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Domination Edge Integrity of Corona Products of P_n with $P_m, C_m, K_{1,m}$

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Abstract

Vulnerability is the most important concept in analysis of communication networks to disruption. Any network can be modelled by graphs. So measures defined on graphs gives an idea in design. Integrity is one of the well-known vulnerability measures interested in remaining structure of a graph after any failure. Domination is also an another popular concept in network design. Nowadays new vulnerability measures take a great role in network design. Recently designers take into account of any failure not only on nodes also on links which have special properties. A new measure edge domination integrity of a connected and undirected graph was defined by E. Kılıç and A. Beşirik such as $DI^{'}(G) = min\{ |S| + m(G-S) : S \subseteq E(G)\}$ where $m(G-S)$ is the order of a maximum component of $G-S$ and S is an edge dominating set. In this paper some results concerning this parameter on corona products of graph structures $P_n \odot P_m$, $P_n \odot C_m$, $P_n \odot K_{1,m}$ are presented.

1. Introduction

A communication network can be modeled by a graph G where nodes are represented by vertices $V(G)$ and links are represented by edges such as $E(G)$ respectively. Any communication network can be considered to be highly vulnerable to any disruption on its nodes or links. All graphs considered in this paper are connected, undirected, do not contain loops and multiple edges. First simple vulnerability measures are connectivity or edge connectivity which shows how easily a graph can be broken apart [1]. Later on, it is observed that these measures are not enough to compare the stability of network structures which have the same order. Most network designers are interested in what happens in the remaining part of the network after failures such as, how many nodes or links are still connected to each other and what is the communication between remaining parts. Integrity and the edge integrity concepts are interested in these questions. Both types of integrity were introduced by C. A. Barefoot et al. [2] and W. Goddard and H.C. Swart [3] has great contributions for this area. Integrity or edge integrity have been widely studied on specific graph families and relationships with other parameters and bounds were obtained K. S. Bagga et al. have presented many results about edge integrity in [4].

The order of a graph G will generally be denoted by n. For a real number x; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Domination is another important concept widely studied in graph theory. A subset S of V is called a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number $\gamma(G)$ (or γ for short) of G is the minimum cardinality taken over all dominating sets of G [5].
S. Mitchell and S.T. Hedetniemi [6] have introduced the concept of edge domination. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G. Later on S.Arumugamm [7] did some contributions to topic.

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Domination and integrity were examined together and many new vulnerability measures were defined. Some of them are domination integrity [8] [9] , domination edge integrity [10], and total domination integrity [11].

The concept of domination edge integrity of a connected graph as a new vulnerability parameter was defined by E. Kılıç and A. Beşirik [10] as follows.

Definition 1.1. *The domination edge integrity of a connected graph G is denoted by $DI^{'}(G)$ and is defined by*

$$DI^{'}(G) = \min\{|S| + m(G-S) \ : S \text{ is an edge dominating set }\}$$

where $m(G-S)$ is the order of a maximum component of $G-S$.

Definition 1.2. *A subset S of $E(G)$ is a $DI^{'}$-set if $DI^{'}(G) = \min\{|S| + m(G-S) \ : S \subseteq E(G)\}$ where S is an edge dominating set of G.*

$DI^{'}$ values of P_n, C_n, $K_{1,n}$, $K_{m,n}$ were presented and some properties for domination edge integrity value of a connected graph were determined in [10].

2. DI' of corona products P_n with some graphs

$DI^{'}$ values of some resulting graphs after corona operation of P_n with P_m, C_m, $K_{1,m}$ are found as follows.

Definition 2.1. *The corona $G_1 \odot G_2$ is defined as G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i'th node of G_1 to every node in the i'th copy of G_2 [1].*

Proposition 2.2. *Let n be an integer, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.*

Proof. There are 2 cases for integer n.

Case 1: Let n is an even integer. Then $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = \frac{n}{2}$. Hence; $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \frac{n}{2} + \frac{n}{2} = n$.

Case 2: Let n is an odd integer. Then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ (since n is odd, then $n-1$ is even) and $\lceil \frac{n}{2} \rceil = \frac{n-1}{2} + 1$. Hence, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \frac{n-1}{2} + \frac{n-1}{2} + 1 = n$.

□

In proof of all theorems, for graph G of order m, edge dominating sets X_1 and X_2 are taken which satisfies, $m(P_n \odot G - X_1) = 2(m+1)$ and $m(P_n \odot G - X_2) = m+1$ respectively (Figure 2.1). There is no other possible selection of edge dominating sets which gives $DI^{'}$ to be minimum. If X_3 is taken to be another edge dominating set, cardinality of X_3 is greater than both X_1 and X_2. It is easy to observe from structure of corona product of P_n with G. And also $m(P_n \odot G - X_1) > 2(m+1)$ since more edges are added. This selection does not give a minimum result.

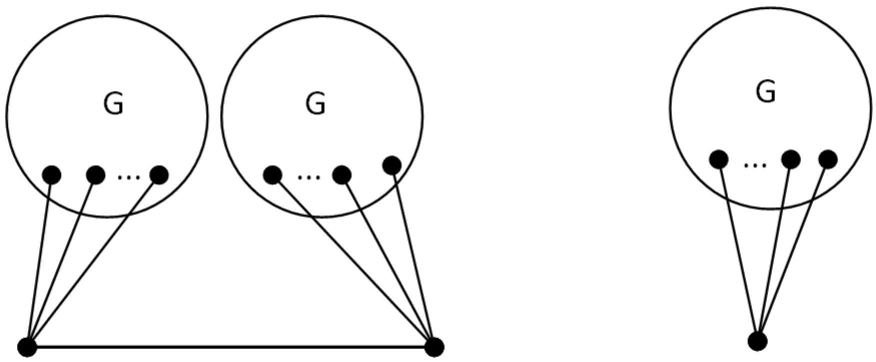

Figure 2.1: Maximum components of $(P_n \odot G) - X_1$ and $(P_n \odot G) - X_2$

For $n < 3$, $DI^{'}$ values of corona products of P_n with P_m, C_m, $K_{1,m}$ are obvious.

Theorem 2.3. *For $n \geq 3$ and $m \geq 2$, let n to be odd and*

$$A = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + 2m + 3,$$

$$B = n + n. \left\lceil \frac{m-1}{3} \right\rceil + m.$$

Then, $DI^{'}(P_n \odot P_m)$ is obtained as follows,

$$DI^{'}(P_n \odot P_m) = \begin{cases} A, & if\ m+2 < \frac{n-1}{2} \\ B, & if\ m+2 > \frac{n-1}{2} \\ A = B, & if\ m+2 = \frac{n-1}{2} \end{cases}$$

Proof. Let $V(P_n)=\{v_1,v_2,...,v_n\}$ and $V(P_m)=\{u_1,u_2,...,u_m\}$ for path graph P_n and P_m. Let $E(P_n)=\{v_1v_2,v_2v_3,...,v_{n-1}v_n\}$ and $E(P_m)=\{u_1u_2,...,u_{m-1}u_m\}$.
For $n\geq 3$ and n is odd, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n\odot P_m$ and $m(P_n\odot P_m-X_1)=2(m+1)$. X_1 is obtained as follows.
Let $S_1=\{v_2v_3,v_4v_5,...,v_{n-1}v_n\}\subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1|=\left\lfloor\frac{n-1}{2}\right\rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}|=\left\lceil\frac{m-1}{3}\right\rceil$ and $S_2=S_{2_1}\cup S_{2_2}\cup...\cup S_{2_n}$.
$S_1\cup S_2$ is not an edge dominating set of $P_n\odot P_m$ because some edges between v_1 and vertices of 1st copy of P_m are not dominated by any edges in $S_1\cup S_2$. So one of these edges (called e) is added $S_1\cup S_2$, $X_1=S_1\cup S_2\cup\{e\}$ is an edge dominating set of $P_n\odot P_m$. Therefore, $|X_1|=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+1$ and $m(P_n\odot P_m-X_1)=2(m+1)$. Thus,

$$DI'(P_n\odot P_m)\leq|X_1|+m(P_n\odot P_m-X_1)=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+3=DI'(P_n\odot P_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n\odot P_m$ and $m(P_n\odot P_m-X_2)=m+1$. X_2 is obtained as follows.
Let $S_1'=E(P_n)$. S_1' is an edge dominating set of P_n and $\left|S_1'\right|=n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}|=\left\lceil\frac{m-1}{3}\right\rceil$ and $S_2=S_{2_1}\cup S_{2_2}\cup...\cup S_{2_n}$.
$X_2=S_1'\cup S_2$ is an edge dominating set of $P_n\odot P_m$.Therefore, $|X_2|=n-1+n\left\lceil\frac{m-1}{3}\right\rceil$ and $m(P_n\odot P_m-X_2)=m+1$. Thus,

$$DI'(P_n\odot P_m)\leq|X_2|+m(P_n\odot P_m-X_2)=n+n\left\lceil\frac{m-1}{3}\right\rceil+m=DI'(P_n\odot P_m)_{X_2}.$$

Because of definition of DI', the relationship between $DI'(P_n\odot P_m)_{X_1}$ and $DI'(P_n\odot P_m)_{X_2}$ must be examined as follows.

1. If $m+2<\frac{n-1}{2}$, then we have

$$\begin{aligned}DI'(P_n\odot P_m)_{X_1} &= \left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+3=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+m+2+m+1\\ &< \frac{n-1}{2}+n\left\lceil\frac{m-1}{3}\right\rceil+\frac{n-1}{2}+m+1\\ &= n-1+n\left\lceil\frac{m-1}{3}\right\rceil+m+1=n+n\left\lceil\frac{m-1}{3}\right\rceil+m=DI'(P_n\odot P_m)_{X_2}.\end{aligned}$$

Since $DI'(P_n\odot P_m)_{X_1}<DI'(P_n\odot P_m)_{X_2}$, then $DI'(P_n\odot P_m)=DI'(P_n\odot P_m)_{X_1}=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+3$.

2. If $m+2>\frac{n-1}{2}$, then we have $DI'(P_n\odot P_m)_{X_2}<DI'(P_n\odot P_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI'(P_n\odot P_m)=DI'(P_n\odot P_m)_{X_2}=n+n\left\lceil\frac{m-1}{3}\right\rceil+m$.

3. If $m+2=\frac{n-1}{2}$, then we have $DI'(P_n\odot P_m)_{X_1}=DI'(P_n\odot P_m)_{X_2}$. Hence, $DI'(P_n\odot P_m)=DI'(P_n\odot P_m)_{X_1}=DI'(P_n\odot P_m)_{X_2}$.

□

Theorem 2.4. *For $n\geq 4$ and $m\geq 2$, let n to be even and*

$$A=\left\lfloor\frac{n-1}{2}\right\rfloor+n\left\lceil\frac{m-1}{3}\right\rceil+2m+4,$$

$$B=n+n\left\lceil\frac{m-1}{3}\right\rceil+m.$$

Then $DI'(P_n\odot P_m)$ is obtained as follows,

$$DI'(P_n\odot P_m)=\begin{cases}A, & if\ m+3<\left\lceil\frac{n-1}{2}\right\rceil\\ B, & if\ m+3>\left\lceil\frac{n-1}{2}\right\rceil\\ A=B, & if\ m+3=\left\lceil\frac{n-1}{2}\right\rceil\end{cases}$$

Proof. Let $V(P_n)=\{v_1,v_2,...,v_n\}$ and $V(P_m)=\{u_1,u_2,...,u_m\}$ for path graph P_n and P_m. Let $E(P_n)=\{v_1v_2,v_2v_3,...,v_{n-1}v_n\}$ and $E(P_m)=\{u_1u_2,...,u_{m-1}u_m\}$. For $n\geq 2$ and n is even, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n\odot P_m$ and $m(P_n\odot P_m-X_1)=2(m+1)$. X_1 is obtained as follows.

Let $S_1 = \{v_2v_3, v_4v_5, ..., v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \lfloor \frac{n-1}{2} \rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}| = \lceil \frac{m-1}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup ... \cup S_{2_n}$.
$S_1 \cup S_2$ is not an edge dominating set of $P_n \odot P_m$ because some edges between v_1 and vertices of 1st copy of P_m and v_n and vertices of nth copy of P_m are not dominated by any edges in $S_1 \cup S_2$. So one of edges between v_1 and vertices of 1st copy of P_m (called e_1) and one of edges between v_n and vertices of nth copy of P_m (called e_2) are added $S_1 \cup S_2$, $X_1 = S_1 \cup S_2 \cup \{e_1, e_2\}$ is an edge dominating set of $P_n \odot P_m$. Therefore, $|X_1| = \lfloor \frac{n-1}{2} \rfloor + n \lceil \frac{m-1}{3} \rceil + 2$ and $m(P_n \odot P_m - X_1) = 2(m+1)$. Thus,

$$DI^{'}(P_n \odot P_m) \leq |X_1| + m(P_n \odot P_m - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + 2m + 4 = DI^{'}(P_n \odot P_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot P_m$ and $m(P_n \odot P_m - X_2) = m+1$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of P_m and $|S_{2_i}| = \lceil \frac{m-1}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup ... \cup S_{2_n}$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot P_m$.Therefore, $|X_2| = n-1+n\lceil \frac{m-1}{3} \rceil$ and $m(P_n \odot P_m - X_2) = m+1$. Thus,

$$DI^{'}(P_n \odot P_m) \leq |X_2| + m(P_n \odot P_m - X_2) = n + n \left\lceil \frac{m-1}{3} \right\rceil + m = DI^{'}(P_n \odot P_m)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}(P_n \odot P_m)_{X_1}$ and $DI^{'}(P_n \odot P_m)_{X_2}$ must be examined as follows.

1. If $m+3 < \lceil \frac{n-1}{2} \rceil$, then we have

$$\begin{aligned} DI^{'}(P_n \odot P_m)_{X_1} &= \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + 2m + 4 = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + m + 3 + m + 1 \\ &< \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m-1}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil + m + 1, \text{ (by Proposition 2.2)} \\ &= n-1+n \left\lceil \frac{m-1}{3} \right\rceil + m + 1 = n + n \left\lceil \frac{m-1}{3} \right\rceil + m = DI^{'}(P_n \odot P_m)_{X_2}. \end{aligned}$$

Since $DI^{'}(P_n \odot P_m)_{X_1} < DI^{'}(P_n \odot P_m)_{X_2}$, then $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_1} = \lfloor \frac{n-1}{2} \rfloor + n \lceil \frac{m-1}{3} \rceil + 2m + 4$.

2. If $m+3 > \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot P_m)_{X_2} < DI^{'}(P_n \odot P_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_2} = n + n\lceil \frac{m-1}{3} \rceil + m$.

3. If $m+3 = \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot P_m)_{X_1} = DI^{'}(P_n \odot P_m)_{X_2}$. Hence, $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_1} = DI^{'}(P_n \odot P_m)_{X_2}$.

□

Theorem 2.5. *For $n \geq 3$ and $m \geq 3$, let n to be odd and*

$$A = \left\lfloor \frac{n-1}{2} \right\rfloor + n \left\lceil \frac{m}{3} \right\rceil + 2m + 3,$$

$$B = n + n \left\lceil \frac{m}{3} \right\rceil + m.$$

Then, $DI^{'}(P_n \odot C_m)$ is obtained as follows,

$$DI^{'}(P_n \odot C_m) = \begin{cases} A, & if\ m+2 < \frac{n-1}{2} \\ B, & if\ m+2 > \frac{n-1}{2} \\ A = B, & if\ m+2 = \frac{n-1}{2} \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $V(C_m) = \{u_1, u_2, ..., u_m\}$ for path graph P_n and cycle graph C_m. Let $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$ and $E(C_m) = \{u_1u_2, ..., u_{m-1}u_m, u_mu_1\}$.
For $n \geq 3$ and n is odd, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. X_1 is obtained as follows.
Let $S_1 = \{v_2v_3, v_4v_5, ..., v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \lfloor \frac{n-1}{2} \rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \lceil \frac{m}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup ... \cup S_{2_n}$.
$S_1 \cup S_2$ is not an edge dominating set of $P_n \odot C_m$ because some edges between v_1 and vertices of 1st copy of C_m are not dominated by any

edges in $S_1 \cup S_2$. So one of these edges (called e) is added $S_1 \cup S_2$, $X_1 = S_1 \cup S_2 \cup \{e\}$ is an edge dominating set of $P_n \odot C_m$. Therefore, $|X_1| = \lfloor \frac{n-1}{2} \rfloor + n \lceil \frac{m}{3} \rceil + 1$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_1| + m(P_n \odot C_m - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 3 = DI^{'}(P_n \odot C_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_2) = m+1$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \lceil \frac{m}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup \ldots \cup S_{2_n}$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot C_m$.Therefore, $|X_2| = n-1+n\lceil \frac{m}{3} \rceil$ and $m(P_n \odot C_m - X_2) = m+1$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_2| + m(P_n \odot C_m - X_2) = n + n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}(P_n \odot C_m)_{X_1}$ and $DI^{'}(P_n \odot C_m)_{X_2}$ must be examined as follows.

1. If $m+2 < \frac{n-1}{2}$, then we have

$$\begin{aligned} DI^{'}(P_n \odot C_m)_{X_1} &= \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 3 = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + m + 2 + m + 1 \\ &< \frac{n-1}{2} + n\left\lceil \frac{m}{3} \right\rceil + \frac{n-1}{2} + m + 1 \\ &= n - 1 + n\left\lceil \frac{m}{3} \right\rceil + m + 1 = n + n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}. \end{aligned}$$

Since $DI^{'}(P_n \odot C_m)_{X_1} < DI^{'}(P_n \odot C_m)_{X_2}$, then $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_1} = \lfloor \frac{n-1}{2} \rfloor + n\lceil \frac{m}{3} \rceil + 2m + 3$.

2. If $m+2 > \frac{n-1}{2}$, then we have $DI^{'}(P_n \odot C_m)_{X_2} < DI^{'}(P_n \odot C_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_2} = n + n\lceil \frac{m}{3} \rceil + m$.

3. If $m+2 = \frac{n-1}{2}$, then we have $DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$. Hence, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$.

□

Theorem 2.6. *For $n \geq 4$ and $m \geq 3$, let to n be even and*

$$A = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 4,$$

$$B = n + n\left\lceil \frac{m}{3} \right\rceil + m.$$

Then, $DI^{'}(P_n \odot C_m)$ is obtained as follows,

$$DI^{'}(P_n \odot C_m) = \begin{cases} A, & if\ m+3 < \lceil \frac{n-1}{2} \rceil \\ B, & if\ m+3 > \lceil \frac{n-1}{2} \rceil \\ A = B, & if\ m+3 = \lceil \frac{n-1}{2} \rceil \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $V(C_m) = \{u_1, u_2, ..., u_m\}$ for path graph P_n and cycle graph C_m. Let $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$ and $E(C_m) = \{u_1u_2, ..., u_{m-1}u_m, u_mu_1\}$.
For $n \geq 2$ and n is even, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. X_1 is obtained as follows.
Let $S_1 = \{v_2v_3, v_4v_5, ..., v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \lfloor \frac{n-1}{2} \rfloor$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \lceil \frac{m}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup \ldots \cup S_{2_n}$.
$S_1 \cup S_2$ is not an edge dominating set of $P_n \odot C_m$ because some edges between v_1 and vertices of 1st copy of C_m and v_n and vertices of nth copy of C_m are not dominated by any edges in $S_1 \cup S_2$. So one of edges between v_1 and vertices of 1st copy of C_m (called e_1) and one of edges between v_n and vertices of nth copy of C_m (called e_2) are added $S_1 \cup S_2$, $X_1 = S_1 \cup S_2 \cup \{e_1, e_2\}$ is an edge dominating set of $P_n \odot C_m$. Therefore, $|X_1| = \lfloor \frac{n-1}{2} \rfloor + n\lceil \frac{m}{3} \rceil + 2$ and $m(P_n \odot C_m - X_1) = 2(m+1)$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_1| + m(P_n \odot C_m - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 4 = DI^{'}(P_n \odot C_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot C_m$ and $m(P_n \odot C_m - X_2) = m+1$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
Let S_{2_i} is a minimum edge dominating set of ith copy of C_m and $|S_{2_i}| = \lceil \frac{m}{3} \rceil$ and $S_2 = S_{2_1} \cup S_{2_2} \cup \ldots \cup S_{2_n}$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot C_m$. Therefore, $|X_2| = n-1+n\lceil \frac{m}{3} \rceil$ and $m(P_n \odot C_m - X_2) = m+1$. Thus,

$$DI^{'}(P_n \odot C_m) \leq |X_2| + m(P_n \odot C_m - X_2) = n + n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}(P_n \odot C_m)_{X_1}$ and $DI^{'}(P_n \odot C_m)_{X_2}$ must be examined as follows.

1. If $m+3 < \lceil \frac{n-1}{2} \rceil$, then we have

$$\begin{aligned} DI^{'}(P_n \odot C_m)_{X_1} &= \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + 2m + 4 = \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + m + 3 + m + 1 \\ &< \left\lfloor \frac{n-1}{2} \right\rfloor + n\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil + m + 1, \text{ (by Proposition 2.2)} \\ &= n - 1 + n\left\lceil \frac{m}{3} \right\rceil + m + 1 = n + n\left\lceil \frac{m}{3} \right\rceil + m = DI^{'}(P_n \odot C_m)_{X_2}. \end{aligned}$$

Since $DI^{'}(P_n \odot P_m)_{X_1} < DI^{'}(P_n \odot P_m)_{X_2}$, then $DI^{'}(P_n \odot P_m) = DI^{'}(P_n \odot P_m)_{X_1} = \lfloor \frac{n-1}{2} \rfloor + n\lceil \frac{m-1}{3} \rceil + 2m + 4$.

2. If $m+3 > \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot C_m)_{X_2} < DI^{'}(P_n \odot C_m)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_2} = n + n\lceil \frac{m}{3} \rceil + m$.

3. If $m+3 = \lceil \frac{n-1}{2} \rceil$, then we have $DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$. Hence, $DI^{'}(P_n \odot C_m) = DI^{'}(P_n \odot C_m)_{X_1} = DI^{'}(P_n \odot C_m)_{X_2}$.

□

Theorem 2.7. *For $n \geq 3$, let $A = \lfloor \frac{n-1}{2} \rfloor + n + 2m + 4$ and $B = 2n + m + 1$. Then, $DI^{'}\left(P_n \odot K_{1,m}\right)$ is obtained as follows,*

$$DI^{'}\left(P_n \odot K_{1,m}\right) = \begin{cases} A, & if\ m+2 < \lceil \frac{n-1}{2} \rceil \\ B, & if\ m+2 > \lceil \frac{n-1}{2} \rceil \\ A = B, & if\ m+2 = \lceil \frac{n-1}{2} \rceil \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ for path graph P_n and for each ith copy of $K_{1,m}$ vertex set $V_i(K_{1,m}) = \{u_{1_i}, u_{2_i}, ..., u_{m_i}, u_{m+1_i}\}$.(u_{1_i} is central vertex of ith copy of $K_{1,m}$.) Let $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$ and $E_i(K_{1,m}) = \{u_{1_i}u_{2_i}, ..., u_{1_i}u_{m_i}, u_{1_i}u_{m+1_i}\}$.
For $n \geq 2$ and n is even, we have 2 cases as follows.

Case 1: Let X_1 is an edge dominating set of $P_n \odot K_{1,m}$ and $m(P_n \odot K_{1,m} - X_1) = 2(m+2)$. X_1 is obtained as follows.
Let $S_1 = \{v_2v_3, v_4v_5, ..., v_{n-1}v_n\} \subset E(P_n)$. S_1 is an edge dominating set of P_n and $|S_1| = \lfloor \frac{n-1}{2} \rfloor$.
$S_2 = \{v_1u_{1_1}, v_2u_{1_2}, ..., v_nu_{1_n}\}$ is a minimum edge dominating set of $P_n \odot K_{1,m} - E(P_n)$ and $|S_2| = n$.
$S_1 \cup S_2$ is an edge dominating set of $P_n \odot K_{1,m}$. Therefore, $|X_1| = \lfloor \frac{n-1}{2} \rfloor + n$ and $m(P_n \odot P_m - X_1) = 2(m+2)$. Thus,

$$DI^{'}\left(P_n \odot K_{1,m}\right) \leq |X_1| + m(P_n \odot K_{1,m} - X_1) = \left\lfloor \frac{n-1}{2} \right\rfloor + n + 2m + 4 = DI^{'}(P_n \odot P_m)_{X_1}.$$

Case 2: Let X_2 is an edge dominating set of $P_n \odot K_{1,m}$ and $m(P_n \odot K_{1,m} - X_2) = m+2$. X_2 is obtained as follows.
Let $S_1^{'} = E(P_n)$. $S_1^{'}$ is an edge dominating set of P_n and $\left|S_1^{'}\right| = n-1$.
$S_2 = \{v_1u_{1_1}, v_2u_{1_2}, ..., v_nu_{1_n}\}$ is a minimum edge dominating set of $P_n \odot K_{1,m} - E(P_n)$ and $|S_2| = n$.
$X_2 = S_1^{'} \cup S_2$ is an edge dominating set of $P_n \odot K_{1,m}$. Therefore, $|X_2| = n-1+n = 2n-1$ and $m(P_n \odot K_{1,m} - X_2) = m+2$. Thus,

$$DI^{'}\left(P_n \odot K_{1,m}\right) \leq |X_2| + m(P_n \odot K_{1,m} - X_2) = 2n + m + 1 = DI^{'}\left(P_n \odot K_{1,m}\right)_{X_2}.$$

Because of definition of $DI^{'}$, the relationship between $DI^{'}\left(P_n \odot K_{1,m}\right)_{X_1}$ and $DI^{'}\left(P_n \odot K_{1,m}\right)_{X_2}$ must be examined as follows.

1. If $m+2 < \lceil \frac{n-1}{2} \rceil$, then we have

$$
\begin{aligned}
DI^{'}\left(P_n\odot K_{1,m}\right)_{X_1} &= \left\lfloor\frac{n-1}{2}\right\rfloor+n+2m+4=\left\lfloor\frac{n-1}{2}\right\rfloor+n+m+2+m+2\\
&< \left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lceil\frac{n-1}{2}\right\rceil+m+2,\ \text{(by Proposition2.2)}\\
&= n-1+n+m+2=2n+m+1=DI^{'}\left(P_n\odot K_{1,m}\right)_{X_2}.
\end{aligned}
$$

Since $DI^{'}\left(P_n\odot K_{1,m}\right)_{X_1}<DI^{'}\left(P_n\odot K_{1,m}\right)_{X_2}$, then $DI^{'}\left(P_n\odot K_{1,m}\right)=DI^{'}\left(P_n\odot K_{1,m}\right)_{X_1}=\lfloor\frac{n-1}{2}\rfloor+n+2m+4$.

2. If $m+2>\lceil\frac{n-1}{2}\rceil$, then we have $DI^{'}\left(P_n\odot K_{1,m}\right)_{X_2}<DI^{'}\left(P_n\odot K_{1,m}\right)_{X_1}$. It can be proved in similar way as above. Therefore, $DI^{'}\left(P_n\odot K_{1,m}\right)=DI^{'}\left(P_n\odot K_{1,m}\right)_{X_2}=2n+m+1$.

3. If $m+2=\lceil\frac{n-1}{2}\rceil$, then we have $DI^{'}\left(P_n\odot K_{1,m}\right)_{X_1}=DI^{'}\left(P_n\odot K_{1,m}\right)_{X_2}$. Hence, $DI^{'}\left(P_n\odot K_{1,m}\right)=DI^{'}\left(P_n\odot K_{1,m}\right)_{X_1}=DI^{'}\left(P_n\odot K_{1,m}\right)_{X_2}$.

□

3. Conclusion

Domination concept and integrity are very valuable measures for network designers. Due to the changes of network models and design styles based on demands, measures on nodes or links which have specific properties became more important. For example edge domination and integrity are some of these concepts related to these specific properties. Edge domination only gives an idea about the communication on graph model after any faiulers on edges (links) and integrity itself gives information about only the stability of graph model of network based on edges. But our new measure domination edge integrity [10] combines these two important concepts. Corona operation is one of those commonly used operations in network design which has applications. In this paper, domination edge integrity of some graphs under corona operation is examined such as $P_n\odot P_m$, $P_n\odot C_m$, $P_n\odot K_{1,n}$ and results are obtained. For future work we plan to extend our results on other important classes of graphs under corona operation and generalize the obtained results.

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