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Rotational hypersurfaces satisfying $\Delta^I \mathbf{R} = \mathbf{AR}$ in the four-dimensional Euclidean space

Dört-boyutlu Öklid uzayında $\Delta^I \boldsymbol{R} = \boldsymbol{AR}$ koşulunu sağlayan dönel hiperyüzeyler

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Rotational Hypersurfaces Satisfying $\Delta^{I}R = AR$ in the Four-Dimensional Euclidean Space

Highlights

- *Rotational hypersurface has zero mean curvature iff its Laplace-Beltrami operator vanishing*
- *Each element of the 4×4 order matrix A, which satisfies the condition $\Delta^{I}R = AR$, is zero*
- *Laplace-Beltrami operator of the rotational hypersurface depends on its mean curvature and the Gauss map*

Graphical Abstract

Rotational hypersurfaces in the 4-dimensional Euclidean space are discussed. Some relations of curvatures of hypersurfaces are given, such as the mean, Gaussian, and their minimality and flatness. The Laplace-Beltrami operator has been defined for 4-dimensional hypersurfaces depending on the first fundamental form. In addition, it is indicated that each element of the 4×4 order matrix A, which satisfies the condition $\Delta^{I}R = AR$, is zero, that is, the rotational hypersurface R is minimal.

Aim

We consider the rotational hypersurfaces in $\mathbb{E}^4$ to find its Laplace-Beltrami operator.

Design & Methodology

We indicate fundamental notions of $\mathbb{E}^4$. Considering differential geometry formulas in 3-space, we transform them in 4-space. Moreover, we use straight calculations by hand.

Originality

All findings in the paper are original.

Findings

We define rotational hypersurface using rotation matrix. We calculate curvatures of rotational hypersurface. Defining the Laplace-Beltrami operator (LBo for short), we compute the LBo of rotational hypersurface. Finally, we give the rotational hypersurface satisfying $\Delta^{I}R = AR$.

Conclusion

Rotational hypersurface has zero mean curvature iff its Laplace-Beltrami operator vanishing. Then, each element of the 4×4 order matrix A, which satisfies the condition $\Delta^{I}R = AR$, is zero. Finally, the Laplace-Beltrami operator of the rotational hypersurface depends on its mean curvature and the Gauss map.

Declaration of Ethical Standards

The author of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Rotational Hypersurfaces Satisfying $\Delta^{I}\mathbf{R} = \mathbf{AR}$ in the Four-Dimensional Euclidean Space

Araştırma Makalesi / Research Article

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ABSTRACT

In this study, rotational hypersurfaces in the 4-dimensional Euclidean space are discussed. Some relations of curvatures of hypersurfaces are given, such as the mean, Gaussian, and their minimality and flatness. In addition, Laplace-Beltrami operator has been defined for 4-dimensional hypersurfaces depending on the first fundamental form. Moreover, it is shown that each element of the 4×4 order matrix $\mathbf{A}$, which satisfies the condition $\Delta^{I}\mathbf{R} = \mathbf{AR}$, is zero, that is, the rotational hypersurface $\mathbf{R}$ is minimal.

Keywords: 4-dimensional Euclidean space, Laplace-Beltrami operator, rotational hypersurface, curvature.

Dört-Boyutlu Öklid Uzayında $\Delta^{I}\mathbf{R} = \mathbf{AR}$ Koşulunu Sağlayan Dönel Hiperyüzeyler

ÖZ

Bu çalışmada, 4-boyutlu Öklid uzayındaki dönel hiperyüzeyler ele alınmıştır. Hiperyüzeylerin ortalama, Gauss eğrilikleri hesaplanıp aralarındaki minimal ve düzlemsel olma durumları gibi bazı bağıntılar verilmiştir. Ayrıca, 4-boyutlu hiperyüzeyler için birinci temel forma bağlı olarak Laplace-Beltrami operatörü tanımlanmıştır. Üstelik, dönel yüzeyin $\Delta^{I}\mathbf{R} = \mathbf{AR}$ koşulunu sağlayan 4×4 mertebeli $\mathbf{A}$ matrisinin her elemanının sıfır olduğu, yani $\mathbf{R}$ dönel hiperyüzeyinin minimal olduğu gösterildi.

Anahtar Kelimeler: 4-boyutlu Öklid uzayı, Laplace-Beltrami operatörü, dönel hiperyüzey, eğrilik

1. INTRODUCTION

After Chen [5], finite type submanifold, (i.e. for Laplacian, coordinate functions are finite sum of eigen-functions) has been studied by [1-3,5-16,18,21-23,25-26].

In $\mathbb{E}^3$, Takahashi [24] constructed spheres and minimal surfaces are the only surfaces with $\Delta r = \lambda r$, $r \in \mathbb{R}$. Ferrandez et al [12] found $\Delta H = (A)_{3\times3}H$ which are either an open part of sphere or of a right circular cylinder or minimal. Choi and Kim [8] classified the helicoid depends on the first kind pointwise 1-type Gauss map. Dillen et al [9] gave $\Delta r = (A)_{3\times3}r + (B)_{3\times1}$ which are the circular cylinders, spheres, minimal surfaces. Senoussi and Bekkar [23] introduced helicoidal surfaces depends on three fundamental forms. Lawson [17] revealed general Laplace-Beltrami operator.

General rotational surfaces were originated by Moore [19,20] in $\mathbb{E}^4$. Ganchev and Milousheva [13] gave the counterpart of them in $\mathbb{E}^4_1$.

Arslan et al [2] worked generalized rotation surfaces, Dursun and Turgay [11] considered pseudo umbilical, minimal rotational surfaces. Recently, Altın et al [1] worked Monge hypersurfaces with density in $\mathbb{E}^4$.

We consider the rotational hypersurfaces in $\mathbb{E}^4$. We indicate fundamental notions of $\mathbb{E}^4$ in Section 2. In Section 3, we define rotational hypersurface using rotation matrix. We calculate curvatures of rotational hypersurface. Defining the Laplace-Beltrami operator (LBo for short) in Section 4, we compute the LBo of rotational hypersurface. Finally, we give the rotational hypersurface satisfying $\Delta^{I}\mathbf{R} = \mathbf{AR}$ in Section 5. We give a conclusion in Section 6.

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2. PRELIMINARIES

We introduce shape operator matrix $\mathbf{S}$, Gaussian curvature (GC for short) K, and the mean curvature (MC for short) H of hypersurface (hypface for short) $\mathbf{M}(r, \theta_1, \theta_2)$ in $\mathbb{E}^4$.

Whole work, we identify its transpose with a vector. Let $\mathbf{M}$ be an isometric immersion of a hypface M^3 in $\mathbb{E}^4$.

Definition 1. Inner product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}^4$ is defined by

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

Definition 2. Triple vector product of $\vec{x} = (x_1, x_2, x_3, x_4), \vec{y} = (y_1, y_2, y_3, y_4), \vec{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}^4$ is given by

$$\vec{x} \times \vec{y} \times \vec{z} = det\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

Definition 3. For a hypface $\mathbf{M}(r, \theta_1, \theta_2)$ in $\mathbb{E}^4$,

$$\det I=(EG-F^2)C-A^2G+2ABF-B^2E, \quad (1)$$

$$\det II=(LN-M^2)V-P^2N+2PTM-T^2L, \quad (2)$$

where I and II are fundamental form matrices, respectively, with coefficients:

$$E=\mathbf{M}_r\cdot\mathbf{M}_r,\quad F=\mathbf{M}_r\cdot\mathbf{M}_{\theta_1},\quad G=\mathbf{M}_{\theta_1}\cdot\mathbf{M}_{\theta_1},$$

$$A=\mathbf{M}_r\cdot\mathbf{M}_{\theta_2},\quad B=\mathbf{M}_{\theta_1}\cdot\mathbf{M}_{\theta_2},\quad C=\mathbf{M}_{\theta_2}\cdot\mathbf{M}_{\theta_2},$$

$$L=\mathbf{M}_{rr}\cdot e,\quad M=\mathbf{M}_{r\theta_1}\cdot e,\quad N=\mathbf{M}_{\theta_1\theta_1}\cdot e,$$

$$P=\mathbf{M}_{r\theta_2}\cdot e,\quad T=\mathbf{M}_{\theta_1\theta_2}\cdot e,\quad V=\mathbf{M}_{\theta_2\theta_2}\cdot e.$$

Here, the Gauss map is defined by

$$e=\frac{\mathbf{M}_r\times\mathbf{M}_{\theta_1}\times\mathbf{M}_{\theta_2}}{\left\|\mathbf{M}_r\times\mathbf{M}_{\theta_1}\times\mathbf{M}_{\theta_2}\right\|}. \quad (3)$$

Definition 4. Resulting matrix of $(I)^{-1}.(II)$ indicates shape operator matrix as follows

$$\mathbf{S}=\frac{1}{\det I}\left(s_{ij}\right)_{3\times3}, \quad (4)$$

where

$$s_{11}=(AB-CF)M+(BF-AG)P+(CG-B^2)L,$$
$$s_{12}=(AB-CF)N+(BF-AG)T+(CG-B^2)M,$$
$$s_{13}=(AB-CF)T+(BF-AG)V+(CG-B^2)P,$$
$$s_{21}=(AB-CF)L+(AF-BE)P+(CE-A^2)M,$$
$$s_{22}=(AB-CF)M+(AF-BE)T+(CE-A^2)N,$$
$$s_{23}=(AB-CF)P+(AF-BE)V+(CE-A^2)T,$$
$$s_{31}=(BF-AG)L+(AF-BE)M+(EG-F^2)P,$$
$$s_{32}=(BF-AG)M+(AF-BE)N+(EG-F^2)T,$$
$$s_{33}=(BF-AG)P+(AF-BE)T+(EG-F^2)T.$$

Definition 5. The formulas of the GC and the MC of hypface are, respectively, as follows

$$K=\det(\mathbf{S}), \quad (5)$$

$$H=\frac{1}{3}\mathrm{tr}(\mathbf{S}), \quad (6)$$

where

$$\mathrm{tr}(\mathbf{S})=\frac{1}{\det I}[-2(APG+BTE-ABM-ATF-BPF)$$
$$-A^2N-B^2L+(EN+GL-2FM)C+(EG-F^2)V].$$

When $H=0$ on $\mathbf{M}$, hypface $\mathbf{M}$ is called minimal.

3. ROTATIONAL HYPERSURFACE

Let us $\gamma:\mathrm{I}\subset\mathbb{R}\longrightarrow\Pi$ be a plane curve, ℓ be a line in Π in $\mathbb{E}^4$.

Definition 6. A rotational hypface in $\mathbb{E}^4$ is hypface rotating a profile curve γ about axis ℓ.

ℓ is spanned by $(0,0,0,1)^t$ and orthogonal matrix $Z(\theta_1,\theta_2)$ is defined by

$$\begin{pmatrix}\cos\theta_1\cos\theta_2 & -\sin\theta_1 & -\cos\theta_1\sin\theta_2 & 0\\ \sin\theta_1\cos\theta_2 & \cos\theta_1 & -\sin\theta_1\sin\theta_2 & 0\\ \sin\theta_2 & 0 & \cos\theta_2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad (7)$$

with $\theta_1,\theta_2\in\mathbb{R}$. Matrix Z supplies:

$$Z\ell=\ell,\ Z^tZ=ZZ^t=I_4,\ \det Z=1.$$

Profile curve is given by

$$\gamma(r)=\big(r,0,0,\varphi(r)\big),$$

where $\varphi(r):\mathrm{I}\subset\mathbb{R}\longrightarrow\mathbb{R}$ is C^∞ for all $r\in\mathrm{I}$. Rotational hypface, spanned by the vector $(0,0,0,1)^t$, holds:

$$\mathbf{R}(r,\theta_1,\theta_2)=Z(\theta_1,\theta_2).\gamma(r)^t. \quad (8)$$

In $\mathbb{E}^4$, let us rewrite (8) as follows

$$\mathbf{R}(r,\theta_1,\theta_2)=\begin{pmatrix} r\cos\theta_1\cos\theta_2\\ r\sin\theta_1\cos\theta_2\\ r\sin\theta_2\\ \varphi(r)\end{pmatrix}, \quad (9)$$

where $r\in\mathbb{R}-\{0\}$ and $\theta_1,\theta_2\in[0,2\pi)$.

Using derivatives of (9) depends on r,θ_1 and θ_2, get

$$I=\begin{pmatrix}1+\varphi'^2 & 0 & 0\\ 0 & r^2\cos^2\theta_2 & 0\\ 0 & 0 & r^2\end{pmatrix} \quad (10)$$

and

$$II=\begin{pmatrix}\frac{-r^2\varphi''\cos\theta_2}{\sqrt{\det I}} & 0 & 0\\ 0 & \frac{-r^3\varphi'\cos^3\theta_2}{\sqrt{\det I}} & 0\\ 0 & 0 & \frac{-r^3\varphi'\cos\theta_2}{\sqrt{\det I}}\end{pmatrix} \quad (11)$$

with $\varphi=\varphi(r)$, $\varphi'=\frac{d\varphi}{dr}$,

$$\det I=r^4(1+\varphi'^2)\cos^2\theta_2.$$

Using (3) on (9), we find

$$e_{\mathbf{R}}=\frac{1}{\sqrt{\det I}}\begin{pmatrix}r^2\varphi'\cos\theta_1\cos^2\theta_2\\ r^2\varphi'\sin\theta_1\cos^2\theta_2\\ r^2\varphi'\sin\theta_2\cos\theta_2\\ -r^2\cos\theta_2\end{pmatrix}, \quad (12)$$

Hence, we have

$$\mathbf{S}=\begin{pmatrix}\frac{-\varphi''}{W^{3/2}} & 0 & 0\\ 0 & \frac{-\varphi'}{rW^{1/2}} & 0\\ 0 & 0 & \frac{-\varphi'}{rW^{1/2}}\end{pmatrix},$$

where $W=1+\varphi'^2$. Using (5) and (6), respectively, we find followings

$$K=-\frac{\varphi'^2\varphi''}{r^2W^{5/2}},\ \ H=-\frac{r\varphi''+2\varphi'^3+2\varphi'}{3rW^{3/2}}.$$

Corollary 1. Let $\mathbf{R}:M^3\longrightarrow\mathbb{E}^4$ be an immersion in (9). Then, following results holds:

(*a*) M^3 has CGC iff

$$(\varphi')^4(\varphi'')^2-cr^4\left(1+\varphi'^2\right)^5=0.$$

(*b*) M^3 has CMC iff

$$(r\varphi''+2\varphi'^3+2\varphi')^2-9cr^2\left(1+\varphi'^2\right)^3=0.$$

(*c*) M^3 is flat iff

$$\varphi(r)=c_1r+c_2.$$

Proof. Calculating eq. $K=0$, i.e. $\varphi'^2\varphi''=0$, we find the solution.

(*d*) M^3 has ZMC iff

$$\varphi(r)=\pm\int\frac{dr}{\sqrt{c_1r^4-1}}+c_2=\mathrm{Elliptic}F(ir,i),$$

where $\mathrm{Elliptic}F(\phi,m)=\int_0^\phi(1-m\sin^2\theta)^{-1/2}d\theta$ is elliptic integral, and $\phi\in[-\pi/2,\pi/2]$.

Proof. Solving eq. $r\varphi''+2\varphi'^3+2\varphi'=0$, we see solution.

4. LAPLACE-BELTRAMI OPERATOR

Definition 7. For smooth $\phi=\phi(x_1,x_2,x_3)|_{\mathbf{D}\subset\mathbb{R}^3}$ of class C^3 of hypface $\mathbf{M}$, the LBo is defined by

$$\Delta^I\phi=\frac{1}{\sqrt{g}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial\phi}{\partial x^j}\right). \tag{13}$$

where $(g^{ij})=(g_{kl})^{-1}$ and $g=\det(g_{ij})$.

Then, the LBo $\Delta^I\mathbf{R}$ of $\mathbf{R}=\mathbf{R}(r,\theta_1,\theta_2)$ is as follows

$$\frac{1}{\sqrt{\det I}}\left\{\begin{array}{c}\frac{\partial}{\partial r}\left(\frac{(CG-B^2)\mathbf{R}_r-(AB-CF)\mathbf{R}_{\theta_1}+(BF-AG)\mathbf{R}_{\theta_2}}{\sqrt{\det I}}\right)\\-\frac{\partial}{\partial\theta_1}\left(\frac{(AB-CF)\mathbf{R}_r-(CE-A^2)\mathbf{R}_{\theta_1}+(AF-BE)\mathbf{R}_{\theta_2}}{\sqrt{\det I}}\right)\\\frac{\partial}{\partial\theta_2}\left(\frac{(BF-AG)\mathbf{R}_r-(AF-BE)\mathbf{R}_{\theta_1}+(EG-F^2)\mathbf{R}_{\theta_2}}{\sqrt{\det I}}\right)\end{array}\right\}. \tag{14}$$

Hence, the LBo of (9) is given by

$$\Delta^I\mathbf{R}=\frac{1}{\sqrt{\det I}}\left(\frac{\partial}{\partial r}\mathbf{U}-\frac{\partial}{\partial\theta_1}\mathbf{V}+\frac{\partial}{\partial\theta_2}\mathbf{W}\right).$$

By derivatives of r,θ_1,θ_2 on $\mathbf{U},\mathbf{V},\mathbf{W}$, respectively, we get

$$\Delta^I\mathbf{R}=\frac{r\varphi''+2\varphi'^3+2\varphi'}{rW^2}\begin{pmatrix}-\varphi'\cos\theta_1\cos\theta_2\\-\varphi'\sin\theta_1\cos\theta_2\\-\varphi'\sin\theta_2\\1\end{pmatrix}.$$

Remark 1. When $\Delta^I\mathbf{R}=0$, $r\neq0$, the system of equation are as follows

$$\begin{aligned}-\varphi'(r\varphi''+2\varphi'^3+2\varphi')\cos\theta_1\cos\theta_2&=0,\\-\varphi'(r\varphi''+2\varphi'^3+2\varphi')\sin\theta_1\cos\theta_2&=0,\\-\varphi'(r\varphi''+2\varphi'^3+2\varphi')\sin\theta_2&=0,\\r\varphi''+2\varphi'^3+2\varphi'&=0.\end{aligned}$$

By using Remark 1, we find following corollaries:

Corollary 2. While $\cos\theta_i\neq0$, and $\sin\theta_i\neq0$, then $\varphi'=0$. Hence, we obtain

$$\varphi=c=const.\Leftrightarrow\Delta^I\mathbf{R}=0.$$

Corollary 3. When $\varphi'\neq0$, $\cos\theta_i\neq0$, and $\sin\theta_i\neq0$, then we see

$$r\varphi''+2\varphi'^3+2\varphi'=0\ (\text{i.e.}\, H=0)\Leftrightarrow\Delta^I\mathbf{R}=0.$$

5. ROTATIONAL HYPERSURFACES SATISFYING $\Delta^I\mathbf{R}=\mathbf{AR}$ IN $\mathbb{E}^4$

Theorem 1. Assume $\mathbf{R}:M^3\longrightarrow\mathbb{E}^4$ be an immersion by (9). So, $\Delta^I\mathbf{R}=\mathbf{AR}$ iff M^3 has ZMC.

Proof. The Gauss map of (9) is

$$e=\frac{1}{\sqrt{W}}\begin{pmatrix}\varphi'\cos\theta_1\cos\theta_2\\\varphi'\sin\theta_1\cos\theta_2\\\varphi'\sin\theta_2\\-1\end{pmatrix},$$

where $W=1+\varphi'^2$. We use

$$-3He=\mathbf{AR}, \tag{15}$$

then we get

$$\begin{pmatrix}-(\Theta\,\varphi'+a_{11}\,r)\cos\theta_1\cos\theta_2-a_{12}\,r\sin\theta_1\cos\theta_2-a_{13}\,r\sin\theta_2\\-a_{21}\,r\cos\theta_1\cos\theta_2-(\Theta\,\varphi'+a_{22}\,r)\sin\theta_1\cos\theta_2-a_{23}\,r\sin\theta_2\\-a_{31}\,r\cos\theta_1\cos\theta_2-a_{32}\,r\sin\theta_1\cos\theta_2-(\Theta\,\varphi'+a_{33}\,r)\sin\theta_2\\\Theta\end{pmatrix}$$
$$=\begin{pmatrix}a_{14}\varphi\\a_{24}\varphi\\a_{34}\varphi\\a_{41}\,r\cos\theta_1\cos\theta_2+a_{42}\,r\sin\theta_1\cos\theta_2+a_{43}\,r\sin\theta_2+a_{44}\varphi\end{pmatrix},$$

where $\mathbf{A}$ is 4×4 matrix, and $\Theta=\frac{3H}{W}$. The eq. $\Delta^I\mathbf{R}=\mathbf{AR}$ by (10) and (15) gives following ODEs system

$$\begin{aligned}&-(\Theta\varphi'+a_{11}r)\cos\theta_1\cos\theta_2-a_{12}\,r\sin\theta_1\cos\theta_2-a_{13}\,r\sin\theta_2=a_{14}\varphi,\\&-a_{21}\,r\cos\theta_1\cos\theta_2-(\Theta\varphi'+a_{22}r)\sin\theta_1\cos\theta_2-a_{23}\,r\sin\theta_2=a_{24}\varphi,\\&-a_{31}\,r\cos\theta_1\cos\theta_2-a_{32}\,r\sin\theta_1\cos\theta_2-(\Theta\varphi'+a_{33}r)\sin\theta_2=a_{34}\varphi,\\&a_{41}\,r\cos\theta_1\cos\theta_2+a_{42}\,r\sin\theta_1\cos\theta_2+a_{43}\,r\sin\theta_2+a_{44}\varphi=\Theta.\end{aligned}$$

Differentiating ODEs twice with respect to θ_1, we have

$$a_{14}=a_{24}=a_{34}=a_{44}=0,\ \ \Theta=0. \tag{16}$$

From (16), we get

$$\begin{aligned}-a_{11}r\cos\theta_1-a_{12}r\sin\theta_1&=0,\\-a_{21}r\cos\theta_1-a_{22}r\sin\theta_1&=0,\\-a_{31}r\cos\theta_1-a_{32}r\sin\theta_1&=0,\\a_{41}r\cos\theta_1+a_{42}r\sin\theta_1&=0.\end{aligned}$$

Since the functions cos and sin are linear independent on θ_1, the coefficients $a_{ij} = 0$. Considering $\Theta = \frac{3H}{W}$, it is clear $H = 0$. Finally, **R** is a minimal hypface.

6. CONCLUSION

In this study, rotational hypfaces in $\mathbb{E}^4$ are investigated in detail by using its Gauss map, MC and the GC. According to findings, minimality and flatness cases are determined. Taking into account LBo on rotational hypface in four space, $H = 0 \Leftrightarrow \Delta^I \mathbf{R} = 0$ are presented. This means, rotational hypface has ZMC iff its LBo equals to zero. This result can be seen in Corollary 3, clearly. In addition, rotational hypface satisfies the condition $\Delta^I \mathbf{R} = \mathbf{AR}$, when matrix $\mathbf{A}$ occurs only $\mathbf{A} = \mathbf{0}_{4\times 4}$ for 4×4 matrix $\mathbf{A}$. That is, **R** is minimal hypface. This study is important because rotation matrix, rotational hypface, LBo could not defined before, clearly. Also, literature about the topic are limited to reveal and calculate the properties of this kind rotational hypfaces in $\mathbb{E}^4$.

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DECLARATION OF ETHICAL STANDARDS

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS' CONTRIBUTIONS

Erhan GÜLER: Performed the calculations and analyzed the results. Wrote the manuscript.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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