least a solution u in $C(\mathrm{J} \times \mathbb{R}, \mathbb{R})$. Furthermore $\left[\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + \omega\right] L = 0.1083 < 1$. Hence by Theorem 21 the boundary value problem (1.1) with the data (4.1) and (4.3) has a unique solution.

Example 23. *Consider the following boundary value problem of a fractional differential equation:*

$$\begin{cases} {}^{c}_{H}D_{1}^{\frac{7}{4}}u(t) = \frac{1}{2(t+1)^2}\left(u(t) + \sqrt{1+u^2(t)}\right), \ t \in \mathrm{J} := [1,2], \\ u(1) = u'(1) = 0, \quad u(2) = \frac{1}{2}u\left(\frac{3}{2}\right), \end{cases} \tag{4.4}$$

Note that, this problem is a particular case of BVP (1.1), where

$$\alpha = \frac{5}{2},\ a = 1,\ b = 2,\ \eta = \frac{3}{2}, \lambda = \frac{3}{2},\ \psi(t) = \ln t,$$

and $f\colon \mathrm{J}\times\mathbb{R} \longrightarrow \mathbb{R}$ *given by*

$$f(t,u) = \frac{1}{2(t+1)^2}\left(u + \sqrt{1+u^2}\right), \quad \textit{for } t \in \mathrm{J}, u \in \mathbb{R}.$$

It is clear that the function f is continuous. On the other hand, for any $t \in \mathrm{J}, u, v \in \mathbb{R}$ we have

$$\begin{aligned} |f(t,u) - f(t,v)| &= \frac{1}{(t+1)^2}\left|\frac{1}{2}\left(u - v + \sqrt{1+u^2} - \sqrt{1+v^2}\right)\right| \\ &= \frac{1}{(t+1)^2}\left|\frac{1}{2}(u-v)\left(1 + \frac{u+v}{\sqrt{1+u^2} + \sqrt{1+v^2}}\right)\right| \le \frac{1}{4}|u-v|. \end{aligned}$$

Hence condition (H1) holds with $L = \frac{1}{4}$. We shall check that condition (3.9) is satisfied. Indeed using the Matlab program, we can find

$$\left[\frac{(\psi(b) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] L = 0.1190 < 1,$$

Hence by Theorem 21 the boundary value problem (4.4) has a unique solution.

5. Conclusion

We have presented the existence and uniqueness of solutions to a nonlinear boundary value problem of fractional differential equations involving the ψ–Caputo fractional derivative with multi-point boundary conditions. The proof of the existence results is based on a fixed point theorem due to Isaia [21], which was obtained via coincidence degree theory for condensing maps, while the uniqueness of the solution is proved by applying the Banach contraction principle. Moreover, two examples are presented for the illustration of the obtained theory. Our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem.

References

[1] S. Abbas, M. Benchohra, J.R. Graef, J. Henderson, *Implicit Fractional Differential and Integral Equations*, Existence and Stability, de Gruyter, Berlin 2018.

[2] S. Abbas, M. Benchohra, G.M.N' Guérékata, *Topics in Fractional Differential Equations*, Springer, New York 2012.

[3] S. Abbas, M. Benchohra and G M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[4] S. Abbas, M. Benchohra, J. Henderson, J. E. Lazreg, *Weak solutions for a coupled system of partial Pettis Hadamard fractional integral equations*, Adv. Theory Nonl. Anal. Appl. 1 (2017) No. 2, 136–146.

[5] S. Abbas, M. Benchohra, B. Samet, Y. Zhou, *Coupled implicit Caputo fractional q-difference systems*, Adv. Difference Equ. **2019**, Paper No. 527, 19 pp.

[6] S. Abbas, M. Benchohra, N. Hamidi and J. Henderson, *Caputo–Hadamard fractional differential equations in Banach spaces*, Fract. Calc. Appl. Anal. **21** (2018), 1027–1045.

[7] M.S. Abdo, S.K. Panchal, A.M. Saeed, *Fractional boundary value problem with ψ-Caputo fractional derivative*, Proc. Indian Acad. Sci. (Math. Sci), 129:65 (2019).

[8] R.P. Agarwal, D. O'Regan, *Toplogical degree theory and its applications*, Taylor and Francis, 2006.

[9] A. Aghajani, E. Pourhadi, J.J. Trujillo, Application of measure of noncompactness to a Cauchy problem for fractional differential equations in Banach spaces. Fract. Calc. Appl. Anal. **16** (4) (2013), 962–977.

[10] A. Ali, B. Samet, K. Shah and R. A. Khan, *Existence and stability of solution to a coppled systems of differential equations of non-integer order*, Bound. Value Probl., **2017** (2017), Paper No. 16, 13 pp.

[11] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Communications in Nonlinear Science and Numerical Simulation., 2017, 44, 460–481.

[12] R. Almeida, *Fractional Differential Equations with Mixed Boundary Conditions*, Bull. Malays. Math. Sci. Soc. **42** (2019), 1687–1697.

[13] R. Almeida, A.B. Malinowska, M.T.T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Math. Meth. Appl. Sci. **41** (2018), 336–352

[14] R. Almeida, A.B. Malinowska, T. Odzijewicz, *Optimal Leader-Follower Control for the Fractional Opinion Formation Model*, J. Optim. Theory Appl. **182** (2019), 1171–1185

[15] R. Almeida, M. Jleli, B. Samet, A numerical study of fractional relaxation-oscillation equations involving ψ-Caputo fractional derivative. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), 1873–1891

[16] M. Bahadur Zada, K. Shah and R. A. Khan, *Existence theory to a coupled system of higher order fractional hybrid differential equations by topological degree theory*, Int. J. Appl. Comput. Math., **4** (2018), Art. 102, 19 pp.

[17] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal., **71** (2009), 2391–2396.

[18] M. Benchohra, S. Bouriah, J.J. Nieto, *Existence of periodic solutions for nonlinear implicit Hadamard's fractional differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112** (2018), 25–35.

[19] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[20] R. Hilfer, *Application of fractional calculus in physics*, New Jersey: World Scientific, 2001.

[21] F. Isaia, *On a nonlinear integral equation without compactness*, Acta Math. Univ. Comenian. (N.S.) **75** (2006), 233–240.

[22] R.A. Khan and K. Shah, *Existence and uniqueness of solutions to fractional order multi-point boundary value problems*, Commun. Appl. Anal. **19** (2015), 515–526.

[23] A. Khan, K. Shah, Y. Li, T.S. Khan, *Ulam type stability for a coupled system of boundary value problems of nonlinear fractional differential equations*. J. Funct. Spaces 2017, Art. ID 3046013, 8 pp.

[24] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of North-Holland Mathematics Sudies Elsevier Science B.V. Amsterdam the Netherlands, 2006.

[25] K.D. Kucche, A.D. Mali, J.V.C. Sousa, *On the nonlinear Ψ-Hilfer fractional differential equations*, Comput. Appl. Math. **38** (2) (2019), Art. 73, 25 pp

[26] K.B. Oldham, *Fractional differential equations in electrochemistry*. Adv. Eng. Softw., **41** (1) (2010), 9–12.

[27] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1993.

[28] J. Sabatier, O.P. Agrawal, J.A.T. Machado, *Advances in Fractional Calculus-Theoretical Developments and Applications in Physics and Engineering*. Dordrecht: Springer, 2007.

[29] B. Samet, H. Aydi, *Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative*, J. Inequal. Appl. **2018**, Paper No. 286, 11 pp.

[30] K. Shah, A. Ali and R. A. Khan, *Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems*, Bound. Value Probl., **2016**, Paper No. 43, 12 pp.

[31] K. Shah and R. A. Khan, *Existence and uniqueness results to a coupled system of fractional order boundary value problems by topological degree theory*, Numer. Funct. Anal. Optim., **37**(7) (2016), 887–899.

[32] K. Shah and W. Hussain, *Investigating a class of nonlinear fractional differential equations and its Hyers-Ulam stability by means of topological degree theory*, Numer. Funct. Anal. Optim., **40** (2019), no. 12, 1355–1372.

[33] M. Shoaib, K. Shah, R. Ali Khan, *Existence and uniqueness of solutions for coupled system of fractional differential equation by means of topological degree method,* Journal Nonlinear Analysis and Application., **2018** no. 2 (2018) 124–135

[34] S. Smirnov, *Green's function and existence of a unique solution for a third-order three-point boundary value problem*, Math. Model. Anal. **24** (2019), no. 2, 171–178.

[35] V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media.* Springer, Heidelberg & Higher Education Press, Beijing, 2010.

[36] J. Wang, Y. Zhou and W. Wei, *Study in fractional differential equations by means of topological degree methods*, Numer. Funct. Anal. Optim., **33**(2) (2012), 216–238.