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Smarandache Curves of Spatial Quaternionic Bertrand Curve According to Frenet Frame

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Abstract. In this study, Frenet vectors of spatial quaternionic Bertrand curve pair were taken as the position vector. The obtained Smarandache curves from position vector were defined. Frenet vectors, the curvature and torsion of this curve were calculated. This later the Frenet apparatus were expressed in terms of Frenet apparatus of the spatial quaternionic Bertrand curve pair. Example related to the subject was found and their drawings were done with Maple program.

1. Introduction

Quaternion was first introduced by the Irish mathematician William Rowan Hamilton in 1843 in the form of generalized complex numbers. Each quaternion is accompanied by four units $\{1, e_1, e_2, e_3\}$, [9]. In 1987, Bharathi, K. and Nagaraj, M.'s "Quaternion Valued Function of a Real Variable Serret-Frenet Formulae" named article have shed light to many studies related to quaternions. In recent years, many studies have been done on quaternions. These studies are found in [2–4, 6–9, 13, 14, 16]. Many studies have been done on special curves in differential geometry. Studies on one of these, the Bertrand curve, are see in [17, 18]. Some studies of Smarandache curves are available in [1, 9–13, 15].

2. Preliminaries

A real quaternion is defined with q of the form
$\mathbf{Q} = \{q | q = d + ae_1 + be_2 + ce_3, d, a, b, c \in \mathbb{R}, e_1, e_2, e_3 \in \mathbb{R}^3\}$ such that

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1\,, \qquad e_1 \times e_2 = -e_2 \times e_1 = e_3, \\ e_1 &\times e_3 = -e_3 \times e_1 = e_2\,, \quad e_2 \times e_3 = -e_3 \times e_2 = e_1. \end{aligned}$$

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We put $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$. Then a quaternion q can rewrite as

$$q = S_q + V_q$$

where S_q and V_q are the scalar part and vectorial part of q, respectively, [5]. For $q_1 = S_{q_1} + V_{q_1}, q_2 = S_{q_2} + V_{q_2}$ quaternions, quaternionic summation, multiplication and conjugate operations are, respectively

$$\begin{aligned}
&q_1 + q_2 = S_{q_1} + V_{q_1} + S_{q_2} + V_{q_2} = S_{q_1+q_2} + V_{q_1+q_2}\\
&q_1 \times q_2 = S_{q_1}S_{q_2} - \langle V_{q_1}, V_{q_2}\rangle + S_{q_1}V_{q_2} + S_{q_2}V_{q_1} + V_{q_1} \wedge V_{q_2}\\
&\bar{q} = S_{q_1} - V_{q_1}
\end{aligned}$$

These expression the symmetric real-valued, non-degenerate, bilinear form as follows,

$$\langle , \rangle|_{\mathbf{Q}} : \mathbf{Q} \times \mathbf{Q} \to \mathbb{R}, \ \langle q_1, q_2\rangle|_{\mathbf{Q}} = \frac{1}{2}(q_1 \times \bar{q_2} + q_2 \times \bar{q_1}).$$

It is called the quaternionic inner product, [5]. Then the norm of q is

$$N(q) = \sqrt{\langle q, q\rangle|_{\mathbf{Q}}} = \sqrt{q \times \bar{q}},$$

A spatial quaternion set define that $\mathbf{Q}_H = \{q \in \mathbf{Q} | q + \bar{q} = 0\}$, [2]. Let $I = [0, 1]$ be an interval in the real line $\mathbf{R}$ and $s \in I$ be the are-length parameter along the smooth curve, [7]

$$\gamma : [0, 1] \to \mathbf{Q}_H, \ \gamma(s) = \sum_{i=1}^{3} \gamma_i(s)e_i, \ \ (1 \le i \le 3). \tag{1}$$

The tangent vector $\gamma'(s) = t(s)$ has unit length N(t(s))=1 for alls, [2]. Let γ be a differentiable spatial quaternions curve with arc-length parameter s and $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of γ at the point $\gamma(s)$, [6],

$$t(s) = \gamma'(s), \quad n_1(s) = \frac{\gamma''(s)}{N(\gamma''(s))}, \quad n_2(s) = t(s) \times n_1(s), \tag{2}$$

Let $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of $\gamma(s)$. Then Frenet formulae, curvature and the torsion are given by [6]

$$\begin{aligned}
t'(s) &= k(s)n_1(s)\,, \\
n_1'(s) &= -k(s)t(s) + r(s)n_2(s)\,, \\
n_2'(s) &= -r(s)n_1(s)
\end{aligned} \tag{3}$$

where $t(s)$, $n_1(s)$ and $n_2(s)$ are the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve, respectively, [2, 8].

Let $\{k(s), r(s)\}$ be the curvatures of $\gamma(s)$. Then curvature and the torsion are given by[6]

$$\begin{aligned}
k_{\beta_1} &= \frac{N(\beta' \times \beta'')}{N(\beta')^3} \\
r_{\beta_1} &= \frac{\langle \beta' \times \beta'', \beta''' \rangle|_Q}{\left(N(\beta' \times \beta'')\right)^2}.
\end{aligned} \tag{4}$$

Definition 2.1. *Let $\alpha : I \to \mathbf{Q}_H$ unit speed and $\alpha^* : I \to \mathbf{Q}_H$ differentiable two spatial quaternionic curves. If the principal normal vector n_1 of the curve α is linearly dependent on the principal vector n_1^* of the curve α^*, then the pair (α, α^*) is defined to be quaternionic Bertrand curves pair, [7].*

If the curve α^* is Bertrand partner of α and n_1 principal vector of α, then we may write that

$$\alpha^*(s) = \alpha(s) + \lambda n_1(s)\,, \quad \lambda = constant. \tag{5}$$

Theorem 2.2. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. The relations between the Frenet frames $\{t, n_1, n_2\}$ and $\{t^*, n_1^*, n_2^*\}$ are as follows*

$$\begin{aligned} t^*(s) &= \cos\theta t + \sin\theta n_2, \\ {n_1}^*(s) &= n_1, \\ {n_2}^*(s) &= -\sin\theta t + \cos\theta n_2 \end{aligned} \tag{6}$$

where $\angle(t, t^*) = \theta$, [7].

Theorem 2.3. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have*

$$\begin{aligned} k^*(s)\frac{ds^*}{ds} &= \cos\theta k - \sin\theta r\,, \\ r^*(s)\frac{ds^*}{ds} &= \sin\theta k + \cos\theta r, \ [7]. \end{aligned} \tag{7}$$

3. Smarandache Curves of Spatial Quaternionic Bertrand Curve according to Frenet Frame

Frenet vectors of a curve are taken as position vector and a regular curve is defined with this vector. This curve is called as Smarandache curve, [15]. In this study, (α^*, α) will be defined as quaternionic Bertrand curve pair. Curve α^* will be taken as main curve and the other curve α will be taken as Bertrand partner curve of curve α^*. Frenet vectors of α^* curve taken from the curve pair will be taken as position vector. Smarandache curves' Frenet apparatus defined by these position vectors will be calculated. The resulting Frenet apparatus will be expressed Frenet aparatus denominated belonging to α^* curve by using connecting equation between Bertrand curve pair Frenet apparatus.

Definition 3.1. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of curve α^* is shown with $\{t^*, {n_1}^*, {n_2}^*\}$,*

$$\beta_1(s) = \frac{1}{\sqrt{2}}(t^* + {n_1}^*) \tag{8}$$

regular curve drawn by vectors t^* and ${n_1}^*$ is called spatial quaternionic Smarandache curve β_1.

Theorem 3.2. *Frenet vectors of Smarandache curve β_1 are given as follows;*

$$\begin{aligned} t_{\beta_1(s)} &= \frac{-k^*t^* + k^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}}\,, \quad n_{1\beta_1}(s) = \frac{\omega_1 t^* + \phi_1 n_1^* + \sigma_1 n_2^*}{\sqrt{{\omega_1}^2 + {\phi_1}^2 + {\sigma_1}^2}}\,, \\ n_{2\beta_1}(s) &= \frac{(k^*\sigma_1 - r^*\phi_1)t^* + (k^*\sigma_1 + r^*\omega_1)n_1^* + (-k^*\phi_1 - k^*\omega_1)n_2^*}{\sqrt{({\omega_1}^2 + {\phi_1}^2 + \sigma_1^2)(2k^{*2} + r^{*2})}}. \end{aligned} \tag{9}$$

Herein, the coefficients are

$$\begin{aligned} \omega_1 &= -k^{*2}(2k^{*2} + r^{*2}) - r^*(r^*k^{*'} - k^*r^{*'}), \\ \phi_1 &= -k^{*2}(2k^{*2} + 3r^{*2}) - r^*(r^{*3} - r^*k^{*'} + k^*r^{*'}), \\ \sigma_1 &= k^*r^*(2k^{*2} + r^{*2}) - 2k^*(r^*k^{*'} - k^*r^{*'}). \end{aligned} \tag{10}$$

Proof. If derivative according to s_{β_1} arc parameter of curve $\beta_1(s)$ is taken, $t_{\beta_1}(s)$ and $t'_{\beta_1}(s)$ are given, respectively

$$t_{\beta_1(s)} = \frac{-k^*t^* + k^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}}, \quad t'_{\beta_1}(s) = \frac{\sqrt{2}\left(\omega_1 t^* + \phi_1 n_1^* + \sigma_1 n_2^*\right)}{\left(2k^{*2} + r^{*2}\right)^2}. \tag{11}$$

Herein, the coefficients are as seen in (10). From equation (2), principal vector $n_{1\beta_1}$ and binormal vector $n_{2\beta_1}$ are found as in (9). □

Theorem 3.3. *Curvature and torsion belonging to Smarandache curve β_1 are, respectively*

$$k_{\beta_1} = \frac{\sqrt{2({\omega_1}^2 + {\phi_1}^2 + {\sigma_1}^2)}}{\left(2k^{*2} + r^{*2}\right)^2}, r_{\beta_1} = \frac{\sqrt{2}\left(x_1\eta_1 + y_1\theta_1 + z_1\rho_1\right)}{{x_1}^2 + {y_1}^2 + {z_1}^2} \tag{12}$$

where coefficients are

$$\begin{aligned} \eta_1 &= k^{*3} + k^*(r^{*2} - 3k^{*'}) - k^{*''}, \quad \theta_1 = -k^{*3} - k^*(r^{*2} + 3k^{*'}) - 3r^*r^{*'} + k^{*''}, \\ \rho_1 &= -k^{*2}r^* - r^{*3} + 2r^*k^{*'} + k^*r^{*'} + r^{*''}, \\ x_1 &= r^*(2k^{*2} + r^{*2}) + k^*r^{*'} - k^{*'}r^*, \quad y_1 = k^{*'}r^* - k^*r^{*'}, \quad z_1 = 2k^{*3} + k^*r^{*2}. \end{aligned} \tag{13}$$

Proof. First, second and third derivatives of curve β_1 are, respectively

$${\beta_1}' = \frac{-k^*t^* + k^*{n_1}^* + r^*{n_2}^*}{\sqrt{2}}$$

$${\beta_1}'' = \frac{-(k^{*2} + k^{*'})t^* + (k^{*'} - k^{*2} - r^{*2}){n_1}^* + (k^*r^* + r^{*'}){n_2}^*}{\sqrt{2}},$$

$${\beta_1}''' = \frac{\eta_1 t^* + \theta_1 {n_1}^* + \rho_1 {n_2}^*}{\sqrt{2}}$$

where the coefficients are as seen in (13). From (4) equation, curvatures are found as in (12). □

Corollary 3.4. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of Smarandache curve β_1 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$\begin{aligned} t_{\beta_1}(s) &= \frac{-kt + (\cos\theta - \sin\theta)n_1 + rn_2}{\sqrt{k^2 + r^2 + (\cos\theta k - \sin\theta r)^2}}, \quad n_{1\beta_1}(s) = \frac{\bar{\omega}_1 t + \bar{\phi}_1 n_1 + \bar{\sigma}_1 n_2}{\sqrt{\bar{\omega}_1^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2}}, \\ n_{2\beta_1}(s) &= \frac{((k\cos\theta - r\sin\theta)\bar{\sigma}_1 - r\bar{\phi}_1)t + (k\bar{\sigma}_1 + r\bar{\omega}_1)n_1}{\sqrt{\left(k^2 + r^2 + (\cos\theta k - \sin\theta)^2\right)(\bar{\omega}_1^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2)}} \\ &\quad - \frac{(k\bar{\phi}_1 + (k\cos\theta - r\sin\theta)\bar{\omega}_1)n_2}{\sqrt{\left(k^2 + r^2 + (\cos\theta k - \sin\theta)^2\right)(\bar{\omega}_1^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2)}}. \end{aligned} \tag{14}$$

Herein, coefficients are

$$\begin{aligned}\bar{\omega}_1 &= (-k' - k^2\cos\theta + kr\sin\theta)(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2)\\ &\quad + k(k^2 + r^2 + (k\cos\theta - r\sin\theta)',\end{aligned}$$

$$\begin{aligned}\bar{\phi}_1 &= (-k^2 - r^2 + k'\cos\theta - r'\sin\theta)(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2)\\ &\quad -(k\cos\theta - r\sin\theta)(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2)',\end{aligned}$$

$$\begin{aligned}\bar{\sigma}_1 &= (kr\cos\theta - r^2\sin\theta + r')(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2)\\ &\quad -r(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2)'.\end{aligned}$$

Proof. If expression (6) instead of t^* and ${n_1}^*$ in curve β_1 is written, we have

$$\beta_1(s) = \frac{1}{\sqrt{2}}\Big(\cos\theta t(s) + n_1(s) + \sin\theta n_2(s)\Big).$$

If equations (6) and (7) into equation (9) and (25) are written, the proof is completed. □

Corollary 3.5. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_1 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$k_{\beta_1} = \frac{\sqrt{{\bar{\omega}_1}^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2}}{\Big(k^2 + r^2 + (\cos\theta k - \sin\theta)^2\Big)^{\frac{3}{2}}}, \quad r_{\beta_1} = \sqrt{2}\frac{\bar{x}_1\bar{\eta}_1 + \bar{y}_1\bar{\theta}_1 + \bar{z}_1\bar{\rho}_1}{\bar{x}_1^2 + \bar{y}_1^2 + \bar{z}_1^2}. \tag{15}$$

Herein, coefficients are

$$\bar{\eta}_1 = (-k' - k^2\cos\theta + kr\sin\theta)' - k(-k^2 - r^2 + (k\cos\theta - r\sin\theta)'),$$

$$\begin{aligned}\bar{\theta}_1 &= k(-k' - k^2\cos\theta + kr\sin\theta) + (-k^2 - r^2 + (k\cos\theta - r\sin\theta)')'\\ &\quad -r(kr\cos\theta - r^2\sin\theta + r'),\end{aligned}$$

$$\bar{\rho}_1 = r(-k^2 - r^2 + (k\cos\theta - r\sin\theta)') + (kr\cos\theta - r^2\sin\theta + r')',$$

$$\bar{x}_1 = (k\cos\theta - r\sin\theta)(kr\cos\theta - r^2\sin\theta + r') - r(-k^2 - r^2 + (k\cos\theta - r\sin\theta)'),$$

$$\bar{y}_1 = k(kr\cos\theta - r^2\sin\theta + r') + r(-k' - k^2\cos\theta + kr\sin\theta),$$

$$\begin{aligned}\bar{z}_1 &= -\Big(k(-k^2 - r^2 + (k\cos\theta - r\sin\theta)') + k(k\cos\theta - r\sin\theta)\\ &\quad .(-k' - k^2\cos\theta + kr\sin\theta)\Big).\end{aligned}$$

Proof. If equations (6) and (7) into equation (12) and (13) are written, the proof is completed. □

Definition 3.6. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of curve α^* is shown with $\{t^*, {n_1}^*, {n_2}^*\}$,*

$$\beta_2(s) = \frac{(n_1^* + {n_2}^*)}{\sqrt{2}} \tag{16}$$

regular curve drawn by vectors ${n_1}^*$ and ${n_2}^*$ is called spatial quaternionic Smarandache curve β_2.

Theorem 3.7. *The Frenet vectors of Smarandache curve β_2 are given as follows:*

$$t_{\beta_2}(s) = \frac{-kt^* - rn_1^* + rn_2^*)}{\sqrt{2r_2^* + k_2^*}}, \quad n_{1\beta_2}(s) = \frac{\omega_2 t^* + \phi_2 n_1^* + \sigma_2 n_2^*}{\sqrt{\omega_2{}^2 + \phi_2{}^2 + \sigma_2{}^2}},$$
$$n_{2\beta_2}(s) = \frac{-r^*(\sigma_2 + \phi_2)t^* + (r^*\omega_2 + k^*\sigma_2)n_1^* + (-k^*\phi_2 + r^*\omega_2)n_2^*}{\sqrt{(\omega_2{}^2 + \phi_2{}^2 + \sigma_2^2)(2r^{*2} + k^{*2})}}. \tag{17}$$

Herein, the coefficients are

$$\begin{aligned} \omega_2 &= 2r^{*2}(-k^{*'} + r^*r^*) + k^*r^*(k^{*2} + 2r^{*'}), \\ \phi_2 &= k^*(-k^{*3} - r^{*'}k^* + r^*k^{*'}) - r^{*2}(3k^{*2} + 2r^{*2}), \\ \sigma_2 &= k^{*2}(r^{*'} - r^{*2}) - r^*(2r^{*3} + k^*k^{*'}). \end{aligned} \tag{18}$$

Proof. If derivative is taken according to s_{β_2} arc parameter of curve $\beta_2(s)$, $t_{\beta_2}(s)$ and $t'_{\beta_2}(s)$ are given, respectively

$$t_{\beta_2(s)} = \frac{-k^*t^* - r^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}}, \quad t'_{\beta_1}(s) = \frac{\sqrt{2}\left(\omega_2 t^* + \phi_2 n_1^* + \sigma_2 n_2^*\right)}{\left(2k^{*2} + r^{*2}\right)^2}.$$

Herein, the coefficients are as seen in (18). From equation (2), principal vector $n_{1\beta_2}$ and binormal vector $n_{2\beta_2}$ are found as in (17). □

Theorem 3.8. *Curvature and torsion belonging to Smarandache curve β_2 are, respectively*

$$k_{\beta_2} = \sqrt{2}\frac{\sqrt{\omega_2{}^2 + \phi_2{}^2 + \sigma_2{}^2}}{\left(k^{*2} + 2r^{*2}\right)^2}, \quad r_{\beta_2} = \sqrt{2}\frac{x_2\eta_2 + y_2\theta_2 + z_2\rho_2}{x_2{}^2 + y_2{}^2 + z_2{}^2} \tag{19}$$

where coefficients are

$$\begin{aligned} \eta_2 &= -r^{*3}k^* + k^{*3} + k^{*'}r^* + 2k^*r^{*'} - k^{*''}, \\ \theta_2 &= r^{*3} - r^*k^{*2} - 3k^*k^{*'} + 3r^{*2}r^{*'} - r^{*'}, \\ \rho_2 &= r^{*3} + r^*k^{*2} - 3r^*r^{*'} - r^*r^{*''}, \\ x_2 &= r^*(2r^{*2} + k^{*2}), \quad y_2 = k^*r^{*'} - r^*k^{*'}, \quad z_2 = k^*(k^{*2} + 2r^{*2} + r^{*'}) - r^*k^{*'}. \end{aligned} \tag{20}$$

Proof. First, second and third derivatives of curve β_2 are, respectively

$$\beta_2{}' = \frac{-k^*t^* - r^*n_1{}^* + r^*n_2{}^*}{\sqrt{2}}$$

$$\beta_2'' = \frac{(-k^* + r^*k^*)t^* - (k^{*2} - k^{*2} + r^{*2} + r^{*'})n_1{}^* + (r^{*'} - r^{*2})n_2{}^*}{\sqrt{2}},$$

$$\beta_2''' = \frac{\eta_2 t^* + \theta_2 n_1{}^* + \rho_2 n_2{}^*}{\sqrt{2}}$$

where the coefficients are as seen in (20). From (4) equation, curvatures are found as in (19). □

Corollary 3.9. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of Smarandache curve β_2 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$\begin{aligned} t_{\beta_2}(s) &= \frac{-kt-(\sin\theta+\cos\theta r)n_1+rn_2}{\sqrt{k^2+r^2+(\sin\theta+\cos\theta r)^2}}, \quad n_{1\beta_2}(s)=\frac{\bar{\omega}_1 t+\bar{\phi}_1 n_1+\bar{\sigma}_1 n_2}{\sqrt{\bar{\omega}_1^2+\bar{\phi}_1^2+\bar{\sigma}_1^2}}, \\ n_{2\beta_2}(s) &= \frac{-(\bar{\sigma}_2(k\sin\theta+r\cos\theta)+r\bar{\phi}_2)t+(k\bar{\sigma}_2+r\bar{\omega}_2)n_1}{\sqrt{\big(k^2+r^2+(k\sin\theta+r\cos\theta)^2\big)(\bar{\omega}_2^2+\bar{\phi}_2^2+\bar{\sigma}_2^2)}} \\ &\quad+\frac{(-k\bar{\phi}_2+\big(k\sin\theta+r\cos\theta)\bar{\omega}_2\big)n_2}{\sqrt{\big(k^2+r^2+(k\sin\theta+r\cos\theta)^2\big)(\bar{\omega}_2^2+\bar{\phi}_2^2+\bar{\sigma}_2^2)}}. \end{aligned} \tag{21}$$

Herein, the coefficients are

$$\begin{aligned} \bar{\omega}_2 &= (-k'+k(k\sin\theta+r\cos\theta))(k^2+r^2+(k\sin\theta+r\cos\theta)^2) \\ &\quad +k(k^2+r^2+(k\sin\theta+r\cos\theta)^2)', \\ \bar{\phi}_2 &= (-k^2-r^2-(k'\sin\theta-r'\cos\theta))(k^2+r^2+(k\sin\theta+r\cos\theta)^2) \\ &\quad +(k\sin\theta+r\cos\theta)(k^2+r^2+(k\sin\theta+r\cos\theta)^2)', \\ \bar{\sigma}_2 &= (-r(k\sin\theta+r\cos\theta)+r')(k^2+r^2+(k\sin\theta+r\cos\theta)^2) \\ &\quad -r(k^2+r^2+(k\sin\theta+r\cos\theta)^2)'. \end{aligned}$$

Proof. If expression (6) instead of $n_1{}^*$ and $n_2{}^*$ in curve β_2 is written, we have

$$\beta_2(s)=\frac{1}{\sqrt{2}}\Big(-\sin\theta t+n_1+\cos\theta n_2\Big).$$

If equations (6) and (7) into equation (17) and (18) are written, the proof is completed. □

Corollary 3.10. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_2 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$k_{\beta_2} = \sqrt{2}\frac{\sqrt{\bar{\omega}_2{}^2+\bar{\phi}_2^2+\bar{\sigma}_2^2}}{\Big(k^2+r^2+\Big(\sin\theta k+\cos\theta r\Big)^2\Big)^{\frac{3}{2}}}, \quad r_{\beta_2} = \sqrt{2}\frac{\bar{x}_2\bar{\eta}_2+\bar{y}_2\bar{\theta}_2+\bar{z}_2\bar{\rho}_2}{\bar{x}_2^2+\bar{y}_2^2+\bar{z}_2^2}. \tag{22}$$

Herein, the coefficients are

$$\begin{aligned} \bar{\eta}_2 &= \Big(-k'+k(k\sin\theta+r\cos\theta)\Big)'+k\Big(k^2+r^2-(k'\sin\theta+r'\cos\theta)\Big), \\ \bar{\theta}_2 &= -kk'+(k^2+r^2)(k\sin\theta+r\cos\theta)-(2kk'+2rr'+(k''\sin\theta+r''\cos\theta))-rr', \\ \bar{\rho}_2 &= r\Big(-k'+k(k\sin\theta+r\cos\theta)\Big)+\Big(-r(k\sin\theta+r\cos\theta)+r'\Big), \\ \bar{x}_2 &= (k\sin\theta+r\cos\theta)(r(\sin\theta+r\cos\theta)-r')+r\Big(k^2+r^2-(k'\sin\theta+r'\cos\theta)\Big), \\ \bar{y}_2 &= k(-r(k\sin\theta+r\cos\theta)+r')+r\Big(-k'+k(k\sin\theta+r\cos\theta)\Big), \\ \bar{z}_2 &= k\Big(k^2+r^2-(k'\sin\theta+r'\cos\theta\Big)+(k\sin\theta+r\cos\theta)\Big(-k'+k(k\sin\theta+r\cos\theta)\Big). \end{aligned}$$

Proof. If equations (6) and (7) into equation (19) and (20) are written, the proof is completed. □

Definition 3.11. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of α^* curve is shown with $\{t^*, {n_1}^*, {n_2}^*\}$,*

$$\beta_3(s) = \frac{(t^* + n_2^*)}{\sqrt{2}} \tag{23}$$

regular curve drawn by vectors t^* and n_2^* is called spatial quaternionic Smarandache curve β_3.

Theorem 3.12. *Frenet vectors of Smarandache curve β_3 are given as follows:*

$$t_{\beta_3}(s) \;=\; {n_1}^*\,, \quad n_{1\beta_3}(s) = \frac{-k^*t^* + r^*{n_2}^*}{k^{*2} + r^{*2}}\,, \quad n_{2\beta_3}(s) = \frac{r^*t^* + k^*{n_2}^*}{\sqrt{k^{*2} + r^{*2}}}. \tag{24}$$

Proof. If derivative is taken according to s_{β_3} arc parameter of curve $\beta_3(s)$, $t_{\beta_3}(s)$ and $t'_{\beta_3}(s)$ are given, respectively

$$t_{\beta_3(s)} \;=\; \frac{(k^* - r^*)n_1^*}{\sqrt{2k^{*2} + r^{*2}}}\,, \quad {t_{\beta_3}}'(s) = \frac{\sqrt{2}\Big(-k^*t^* + r^*n_2^*\Big)}{k^* - r^*}.$$

From equation (2), principal vector $n_{1\beta_3}$ and binormal vector $n_{2\beta_3}$ are found as in (24). □

Theorem 3.13. *Curvature and torsion belonging to Smarandache curve β_3 are, respectively*

$$k_{\beta_3} = \frac{\sqrt{2(k^{*2} + r^{*2})}}{k^* - r^*}, \quad r_{\beta_3} \;=\; \sqrt{2}\frac{x_3\eta_3 + z_3\varphi_3}{{\eta_3}^2 + {\varphi_3}^2} \tag{25}$$

where coefficients are

$$\eta_3 \;=\; -3k^*k^{*'} + 2k^*r^{*'} + k^{*'}r^*\,, \quad \theta_3 = -k^{*3} + r^*k^{*2} - k^*r^{*2} + r^{*3} + k^{*''} - r^{*''},$$

$$\varphi_3 \;=\; k^*r^{*'} + 2k^{*'}r^* - 3r^*r^{*'}\,, \quad x_3 = r^*(k^* - r^*)^2\,, \quad z_3 = k^*(k^* - r^*)^2. \tag{26}$$

Proof. First, second and third derivatives of curve β_3 are, respectively

$${\beta_3}' = \frac{(k^* - r^*){n_1}^*}{\sqrt{2}}$$

$${\beta_3}'' = \frac{(-k^{*2} + k^*r^*)t^* + (k^{*'} - r^{*'}){n_1}^* + (k^*r^* - r^{*2}){n_2}^*}{\sqrt{2}}\,,$$

$${\beta_3}''' = \frac{\eta_3 t^* + \theta_3{n_1}^* + \rho_3{n_2}^*}{\sqrt{2}}.$$

From (4) equation, curvatures are found as in (25). □

Corollary 3.14. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of Smarandache curve β_3 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$t_{\beta_3}(s) \;=\; n_1\,, \quad n_{1\beta_3}(s) = \frac{\bar{\omega}_3 t + \bar{\phi}_3 n_1 + \bar{\sigma}_3 n_2}{\sqrt{{\bar{\omega}_3}^2 + {\bar{\phi}_3}^2 + {\bar{\sigma}_3}^2}}\,,$$

$$n_{2\beta_3}(s) \;=\; \frac{(\bar{\omega}_3 t - \bar{\sigma}_3 n_2)[k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)]}{\sqrt{(k^2 - r^2 - (k^2 - r^2)\sin 2\theta)({\bar{\omega}_3}^2 + {\bar{\phi}_3}^2 + {\bar{\sigma}_3}^2)}}. \tag{27}$$

Herein, coefficients are

$$\begin{aligned}
\bar{\omega_3} &= -k\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta},\\
\bar{\phi_3} &= \Big(k'(\cos\theta-\sin\theta)-r'(\cos\theta+\sin\theta)\Big)\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta}\\
&\quad-\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\Big(\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta}\Big)',\\
\bar{\sigma_3} &= \Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta}.
\end{aligned}$$

Proof. If expression (6) instead of t^* and ${n_2}^*$ in curve β_3 is written, we have

$$\beta_3(s)=\frac{1}{\sqrt{2}}\Big((\cos\theta-\sin\theta)t+(\sin\theta+\cos\theta)n_2\Big).$$

If equations (6) and (7) into equation (21) and (22) are written, the proof is completed. □

Corollary 3.15. *Let (α,α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_3 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$k_{\beta_3}=\sqrt{2}\frac{\sqrt{\bar{\omega_3}^2+\bar{\phi_3}^2+\bar{\sigma_3}^2}}{k^2-r^2-(k^2-r^2)\sin 2\theta},\quad r_{\beta_3}=\sqrt{2}\frac{\bar{x_3}\bar{\eta_3}+\bar{z_3}\bar{\varphi_3}}{\bar{\eta_3}^2+\bar{\varphi_3}^2}.$$

Herein, the coefficients are

$$\begin{aligned}
\bar{\eta_3} &= -k'\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)-2k\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)',\\
\bar{\theta_3} &= -k^2\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)-r^2\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\\
&\quad+\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)'',\\
\bar{\varphi_3} &= r'\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)+2r\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)',\\
\bar{x_3} &= k(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)))^2,\quad \bar{z_3}=r((\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2.
\end{aligned}$$

Proof. If equations (6) and (7) into equation (25) and (26) are written, the proof is completed. □

Definition 3.16. *Let (α,α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of α^* curve is shown with $\{t^*,{n_1}^*,{n_2}^*\}$,*

$$\beta_4(s)=\frac{(t^*+n_1^*+{n_2}^*)}{\sqrt{2}} \tag{28}$$

regular curve drawn by vectors t^*, ${n_1}^*$ and ${n_2}^*$ is called spatial quaternionic Smarandache curve β_4.

Theorem 3.17. *Frenet vectors of Smarandache curve β_4 are given as follows:*

$$\begin{aligned}
t_{\beta_4}(s) &= \frac{k^*t^*+(k^*-r^*){n_1}^*+r^*{n_2}^*}{\sqrt{2(k^*+r^*-k^*r^*)}},\quad {n_1}_{\beta_4}(s)=\frac{\omega_4t^*+\phi_4{n_1}^*+\sigma_4{n_2}^*}{\sqrt{{\omega_4}^2+{\phi_4}^2+{\sigma_4}^2}},\\
{n_2}_{\beta_4}(s) &= \frac{((k^*-r^*)\sigma_4-r^*\phi_4)t^*+(r^*\omega_4+k^*\sigma_4){n_1}^*}{\sqrt{(2{k^*}^2+2{r^*}^2-2k^*r^*)({\omega_4}^2+{\phi_4}^2+{\sigma_4}^2)}}\\
&\quad-\frac{(k^*\phi_4+(k^*-r^*)\omega_4){n_2}^*}{\sqrt{(2{k^*}^2+2{r^*}^2-2k^*r^*)({\omega_4}^2+{\phi_4}^2+{\sigma_4}^2)}}.
\end{aligned} \tag{29}$$

Herein, the coefficients are

$$
\begin{aligned}
\omega_4 &= k^{*2}(-2k^{*2} - 4r^{*2} + 4r^*k^* - k^{*2}r^{*'}) + k^*r^*(k^{*'} + 2r^{*2} + 2r^{*'}) - 2k^{*'}r^{*2},\\
\phi_4 &= k^{*2}(-2k^{*2} - 4r^{*2} + 2k^*r^* - r^{*'}) + r^{*2}(-2r^{*2} + 2k^*r^* + k^{*'}) + k^*r^*(k^{*'} - r^{*'}),\\
\sigma_4 &= 2k^{*2}(k^*r^* - 2r^{*2} + r^{*'}) + r^{*2}(4k^*r^* - 2r^{*2} + k^{*'}) - k^*r^*(r^{*'} + 2k^{*'}).
\end{aligned}
$$

Proof. If derivative is taken according to s_{β_4} arc parameter of curve $\beta_4(s)$, $t_{\beta_4}(s)$ and $t'_{\beta_4}(s)$ are given, respectively

$$
\begin{aligned}
t_{\beta_4(s)} &= \frac{-k^*t^* + (k^* - r^*)n_1^* + r^*n_2^*}{2(\sqrt{k^{*2} + r^{*2} - k^*r^*})^2}\\
t'_{\beta_4}(s) &= \frac{\sqrt{3}\left(\omega_4 t^* + \phi_4 n_1^* + \sigma_4 n_2^*\right)}{4\left(2k^{*2} + r^{*2}\right)^2}.
\end{aligned}
$$

Herein, the coefficients are as seen in (30). From equation (2), principal vector $n_{1\beta_4}$ and binormal vector $n_{2\beta_4}$ are found as in (29). □

Theorem 3.18. *Curvature and torsion belonging to Smarandache curve β_4 are, respectively*

$$
k_{\beta_4} = \frac{\sqrt{3}}{4}\frac{\sqrt{\omega_4^2 + \phi_4^2 + \sigma_4^2}}{(k^{*2} + r^{*2} - k^*r^*)^2}, \quad r_{\beta_4} = \frac{\sqrt{3}\left[\eta_4 x_4 + \theta_4 y_4 + \rho_4 z_4\right]}{x_4{}^2 + y_4{}^2 + z_4{}^2} \tag{30}
$$

where coefficients are

$$
\begin{aligned}
\eta_4 &= k^{*'}r^* - k^{*''} - 3k^*k^{*'} + 2k^*r^{*'} + k^{*3} + k^*r^{*2},\\
\theta_4 &= r^{*3} - k^{*3} - 3(k^*k^{*'} + r^*r^{*'}) - (-k^{*''} + r^{*''}) + k^*r^*(k^* - r^*),\\
\rho_4 &= r^{*''} - k^{*2}r^* - 3r^*r^{*'} - r^{*3} + 2r^*k^{*'} + k^*r^{*'},\\
x_4 &= 2k^*r^*(k^* - r^*) + k^*r^{*'} - r^*k^{*'} + 2r^{*3},\\
y_4 &= k^*r^{*'} - r^*k^{*'}, \quad z_4 = 2k^{*3} + k^*r^{*'} + 2k^*r^{*2} - 2k^{*2}r^* - k^{*'}r^*.
\end{aligned} \tag{31}
$$

Proof. First, second and third derivatives of curve β_4 are, respectively

$$
\beta_4{}' = \frac{-k^*t^* + (k^* - r^*)n_1{}^* + r^*n_2{}^*}{\sqrt{3}}
$$

$$
\beta_4'' = \frac{(-k^{*'} - k^{*2} + k^*r^*)t^* - (k^{*2} - k^{*'} + r^{*'} + r^{*2})n_1{}^* + (k^*r^* - r^{*2} + r^{*'})n_2{}^*}{\sqrt{3}},
$$

$$
\beta_4''' = \frac{\eta_4 t^* + \theta_4 n_1{}^* + \rho_4 n_2{}^*}{\sqrt{2}}
$$

where the coefficients are as seen in (31). From (4) equation, curvatures are found as in (30). □

Corollary 3.19. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of*

Smarandache curve β_4 in terms of Frenet apparatus of Bertrand partner curve are as follows:

$$t_{\beta_4}(s) = \frac{1}{\sqrt{2}}\frac{-kt+\big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\big)+rn_2}{\sqrt{(-2\cos\theta kr-\cos\theta\sin\theta k^2+\cos\theta\sin\theta r^2+k^2+kr+r^2)}},$$

$$n_{1\beta_4}(s) = \frac{\bar{\omega}_4 t+\bar{\phi}_4 n_1+\bar{\sigma}_4 n_2}{\sqrt{\bar{\omega}_4{}^2+\bar{\phi}_4{}^2+\bar{\sigma}_4{}^2}},$$

$$\begin{aligned} n_{2\beta_4}(s) &= \frac{\big((k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))\bar{\sigma}_4-r\bar{\phi}_4\big)t}{\sqrt{k^2+r^2+[k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)]^2(\bar{\omega}_4{}^2+\bar{\phi}_4{}^2+\bar{\sigma}_4{}^2)}} \\ &+\frac{(k\bar{\sigma}_4+r\bar{\omega}_4)n_1}{\sqrt{k^2+r^2+[k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)]^2(\bar{\omega}_4{}^2+\bar{\phi}_4{}^2+\bar{\sigma}_4{}^2)}} \\ &+\frac{-(k\bar{\phi}_4+(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))\bar{\omega}_4)n_2}{\sqrt{k^2+r^2+[k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)]^2(\bar{\omega}_4{}^2+\bar{\phi}_4{}^2+\bar{\sigma}_4{}^2)}}. \end{aligned} \tag{32}$$

Herein, the coefficients are

$$\begin{aligned} \bar{\omega}_4 &= \big(-k'-k(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))\big).\big(k^2+r^2+(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2\big) \\ &\quad +k\Big((k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2\Big)', \\ \bar{\phi}_4 &= \big(-k^2-r^2+(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))'\big) \\ &\quad .\big(k^2+r^2+(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2\big) \\ &\quad -(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)).\big(k^2+r^2+(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2\big)', \\ \bar{\sigma}_4 &= (r(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))+r').\big(k^2+r^2+(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2\big) \\ &\quad -r\big(k^2+r^2+(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2\big)'. \end{aligned}$$

Proof. If expression (6) instead of t^*, ${n_1}^*$ and ${n_2}^*$ in curve β_4 is written, we have

$$\beta_4 = \frac{1}{\sqrt{3}}\big((\cos\theta-\sin\theta)t+n_1+(\sin\theta+\cos\theta)n_2\big).$$

If equations (6) and (7) into (29) and (30) equations are written, the proof is completed. □

Corollary 3.20. *Let (α,α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_4 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$\begin{aligned} k_{\beta_4} &= \frac{\sqrt{3}\sqrt{\bar{\omega}_4{}^2+\bar{\phi}_4{}^2+\bar{\sigma}_4{}^2}}{\big((k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2+k^2+r^2\big)^{\frac{3}{2}}}, \\ r_{\beta_4} &= \sqrt{2}\frac{\bar{x}_4\bar{\eta}_4+\bar{y}_4\bar{\theta}_4+\bar{z}_4\bar{\rho}_4}{\bar{x}_4^2+\bar{y}_4^2+\bar{z}_4^2}. \end{aligned} \tag{33}$$

Herein, the coefficients are

$$\begin{aligned}
\bar{\eta_4} &= \Big(-k' - k(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))\Big)' \\
&\quad -k\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big), \\
\bar{\theta_4} &= k\Big(-k' - k(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big) - r\Big((rk - r^2)(\cos\theta - \sin\theta) + r'\Big) \\
&\quad +\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big)', \\
\bar{\rho_4} &= r\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big) + \Big((rk - r^2)(\cos\theta - \sin\theta) + r'\Big)', \\
\bar{x_4} &= \Big(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)\Big).\Big(r + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)) + r'\Big) \\
&\quad -r\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big), \\
\bar{y_4} &= k\Big(r(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)) + r'\Big) + r\Big(-k' - (k^2 + kr)(\cos\theta - \sin\theta)\Big), \\
\bar{z_4} &= k\Big(-k^2 - r^2 + [k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)]'\Big) \\
&\quad \Big((k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))\Big).\Big(-k' - k(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))\Big).
\end{aligned}$$

Proof. If equations (6) and (7) into equation (30) and (31) are written, the proof is completed. □

Example. Let be spatial quaternionic curve

$$\alpha(s) = \Big(\frac{\sqrt{2}}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big) + \frac{\sqrt{2}}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big), -\frac{2\sqrt{5}}{5}s, \frac{-\sqrt{2}}{2}\cos(\frac{\sqrt{5}}{5}s) + \frac{\sqrt{2}}{2}\sin(\frac{\sqrt{5}}{5}s)\Big)$$

and if taken as $\lambda = 1$, Bertrand partner belonging to this curve,

$$\alpha^*(s) = \Big(0, \frac{-2\sqrt{5}}{5}s, 0\Big).$$

In terms of definition, we obtain special Smarandache curves β_1, β_2, β_3 and β_4 according to Frenet frame of spatial quaternionic curve, (Figure 1).

$$\begin{aligned}
\beta_1(s) &= \Big(-\frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big), -\frac{\sqrt{2}}{2}, \frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big), \\
\beta_2(s) &= \Big(-\cos\Big(\frac{\sqrt{5}}{5}s\Big), 0, -\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big), \\
\beta_3(s) &= \Big(\frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big), -\frac{\sqrt{2}}{2}, -\frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big), \\
\beta_4(s) &= \Big(-\frac{\sqrt{6}}{3}\cos\Big(\frac{\sqrt{5}}{5}s\Big), -\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big)
\end{aligned}$$

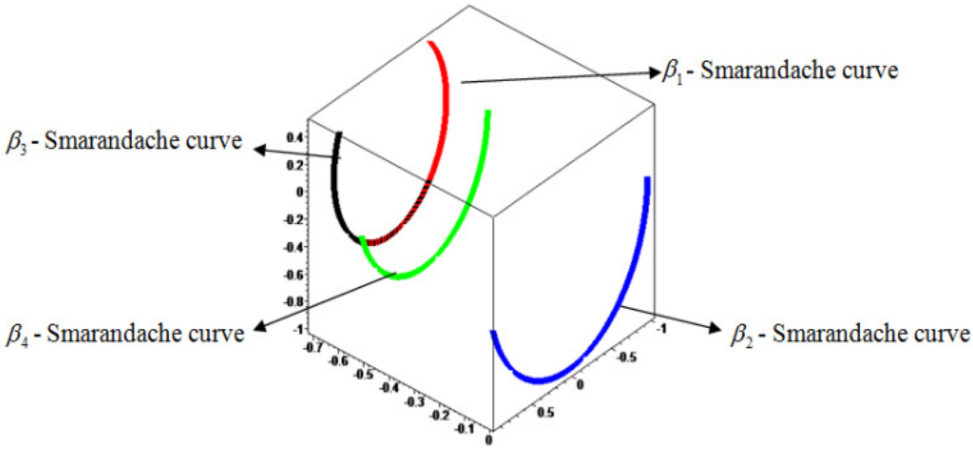

Figure 1: Smarandache Curves of Quaternionic Bertrand Curve

4. Conclusion

In this study, We have calculated the Smarandache curves of the Bertrand curve pairs. To put it simply, we derived curves from a curve according to a method. We found the Frenet frames and curvatures of these curves, which we call Smarandache curves. Finally, we found these results depending on the Frenet frames of the Bertrand curve pair. We saw that we could switch between Frenet frames. It is possible to examine whether these obtained curves are included in special curves.

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