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On Riemannian Manifolds Admitting a Type of Semi-Symmetric Pseudo Symmetric-Connection

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Abstract

The object of the present paper is to study a special type of semi-symmetric pseudo symmetric-connection on a Riemannian manifold. Finally, we has been studied some properties on Riemannian manifold with respect to a special type of semi-symmetric pseudo symmetric connection.

Keywords: *semi-symmetric pseudo symmetric connection; Levi-Civita connection; recurrent; cyclic Ricci tensor; Einstein's equation.*
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1. Introduction

In 1924, Friedmann and Schouten [7] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\bar{\nabla}$ satisfies $T(X,Y) = \pi(Y)X - \pi(X)Y$, where π is a $1-$ form and ρ is a vector field defined by $\pi(X) = g(X,\rho)$ for all vector fields $X,Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M and g be the Riemannian metric.
A semi-symmetric connection $\bar{\nabla}$ satisfying $\bar{\nabla}g \neq 0$, was initiated by Prvanovic [11] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [14]. The semi-symmetric connection $\bar{\nabla}$ is said to be a semi-symmetric non-metric connection.
After Prvanovic [11] and Andonie [14] continued the systematic study of semi-symmetric non-metric connection by Agashe and Chafle [13], Barua and Mukhopadhyay [10], Liang [15], Barman ([1], [2], [3], [4]), Barman and De [8], Barman and Ghosh [6] and many others.
A non-flat Riemannian manifold (M^n,g), $n \geq 2$ is said to be a pseudo symmetric manifold [12] if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = {} & 2\pi(X)R(Y,Z)W + \pi(Y)R(X,Z)W \\ & + \pi(Z)R(Y,X)W + \pi(W)R(Y,Z)X \\ & + g(R(Y,Z)W,X)\rho.\end{aligned}$$

After introduction in section 2, we define a special type of semi-symmetric pseudo symmetric-connection on Riemannian manifolds. Section 3 is devoted to establish the relation between the curvature tensors with respect to the special type of the semi-symmetric pseudo symmetric-connection and the Levi-Civita connection on Riemannian manifolds. Finally, we have been discussed some properties on Riemannian manifold with respect to a special type of semi-symmetric pseudo symmetric-connection.

2. Semi-symmetric pseudo symmetric-connection on Riemannian manifolds

This section deals with a type of semi-symmetric pseudo symmetric connection on a Riemannian manifold. Let (M,g) be a Riemannian Manifold with the Levi-Civita connection ∇ and we define a linear connection $\bar{\nabla}$ on M by

$$\bar{\nabla}_X Y = \nabla_X Y - \pi(X)Y - g(X,Y)\rho. \tag{2.1}$$

Using (2.1), the torsion tensor T of M with respect to the connection $\bar{\nabla}$ is given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = \pi(Y)X - \pi(X)Y. \tag{2.2}$$

The linear connection $\bar{\nabla}$ satisfying (2.2) is a semi-symmetric connection. So the equation (2.1) turns into

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y,Z) - g(Y,\bar{\nabla}_X Z) = {} & 2\pi(X)g(Y,Z) \\ & + \pi(Y)g(X,Z) + \pi(Z)g(X,Y) \neq 0.\end{aligned} \tag{2.3}$$

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The linear connection $\bar{\nabla}$ satisfying (2.2) and (2.3) is called a semi-symmetric non-metric connection. Since the manifold is satisfied $(\bar{\nabla}_X g)(Y,Z) = 2\pi(X)g(Y,Z) + \pi(Y)g(X,Z) + \pi(Z)g(X,Y)$, then the manifold is called pseudo symmetric manifolds. Therefore, the semi-symmetric non-metric connection will be semi-symmetric pseudo symmetric-connection.
The linear connection $\bar{\nabla}$ define by (2.1) satisfying (2.2) and (2.3) is a type of semi-symmetric pseudo symmetric-connection on Riemannian manifolds.
Conversely, we show that a linear connection $\bar{\nabla}$ defined on M satisfying (2.2) and (2.3) is given by (2.1). Let H be a tensor field of type $(1,2)$ and

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y). \tag{2.4}$$

Then we conclude that

$$T(X,Y) = H(X,Y) - H(Y,X). \tag{2.5}$$

Further using (2.4), it follows that

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y,Z) - g(Y,\bar{\nabla}_X Z) = -g(H(X,Y),Z)\\ -g(Y,H(X,Z)).\end{aligned} \tag{2.6}$$

In view of (2.3) and (2.6) yields,

$$g(H(X,Y),Z) + g(Y,H(X,Z)) = -2\pi(X)g(Y,Z) - \pi(Y)g(X,Z) - \pi(Z)g(X,Y). \tag{2.7}$$

Also using (2.7) and (2.5), we derive that

$$\begin{aligned}g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X) = 2g(H(X,Y),Z) + 2\pi(Y)g(X,Z)\\ +2\pi(X)g(Y,Z).\end{aligned}$$

From the above equation yields,

$$\begin{aligned}g(H(X,Y),Z) = \frac{1}{2}[g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X)]\\ -\pi(Y)g(X,Z) - \pi(X)g(Y,Z).\end{aligned} \tag{2.8}$$

Let T' be a tensor field of type $(1,2)$ given by

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{2.9}$$

Adding (2.2) and (2.9), we obtain

$$T'(X,Y) = \pi(X)Y - g(X,Y)\rho. \tag{2.10}$$

From (2.8) we have by using (2.9) and (2.10)

$$\begin{aligned}g(H(X,Y),Z) = \frac{1}{2}[g(T(X,Y),Z) + g(T'(X,Y),Z) + g(T'(Y,X),Z)]\\ -\pi(Y)g(X,Z) - \pi(X)g(Y,Z) = -\pi(X)g(Y,Z)\\ -\pi(Z)g(X,Y).\end{aligned} \tag{2.11}$$

Now contracting Z in (2.11), implies that

$$H(X,Y) = -\pi(X)Y - g(X,Y)\rho. \tag{2.12}$$

Combining (2.4) and (2.12), it follows that

$$\bar{\nabla}_X Y = \nabla_X Y - \pi(X)Y - g(X,Y)\rho.$$

Now, we are in a strong position to state the following theorem:

Theorem 2.1. *The linear connection $\bar{\nabla}_X Y = \nabla_X Y - \pi(X)Y - g(X,Y)\rho$, is a special type of semi-symmetric pseudo symmetric-connection iff the connection $\bar{\nabla}$ satisfying $T(X,Y) = \pi(Y)X - \pi(X)Y$ and $(\bar{\nabla}_X g)(Y,Z) = 2\pi(X)g(Y,Z) + \pi(Y)g(X,Z) + \pi(Z)g(X,Y)$ on Riemannian manifolds.*

3. Curvature tensor of a Riemannian manifold with respect to the semi-symmetric pseudo symmetric-connection

In this section we obtain the expressions of the curvature tensor and the Ricci tensor of M with respect to the semi-symmetric pseudo symmetricconnection defined by (2.1).
Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the semi-symmetric pseudo symmetric -connection $\bar{\nabla}$ by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z, \tag{3.1}$$

where $X,Y,Z \in \chi(M)$, the set of all differentiable vector fields on M.
Using (2.1) in (3.1), we obtain

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z - (\nabla_X\pi)(Y)Z + (\nabla_Y\pi)(X)Z + 2\pi(X)g(Y,Z)\rho \\ -2\pi(Y)g(X,Z)\rho.\end{aligned} \tag{3.2}$$

The torsion tensor is recurrent with respect to the semi-symmetric pseudo symmetric-connection $\bar{\nabla}$, that means

$$(\bar{\nabla}_X T)(Y,Z) = \pi(X)T(Y,Z). \tag{3.3}$$

The equation (3.3) with the help of the equation (2.2) will be

$$(\bar{\nabla}_X C_1^1 T)(Y) = (n-1)\pi(X)\pi(Y). \tag{3.4}$$

From the equation (2.2), we implies that

$$(\bar{\nabla}_X C_1^1 T)(Y) = (n-1)(\bar{\nabla}_X\pi)(Y). \tag{3.5}$$

Combining (3.4) and (3.5), we conclude that

$$(\bar{\nabla}_X\pi)(Y) = \pi(X)\pi(Y). \tag{3.6}$$

So the equation (3.6) with the help of the equation (2.1) turns into

$$(\nabla_X\pi)(Y) = -\pi(\rho)g(X,Y). \tag{3.7}$$

By making use of (3.7) in (3.2), we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + 2\pi(X)g(Y,Z)\rho - 2\pi(Y)g(X,Z)\rho. \tag{3.8}$$

So the equation (3.8) turns into

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z,$$

and

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \tag{3.9}$$

We call (3.9) the first Bianchi identity with respect to a special type semi symmetric pseudo symmetric-connection on Riemannian manifolds.
Taking the contrations of (3.8), it follows that

$$\bar{S}(Y,Z) = S(Y,Z) + 2\pi(\rho)g(Y,Z) - 2\pi(Y)\pi(Z), \tag{3.10}$$

where $\bar{S}$ and S denote the Ricci tensors of M with respect to $\bar{\nabla}$ and ∇ respectively. From (3.10), implies that

$$\bar{S}(Y,Z) = \bar{S}(Z,Y).$$

Summing up all of above equations we can state the following proposition:

Proposition 3.1. *For a Riemannian manifold M with respect to a special type of semi-symmetric pseudo symmetric-connection $\bar{\nabla}$, whose torsion tensor T is recurrent*
(i) The curvature tensor $\bar{R}$ is given by $\bar{R}(X,Y)Z = R(X,Y)Z + 2\pi(X)g(Y,Z)\rho - 2\pi(Y)g(X,Z)\rho$,
(ii) The Ricci tensor $\bar{S}$ is given by $\bar{S}(Y,Z) = S(Y,Z) + 2\pi(\rho)g(Y,Z) - 2\pi(Y)\pi(Z)$,
(iii) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,
(iv) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$,
(v) The Ricci tensor $\bar{S}$ is symmetric.

4. Some properties on Riemannian manifold with respect to a special type of semi-symmetric pseudo symmetric-connection

General relativity flows from Einstein's equation [9] given by

$$S(X,Y)-\frac{1}{2}rg(X,Y)+\lambda g(X,Y)=\kappa T(X,Y), \tag{4.1}$$

where $S(X,Y)$ is the Ricci tensor of type $(0,2)$ of the spacetime, r is the scalar curvature, $T(X,Y)$ is the energy-momentum tensor of type $(0,2)$, λ is the cosmological constant and κ is the gravitational constant. Einstein's equation without cosmological constant is given by

$$S(X,Y)-\frac{1}{2}rg(X,Y)=\kappa T(X,Y). \tag{4.2}$$

The equation (4.1) and (4.2) of Einstein imply that "matter determines the geometry of spacetimes and conversely that the motion of matter is determined by the metric tensor of the space which is not fiat [5]".
The energy-momentum tensor is said to describe a Perfect fluid [9] if

$$T(X,Y)=(\sigma+p)\eta(X)\eta(Y)+\rho g(X,Y), \tag{4.3}$$

where σ is the energy density and p is the isotropic pressure of the fluid.
Combining (4.2) and (4.3), it implies that

$$S(X,Y)-(\frac{1}{2}r+\kappa p)g(X,Y)=\kappa(\sigma+p)\eta(X)\eta(Y). \tag{4.4}$$

Definition 4.1. *A Riemannian manifold (M^n,g) is said to have cyclic Ricci tensor S with respect to the Levi-Civita connection ∇ if its Ricci tensor S [9] satisfies the condition*

$$(4.5)(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=0. \tag{4.5}$$

So the equation (4.5) with the help of (4.4) and (3.7) turn into

$$\begin{aligned}&(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=\\&-4\pi(\rho)\kappa(\sigma+p)[\pi(X)g(Y,Z)+\pi(Y)g(Z,X)\\&\qquad+\pi(Z)g(X,Y)].\end{aligned} \tag{4.6}$$

If $(\sigma+p)=0$, then from the above equation (4.6), it implies that $(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=0$.
In view of above discussions we state the following theorem:

Theorem 4.2. *If a Riemannian manifold admits a semi-symmetric pseudo symmetric connection $\bar{\nabla}$, whose torsion tensor T is recurrent and a perfect fluid spacetime satisfies Einstein's field equation without cosmological constant, then the Ricci tensor of the Riemannian manifold is cyclic provided such a spacetime is isentropic and represents inflation and also the fluid behaves as a cosmological constant.*

If $\bar{S}=0$, then the equation (3.10) will be

$$S(Y,Z)=-2\pi(\rho)g(Y,Z)+2\pi(Y)\pi(Z). \tag{4.7}$$

Combining (4.4) and (4.7), we get

$$[2-\kappa(\sigma+p)]\pi(Y)\pi(Z)-[2\pi(\rho)+\frac{1}{2}r+\kappa p]g(Y,Z)=0. \tag{4.8}$$

Putting $Z=\rho$ in (4.8), we conclude that

$$r=-2\kappa[\pi(\rho)(\sigma+p)+p]. \tag{4.9}$$

Since pressure in such a spacetime without pure matter $\sigma=0$, the above equation (4.9), it follows that

$$r=-2\kappa p[\pi(\rho)+1].$$

Hence, we can state the following theorem:

Theorem 4.3. *If the Ricci tensor of Riemannian manifold admits a semisymmetric pseudo symmetric connection $\bar{\nabla}$, whose torsion tensor T is recurrent is flat and a perfect fluid spacetime satisfies Einstein's field equation without cosmological constant, then the scalar curvature of the Riemannian manifold with respect to the Levi-Civita connection is equal to $-2\kappa p[\pi(\rho)+1]$ provided pressure in such a spacetime without pure matter $\sigma=0$.*

Putting $Z=\rho$ in (4.7), we implies that

$$S(Y,\rho)=0.$$

So, we obtain the following theorem:

Theorem 4.4. *If the Ricci tensor of Riemannian manifold admits a semisymmetric pseudo symmetric connection $\bar{\nabla}$, whose torsion tensor T is recurrent is flat, ρ is eigen vector corresponding to the eigen value is zero.*

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