

## Smarandache curves of Anti-Salkowski curve according to the spherical indicatrix curve of the unit darboux vector

*Anti-Salkowski eğrisinin birim darboux vektöründen elde edilen smarandache eğrileri*

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• Geliş tarihi / Received: 26.04.2020

• Düzeltilerek geliş tarihi / Received in revised form: 06.09.2021

• Kabul tarihi / Accepted: 24.09.2021

### Abstract

In this paper, we have defined special Smarandache curves according to Sabban frame formed by the unit Darboux vector of Anti-Salkowski curve. Next, the Sabban frame belonging to these curves have been constituted. Last, the geodesic curvatures of these Smarandache curves have been calculated and an example for each curve has been illustrated.

**Keywords:** Anti-Salkowski curve, Sabban frame, Smarandache curves, Unit darboux vector

### Öz

*Bu çalışmada, Sabban çatısına göre anti-Salkowski eğrisinin birim Darboux vektörlerinden elde edilen özel Smarandache eğrileri tanımlandı. Daha sonra her bir Smarandache eğrisinin Sabban çatısı oluşturuldu. Son olarak bu Smarandache eğrilerinin geodezik eğrilikleri hesaplandı ve her bir eğriye ait grafikler çizildi.*

**Anahtar kelimeler:** Anti-Salkowski eğrisi, Saban çatısı, Smarandache eğrisi, Birim darboux vektörü

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**1. Introduction**

In 1909, Erich Salkowski defined curve families with non-constant  $\tau$  and constant curvature  $K$  in (Salkowski, 1909). Later J. Monterde constructed a method for closed curves and the properties of Salkowski and anti-Salkowski curve used in (Monterde, 2008). In 1990, the geodesic curve of a spherical curve is calculated by J. Koenderink with the Sabban frame of the spherical indicatrix curves in (Koenderink, 1990).

Then the Smarandache curves obtained from Sabban frame are defined and geodesic curvatures of these curves are given in (Gür and Şenyurt,2010; Şenyurt and Öztürk, 2018a,b; Uzun and Şenyurt, 2020).

In this study, Smarandache curves are defined according to the Sabban frames belonging to the spherical indicatrix curves of each of the  $\{T, N, B\}$ , Frenet vectors of the anti-Salkowski curve. The geodesic curvatures of these curves are then calculated.

**2. Materials and methods**

Let  $\alpha : I \rightarrow E^3$  be a unit speed curve, we defined the quantities of the Frenet apparatus and Frenet formulae, respectively (Sabuncuoğlu, 2006),

$$C_\alpha = \sin(\theta)T_\alpha + \cos(\theta)B_\alpha, \quad \cos(\theta) = \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}, \quad \sin(\theta) = \frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \tag{1}$$

where  $\sphericalangle(B, W) = \theta$  (Fenchel, 1951).

Let  $\gamma : I \rightarrow S^2$  be a unit speed spherical curve. We denote  $s$  as the arc-length parameter of  $\gamma$ . Let us denote by

$$\gamma(s) = \gamma(s), \quad t(s) = \gamma'(s), \quad d(s) = \gamma(s) \wedge t(s) \tag{2}$$

$\{\gamma(s), t(s), d(s)\}$  frame is called the Sabban frame of  $\gamma$  on  $S^2$ . Then we have the following spherical Frenet formulae of  $\gamma$

$$\gamma'(s) = t(s), \quad t'(s) = -\gamma(s) + \kappa_g^\gamma(s)d(s), \quad d'(s) = -\kappa_g^\gamma(s)t(s) \tag{3}$$

where  $\kappa_g$  is called the geodesic curvature of the curve  $\gamma$  on  $S^2$  which is

$$\kappa_g^\gamma(s) = \langle t'(s), d(s) \rangle \text{ (Koenderink,1990).}$$

$$T_\alpha = \alpha', \quad N_\alpha = \frac{\alpha''}{\|\alpha''\|}, \quad B_\alpha = T_\alpha \wedge N_\alpha$$

$$T'_\alpha = \kappa_\alpha N_\alpha, \quad N'_\alpha = -\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha, \quad B'_\alpha = -\tau_\alpha N_\alpha.$$

If the Frenet vectors are computed as arbitrary parameter, we can write

$$T_\alpha = \frac{\alpha'}{\|\alpha'\|}, \quad N_\alpha = T_\alpha \wedge B_\alpha, \quad B_\alpha = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|},$$

$$\kappa_\alpha = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \quad \tau_\alpha = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2},$$

(Sabuncuoğlu, 2006). The Frenet–Serret axis system, moving with the point, has an angular velocity. Dividing this by the (signed) point speed, that is, taking the derivative of the angular position of the axis system with respect to the path position, gives the Darboux vector,  $W_\alpha$  which is given in value by

$$W_\alpha = \tau_\alpha T_\alpha + \kappa_\alpha B_\alpha.$$

The unit Darboux vector is

**Definition 2.1** (Anti-Salkowski curve) For any  $m \neq \pm \frac{1}{\sqrt{3}}, 0$  with  $n = \frac{m}{\sqrt{1+m^2}}, m \in \mathbb{R}$ , let us define the space curve

$$\beta_m(t) = \begin{pmatrix} \frac{n}{2(4n^2-1)m} (n(1-4n^2+3\cos(2nt))\cos(t) + (2n^2+1)\sin(t)\sin(2nt)), \\ \frac{n}{2(4n^2-1)m} (n(1-4n^2+3\cos(2nt))\sin(t) - (2n^2+1)\cos(t)\sin(2nt)), \\ \frac{n^2-1}{4n} (2nt + \sin(2nt)) \end{pmatrix} \tag{4}$$

(Salkowski,1909). Frenet apparatus are

$$\begin{cases} T = -\left( \cos(t)\sin(nt) - n\sin(t)\cos(nt), \sin(t)\sin(nt) + n\cos(t)\cos(nt), \frac{n}{m}\cos(nt) \right) \\ N = n\left( \frac{\sin(t)}{m}, -\frac{\cos(t)}{m}, 1 \right) \\ B = \left( -\cos(t)\cos(nt) - n\sin(t)\sin(nt), -\sin(t)\cos(nt) + n\cos(t)\sin(nt), \frac{n}{m}\sin(nt) \right) \\ \kappa = \tan(nt), \tau = 1, \|\beta'_m(t)\| = \frac{\cos(nt)}{\sqrt{1+m^2}} \end{cases} \tag{5}$$

(Monterde, 2008).

### 3. Results and discussion

If we take  $\kappa = \tan(nt)$  and  $\tau = 1$ , from the equation (1) we can write

$$\begin{aligned} C(t) &= \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B \\ &= \frac{1}{\sqrt{\tan^2(nt) + 1}} T + \frac{\tan(nt)}{\sqrt{\tan^2(nt) + 1}} B \end{aligned}$$

$$C(t) = \cos(nt)T + \sin(nt)B. \tag{6}$$

(6) using the equation (5), we get

$$C(t) = \begin{pmatrix} -2\sin(nt)\cos(nt)\cos(t) + n\sin(t)\cos(2nt), \\ -2\sin(nt)\cos(nt)\sin(t) - n\cos(t)\cos(2nt), -\frac{n}{m}\cos(2nt) \end{pmatrix}. \tag{7}$$

Let  $S^2$  be a unit sphere in Euclidean 3-space and suppose that the unit speed regular curve  $\alpha_C(t) = C(t)$  lying fully on  $S^2$ . Differentiating the equation (6), with respect to  $S$ , we have

$$\frac{d\alpha_C(s)}{ds} \frac{ds}{dt} = -n\sin(nt)T + \cos(nt)T' + n\cos(nt)B + \sin(nt)B'$$

$$T_C \frac{ds}{dt} = n(-\sin(nt)T + \cos(nt)B)$$

where  $\frac{ds}{dt} = n$ . Thus, the tangent vector of curve  $\alpha_C$  is to be

$$T_C = -\sin(nt)T + \cos(nt)B. \tag{8}$$

From the equation (5), we can write

$$T_C = \left( \begin{array}{l} -\cos(t)\cos(2nt) - 2n\sin(nt)\sin(t)\cos(nt), \\ -\sin(t)\cos(2nt) + 2n\cos(t)\cos(nt)\sin(nt), \frac{2n}{m}\cos(nt)\sin(nt) \end{array} \right). \tag{9}$$

Differentiating the equation (8) with respect to  $s$ , we get

$$T'_C = -\cos(nt)T - \frac{\sec(nt)}{n}N - \sin(nt)B. \tag{10}$$

Considering the equations (6) and (8), it easily seen that

$$C \wedge T_C = -N = -n \left( \frac{\sin(t)}{m}, -\frac{\cos(t)}{m}, 1 \right) \tag{11}$$

From the equation (10) and (11), the geodesic curvature of  $\alpha_C$  is

$$\begin{aligned} K_g &= \langle T'_C, C \wedge T_C \rangle, \\ K_g &= \left\langle -\cos(nt)T - \frac{\sec(nt)}{n}N - \sin(nt)B, -N \right\rangle \\ K_g &= \frac{\sec(nt)}{n}. \end{aligned} \tag{12}$$

Let  $\alpha_C(t) = C(t)$  be a unit speed spherical curve. We denote  $s$  as the arc-length parameter of  $\alpha_C$ . Let us denote by  $C = C(t)$ ,  $T_C = C'(t)$ ,  $C \wedge T_C = C(t) \wedge T_C(t)$ .  $\{C, T_C, C \wedge T_C\}$  from is called Sabban frame of  $\alpha_C$  on unit sphere. Then equation (3) we have following spherical Sabban formulae of  $\alpha_C$  curve

$$C' = T_C, \quad T'_C = -C + \kappa_g(C \wedge T_C), \quad (C \wedge T_C)' = -\kappa_g T_C, \tag{13}$$

$$C' = T_C, \quad T'_C = -C + \frac{\sec(nt)}{n}(C \wedge T_C), \quad (C \wedge T_C)' = -\frac{\sec(nt)}{n}T_C, \tag{14}$$

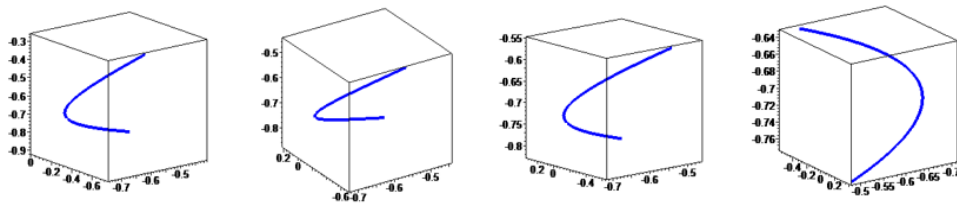
$$\begin{cases} C' = -\sin(nt)T + \cos(nt)B, \\ T_C' = -\cos(nt)T - \frac{\sec(nt)}{n}N - \sin(nt)B, \\ (C \wedge T_C)' = \frac{1}{n}(\tan(nt)T - B). \end{cases} \tag{15}$$

**Definition 3.1.** Let  $\alpha_c(t) = C(t)$  be an anti-Salkowski indicatrix curve and  $\{C, T_C, C \wedge T_C\}$  be the Sabban frame of this curve. Then  $\beta_1(t)$ -Smarandache curve is given by

$$\beta_1(t) = \frac{1}{\sqrt{2}}(C + T_C). \tag{16}$$

Substituting  $C$  and  $T_C$  vectors into the equations (7) and (9), we get the curve  $\beta_1(t)$  as follow (Figure 1.):

$$\beta_1(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(2nt)(n \sin(t) - \cos(t)) - 2 \sin(nt) \cos(nt)(\cos(t) + n \sin(t)), \\ -\cos(2nt)(n \cos(t) + \sin(t)) + 2 \sin(nt) \cos(nt)(-\sin(t) + n \cos(t)), \\ \frac{n}{m}(2 \sin(nt) \cos(nt) - \cos(2nt)) \end{pmatrix} \tag{17}$$



**Figure 1.**  $\beta_1(t)$ -Smarandache curves for  $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ , respectively.

**Theorem 3.1.** The geodesic curvature  $K_g^{\beta_1}$  according to  $\beta_1(t)$ -Smarandache curve is

$$K_g^{\beta_1} = \frac{\sqrt{2}(\sec(nt) + 4n^2 \cos(nt) + 4n^4 \cos^3(nt) + 2n^5 \cos^2(nt) \sin(nt) + n^3 \sin(nt))}{\cos^2(nt)(2n^2 \cos^2(nt) + 1)\sqrt{(2n^2 + \sec^2(nt))}} \tag{18}$$

**Proof:** Differentiating the equation (16), with respect to  $s$ , we have

$$\frac{d\beta_1}{ds} \frac{ds}{dt} = \frac{1}{\sqrt{2}}(C' + T_C'),$$

$$T_{\beta_1} \frac{ds}{dt} = \frac{-C + T_C + K_g(C \wedge T_C)}{\sqrt{2}},$$

$$T_{\beta_1} = \frac{-C + T_C + K_g (C \wedge T_C)}{\sqrt{2 + K_g^2}} \tag{19}$$

where  $\frac{ds}{dt} = \frac{\sqrt{2 + K_g^2}}{\sqrt{2}}$ . If again derivative is taken, we get

$$T'_{\beta_1} \frac{ds}{dt} = \frac{(-C + T_C + K_g (C \wedge T_C))' \sqrt{2 + K_g^2} - (\sqrt{2 + K_g^2})' (-C + T_C + K_g (C \wedge T_C))}{2 + K_g^2},$$

$$T'_{\beta_1} = \frac{\sqrt{2} \left( (-2 - K_g^2 + K_g K'_g) C - (2 + K_g^4 + 3K_g^2 + K_g K'_g) T_C + (2K_g + K_g^3 + 2K'_g) (C \wedge T_C) \right)}{(2 + K_g^2)^2} \tag{20}$$

Using the equations (6) and (19), we easily find

$$C \wedge T_{\beta_1} = \frac{-K_g T_C + C \wedge T_C}{\sqrt{2 + K_g^2}}. \tag{21}$$

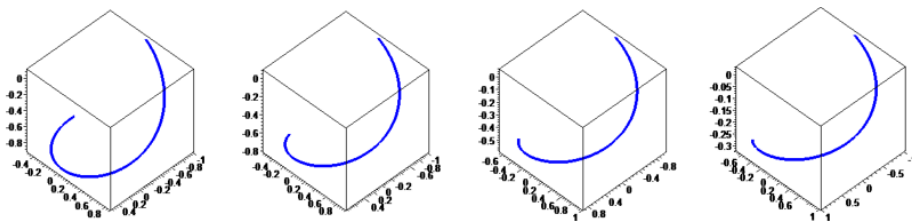
From the equation (12), (20) and (21), the geodesic curvature of  $\beta_1(t)$  is completed.

**Definition 3.2.** Let  $\alpha_c(t) = C(t)$  be an anti-Salkowski indicatrix curve and  $\{C, T_C, C \wedge T_C\}$  be the Sabban frame of this curve. Then Smarandache curve is given by

$$\beta_2(t) = \frac{1}{\sqrt{2}} (T_C(t) + (C \wedge T_C)(t)). \tag{22}$$

Substituting  $T_C$  and  $T_C$  vectors into the equations(9) and (11) we get the curve  $\beta_2(t)$  as follow (Figure 2.):

$$\beta_2(t) = \frac{1}{\sqrt{2}} \left( \begin{array}{l} -\cos(t)\cos(2nt) - 2n\sin(nt)\sin(t)\cos(nt) - \frac{n}{m}\sin(t), \\ -\sin(t)\cos(2nt) + 2n\cos(t)\cos(nt)\sin(nt) + \frac{n}{m}\cos(t), \frac{2n}{m}\cos(nt)\sin(nt) - n \end{array} \right). \tag{23}$$



**Figure 2.**  $\beta_2(t)$ -Smarandache curves for  $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ , respectively.

**Theorem 3.2.** The geodesic curvature  $K_g^{\beta_2}$  according to  $\beta_2(t)$  Smarandache curve is

$$K_g^{\beta_2} = \frac{4\sqrt{2}(\sec^5(nt) + n^2) + \sqrt{2}(2\sin(nt) + \cos^2(nt))}{\cos^3(nt)(n^2 + 2\sec^2(nt))^{\frac{5}{2}}} \tag{24}$$

**Proof:** If we take the derivative of (22) with respect to  $s$ , we have

$$\begin{aligned} \frac{d\beta_2}{ds} \frac{ds}{dt} &= \frac{1}{\sqrt{2}}(T'_C + (C \wedge T)'), \\ T_{\beta_1} \frac{ds}{dt} &= \frac{-C - \kappa_g T_C + \kappa_g (C \wedge T_C)}{\sqrt{2}}, \\ T_{\beta_1} &= \frac{-C - \kappa_g T_C + \kappa_g (C \wedge T_C)}{\sqrt{1 + 2\kappa_g^2}} \end{aligned} \tag{25}$$

where  $\frac{ds}{dt} = \frac{\sqrt{1 + 2K_g^2}}{\sqrt{2}}$ . If again derivative is taken, we get

$$\begin{aligned} \frac{dT_{\beta_1}}{ds} \frac{ds}{dt} &= \frac{(-C - \kappa_g T_C + \kappa_g (C \wedge T_C))' \sqrt{1 + 2\kappa_g^2} - (\sqrt{1 + 2\kappa_g^2})' (-C - \kappa_g T_C + \kappa_g (C \wedge T_C))}{1 + 2\kappa_g^2} \\ T'_{\beta_1} &= \frac{\sqrt{2}((2K_g^3 + K_g + 2K_g K'_g)C - (2K_g^4 + 3K_g^2 + K'_g + 1)T_C + (-2K_g^4 - K_g^2 + K'_g)(C \wedge T_C))}{(1 + 2K_g^2)^2} \end{aligned} \tag{26}$$

Using the equations (6) and (25), we easily find

$$C \wedge T_{\beta_2} = -\frac{K_g(T_C + C \wedge T)_C}{\sqrt{1 + 2K_g^2}}. \tag{27}$$

From the equation (12), (26) and (27), the geodesic curvature of  $\beta_2(t)$  is completed.

**Definiton 3.3.** Let  $\alpha_c(t) = C(t)$  be an anti-Salkowski indicatrix curve and  $\{C, T_C, C \wedge T_C\}$  be the Sabban frame of this curve. Then Smarandache curve is given by

$$\beta_3(t) = \frac{1}{\sqrt{2}}(C(t) + (C \wedge T_C)(t)) \tag{28}$$

Substituting  $C$  and  $C \wedge T_C$  vectors into the equations (7) and (11), we get the curve  $\beta_3(t)$  as follow (Figure 3.):

$$\beta_3(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 \sin(nt) \cos(nt) \cos(t) + n \sin(t) \cos(2nt) - \frac{n}{m} \sin(t), \\ -2 \sin(nt) \cos(nt) \sin(t) - n \cos(t) \cos(2nt) + \frac{n}{m} \cos(t), -\frac{n}{m} \cos(2nt) - n \end{pmatrix}. \tag{29}$$

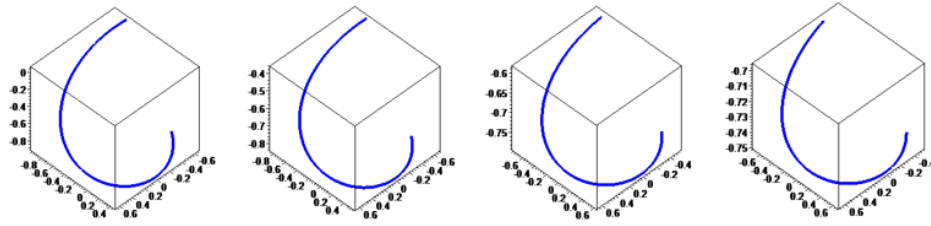


Figure 3.  $\beta_3(t)$ -Smarandache curves for  $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ , respectively.

**Theorem 3.3.** The geodesic curvature  $K_g^{\beta_3}$  according to  $\beta_3(t)$  Smarandache curve is

$$K_g^{\beta_3} = \frac{\sqrt{2} \sec(nt)}{n - \sec(nt)} \tag{30}$$

**Proof:** If we take the derivative of (28) with respect to  $S$ , we have

$$\frac{d\beta_3}{ds} \frac{ds}{dt} = \frac{1}{\sqrt{2}} (C' + (C \wedge T_C)'),$$

$$T_{\beta_3} \frac{ds}{dt} = \frac{1 - \kappa_g}{\sqrt{2}} T_C,$$

$$T_{\beta_1} = T_C \tag{31}$$

where  $\frac{ds}{dt} = \frac{1 - K_g}{\sqrt{2}}$ . If again derivative is taken, we get

$$T'_{\beta_3} = \frac{\sqrt{2} (-C + K_g C \wedge T_C)}{1 - K_g}. \tag{32}$$

Using the equations (6) and (32), we easily find

$$C \wedge T_{\beta_3} = C \wedge T_C. \tag{33}$$

From the equation (12), (32) and (33), the geodesic curvature of  $\beta_3(t)$  is completed.

**Definiton 3.4.** Let  $\alpha_c(t) = C(t)$  be an anti-Salkowski indicatrix curve and  $\{C, T_C, C \wedge T_C\}$  be the Sabban frame of this curve. Then Smarandache curve is given by

$$\beta_4(t) = \frac{1}{\sqrt{3}} (C(t) + T_C(t) + (C \wedge T_C)(t)) \tag{34}$$



Substituting  $C$ ,  $T_C$  and  $C \wedge T_C$  vectors into the equations (7), (9) and (11) we get the curve  $\beta_4(t)$  as follow (Figure 4.):

$$\beta_4(t) = \frac{1}{\sqrt{3}} \begin{pmatrix} \cos(2nt)(n \sin(t) - \cos(t)) - 2 \sin(nt) \cos(nt) (\cos(t) + n \sin(t)) - \frac{n}{m} \sin(t), \\ -\cos(2nt)(n \cos(t) + \sin(t)) + 2 \sin(nt) \cos(nt) (-\sin(t) + n \cos(t)) + \frac{n}{m} \cos(t), \\ \frac{n}{m} (2 \sin(nt) \cos(nt) - \cos(2nt)) - n \end{pmatrix}. \tag{35}$$

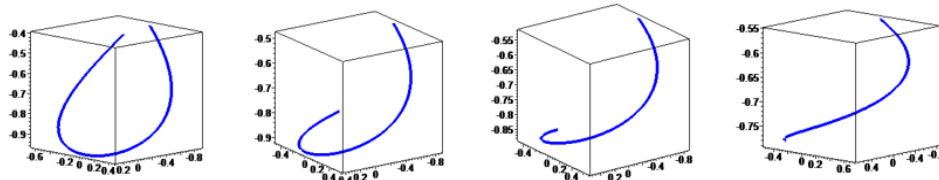


Figure 4.  $\beta_4(t)$ -Smarandache curves for  $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ , respectively.

**Theorem 3.4.** The geodesic curvature  $K_g^{\beta_4}$  according to  $\beta_4(t)$  Smarandache curve is

$$K_g^{\beta_4} = \frac{\sqrt{6} (2 \sec^5(nt) - 4n \sec^4(nt) + 6n^2 \sec^3(nt) - 4n^3 \sec^2(nt) + 2n^4 \sec(nt))}{4(n^2 - n \sec(nt) + \sec^2(nt))^{\frac{5}{2}}} + \frac{\sqrt{6} (n^5 \tan(nt) (\sec^3(nt) - \sec^2(nt) + \sec(nt)))}{4(n^2 - n \sec(nt) + \sec^2(nt))^{\frac{5}{2}}} \tag{36}$$

**Proof:** If we take the derivative of (34) with respect to  $S$ , we have

$$\begin{aligned} \frac{d\beta_4}{ds} \frac{ds}{dt} &= \frac{1}{\sqrt{3}} (C' + T_C' + (C \wedge T_C)'), \\ T_{\beta_4} \frac{ds}{dt} &= \frac{1}{\sqrt{3}} (-C + (1 - K_g) T_C + K_g (C \wedge T_C)) \\ T_{\beta_4} &= \frac{1}{\sqrt{2 - 2K_g + 2K_g^2}} (-C + (1 - K_g) T_C(t) + K_g (C \wedge T_C)(t)) \end{aligned} \tag{37}$$

where  $\frac{ds}{dt} = \frac{\sqrt{2 - 2K_g + 2K_g^2}}{\sqrt{3}}$ . If again derivative is taken, we get

$$\frac{dT_{\beta_4}}{ds} \frac{ds}{dt} = \frac{(-C + (1 - K_g)T_C + K_g(C \wedge T_C))' \sqrt{2 - 2K_g + 2K_g^2}}{2 - 2K_g + 2K_g^2}$$

$$- \frac{(\sqrt{2 - 2K_g + 2K_g^2})' (-C + (1 - K_g)T_C + K_g(C \wedge T_C))}{2 - 2K_g + 2K_g^2}$$

$$T'_{\beta_4} \frac{ds}{dt} = \frac{2K_g^3 - 4K_g^2 + 2K_g K'_g - K'_g + 4K_g - 2}{2\sqrt{2}(1 - K_g + K_g^2)^{\frac{3}{2}}} C$$

$$+ \frac{-2K_g^4 + 2K_g^3 - 4K_g^2 - K_g K'_g - K'_g + 2K_g - 2}{2\sqrt{2}(1 - K_g + K_g^2)^{\frac{3}{2}}} T_C$$

$$+ \frac{-2K_g^4 + 4K_g^3 - 4K_g^2 - K_g K'_g + 2K'_g + 2K_g}{2\sqrt{2}(1 - K_g + K_g^2)^{\frac{3}{2}}} C \wedge T_C$$

$$T'_{\beta_4} = \frac{\sqrt{3}(2K_g^3 - 4K_g^2 + 2K_g K'_g - K'_g + 4K_g - 2)}{4(1 - K_g + K_g^2)^2} C$$

$$+ \frac{\sqrt{3}(-2K_g^4 + 2K_g^3 - 4K_g^2 - K_g K'_g - K'_g + 2K_g - 2)}{4(1 - K_g + K_g^2)^2} T_C$$

$$+ \frac{\sqrt{3}(-2K_g^4 + 4K_g^3 - 4K_g^2 - K_g K'_g + 2K'_g + 2K_g)}{4(1 - K_g + K_g^2)^2} C \wedge T_C$$

Using the equations (6) and (36), we easily find

$$C \wedge T_{\beta_4} = \frac{-K_g T_C + (1 - K_g)(C \wedge T_C)}{\sqrt{2}\sqrt{1 - K_g + K_g^2}} \tag{38}$$

From the equation (12), (37) and (38), the geodesic curvature of  $\beta_4(t)$  is completed.

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