RELATIVE SUBCOPURE-INJECTIVE MODULES

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Abstract. In this paper, copure-injective modules are examined from an alternative perspective. For two modules $A$ and $B$, $A$ is called $B$-subcopure-injective if for every copure monomorphism $f : B \rightarrow C$ and homomorphism $g : B \rightarrow A$, there exists a homomorphism $h : C \rightarrow A$ such that $hf = g$. The class $\mathcal{CP}^{-1}(A) = \{ B : A \text{ is } B\text{-subcopure-injective} \}$ is called the subcopure-injectivity domain of $A$. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains. Since subcopure-injectivity domains clearly contains all copure-injective modules, studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. We refer to these modules as sc-indigent. We studied the properties of subcopure-injectivity domains and of sc-indigent modules and investigated these modules over some certain rings.

1. Introduction and preliminaries

Throughout this paper, $R$ will denote an associative ring with identity, and modules will be unital right $R$-modules, unless otherwise stated. As usual, the category of right $R$-modules is denoted by $\text{Mod} - R$.

Some new studies in module theory have focused on to approach to the injectivity from the point of relative notions. The injectivity domain $\mathcal{In}^{-1}(A)$ for a module $A$, is the class of all modules $B$ such that $A$ is $B$-injective [1]. Given $A$ and $B$ modules, $A$ is called $B$-subinjective if for every monomorphism $f : B \rightarrow C$ and homomorphism $g : B \rightarrow A$, there exists a homomorphism $h : C \rightarrow A$ such that $hf = g$. Instead of using the injectivity domain, in latest articles, authors have proposed to consider an alternative sight so-called subinjectivity domain $\mathcal{In}^{-1}(A)$, contains of modules $B$ such that $A$ is $B$-subinjective (2). It is clear that injectivity of $A$ is equivalent to that $\mathcal{In}^{-1}(A) = \text{Mod} - R$. If $B$ is injective, then $A$ is exactly $B$-subinjective. So by [2] Proposition 2.3), the class of injective modules is the smallest
possible subinjectivity domain. The recent studies of non-injective modules have
been made to figure out the notion of modules that are subinjective only with
respect to the class of injective modules. This kind of non-injective modules are
called indigent in [2]. So far, it is not known whether the existence of indigent
modules for an arbitrary ring, but a positive answer is known for some rings, such
as Noetherian rings ([3, Proposition 3.4]).

A submodule $A$ of a right $R$-module $B$ is said to be pure if for every left $R$-module
$K$ the natural induced map $1_1K : A \otimes K \to B \otimes K$ is a monomorphism. Recall
that a module $A$ is said to be $B$-pure-injective if for every pure monomorphism
$f : C \to B$ and every homomorphism $g : C \to A$, there exists a homomorphism
$h : B \to A$ such that $hf = g$. A module $A$ is said to be pure-injective if it is $B$-pure-
injective for every module $B$. As an analogue to the injectivity profile of [11], the
pure-injectivity profile of a ring is introduced in [5]. The pure-injectivity domain
$\mathfrak{PI}^{-1}(A)$ of a module $A$, consists of those modules $B$ such that $A$ is $B$-pure-injective.

Inspired by the notion of subinjectivity, the notion of pure-subinjectivity introduced in
[11]. A module $A$ is called $B$-pure-subinjective if for every pure monomorphism
$f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$
such that $hf = g$. The pure-subinjectivity domain of a module $A$ is the class $\mathfrak{PIS}^{-1}(A) = \{B : A$ is $B$-pure-subinjective}. If $B$ is pure-injective, then $A$ is exactly
$B$-pure-subinjective. So by [11, Theorem 2.4], for a module $A$, the class $\mathfrak{PIS}^{-1}(A)$
must contain the class of pure-injective modules at least. In [11], modules whose
pure-subinjectivity domain consists of only pure-injective modules is called pure-
subinjectively poor (ps-poor for short).

An $R$-module $A$ is said to be finitely embedded (or cofinitely generated) if $E(A) =
E(S_1) \oplus E(S_2) \oplus \ldots \oplus E(S_n)$, where $S_1, S_2, \ldots, S_n$ are simple $R$-modules (see [16]).
If an $R$-module $A$ is isomorphic to $\prod \{E(S_\alpha) : S_\alpha \text{is a simple right } R\text{-module, } \alpha \in I\}$,
where $I$ is some index set, then $A$ is called a cofree module (see [6]). A right $R$-
module $A$ is said to be cofinitely related if there is an exact sequence $0 \to A \to B \to
C \to 0$ of $R$-modules with $B$ finitely embedded, cofree and $C$ finitely embedded
(see [6]). As a dual notion of purity, by using cofinitely related modules, the notion
of copurity is introduced in [7]. An exact sequence of $R$-modules $0 \to A \to B \to
C \to 0$ is called a copure exact sequence if every cofinitely related right $R$-module
is injective relative to this sequence.

Following idea on pure-injectivity profile of [5], in [15], the copure-injectivity
profile of a ring is introduced. For two modules $A$ and $B$, $A$ is called $B$-copure-
injective if for every copure monomorphism $f : C \to B$ and a homomorphism
g : $C \to A$, there exists a homomorphism $h : B \to A$ such that $hf = g$. A
is copure-injective if it is injective with respect to every copure exact sequences
(see [8]). The copure-injectivity domain $\mathfrak{CPI}^{-1}(A)$ of $A$ is the class of modules
$B$ such that $A$ is $B$-copure-injective. In [15], copure-injectively-poor (shortly copi-
poor) modules introduced as modules with minimal copure-injectivity domain and
studied properties of copi-poor modules. The existence of copi-poor modules are
studied and investigated over some certain rings, but we do not know whether copi-poor modules exist over arbitrary rings (see [15]).

Inspired by the notion of pure-subinjectivity from [11], in this paper we initiate the study of an alternative perspective on the analysis of the copure-injectivity of a module, as we introduce the notions of relative subcopure-injectivity and assign to every module its subcopure-injectivity domain. The aim of this paper is to investigate the viability of obtaining valuable information about a ring $R$ from the perspective of subcopure-injectivity domain.

In Section 2, relative subcopure-injectivity and subcopure-injectivity domains of modules introduced. We investigate the properties of the notion of subcopure-injectivity and we compare subcopure-injectivity domains with (copure-)injectivity domains. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains.

In section 3, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules. We give examples of cc-injective modules and compare cc-injective modules with cotorsion modules in Example [19]. We prove that $R$ is a right V-ring if and only if every cc-injective right $R$-module is injective. We investigate when the class of $B$-subcopure-injective modules is closed under extensions.

An $R$-module is copure-injective if and only if its subcopure-injectivity domain consists of $\text{Mod} - R$. Since subcopure-injectivity domains clearly contain all copure-injective modules, it is reasonable to investigate modules which are subcopure-injective only with respect to the class of copure-injective modules. It is thus to keep in line with [11], we refer to these modules as sc-indigent. In Section 4 of this paper, we studied and investigated sc-indigent modules over some certain rings. We compared sc-indigent modules with indigent modules and ps-poor modules.

2. Relative subcopure-injective modules

In this section, we study the $B$-subcopure-injective modules for a module $B$ and examine its fundamental properties.

Definition 1. For two modules $A$ and $B$, $A$ is called $B$-subcopure-injective if for every copure monomorphism $f : B \rightarrow C$ and homomorphism $g : B \rightarrow A$, there exists a homomorphism $h : C \rightarrow A$ such that $hf = g$. The class $\text{CPI}^{-1}(A) = \{B : A \text{ is } B\text{-subcopure-injective}\}$ is called the subcopure-injectivity domain of $A$.

Hiremath proved in [8, Theorem 7] that every module can be embedded as a copure submodule in a direct product of copure-injective modules. By [8, Proposition 3], every cofinitely related module is copure-injective and every direct product of copure-injective modules is copure-injective. This gives the below result that we use frequently in the sequel.

Lemma 2. For every module $A$, there exists a copure monomorphism $\alpha : A \rightarrow C$ with $C$ is copure-injective.
Our next Lemma gives a characterization of the $B$-subcopure-injective modules for a module $B$.

**Lemma 3.** Let $A$ and $B$ be two modules. The following conditions are equivalent:

1. $A$ is $B$-subcopure-injective.
2. For every homomorphism $g : B \to A$ and every copure monomorphism $\alpha : B \to C$ with $C$ copure-injective, there exists $h : C \to A$ such that $h\alpha = g$.
3. For every homomorphism $g : B \to A$ and every copure monomorphism $\alpha : B \to C$ with $C$ direct product of cofinitely related modules, there exists $h : C \to A$ such that $h\alpha = g$.
4. For every $g : B \to A$ there exist a copure monomorphism $\alpha : B \to C$ with $C$ copure-injective and $h : C \to A$ such that $h\alpha = g$.

**Proof.** (1) $\Rightarrow$ (2) Obvious. (2) $\Rightarrow$ (3) It follows from [8, Proposition 3].

(3) $\Rightarrow$ (4) Let $g : B \to A$ be a homomorphism. By Lemma 2, there exists a copure monomorphism $\alpha : B \to C$ with $C$ copure-injective, whence $C$ is a direct summand of $F$ where $F = \prod_{i \in I} F_i$ with each $F_i$ cofinitely related by [8, Theorem 8]. So $i\alpha : B \to F$ is copure monomorphism where $i : C \to F$. By (3), there exists $h : F \to A$ such that $(hi)\alpha = h(i\alpha) = g$, where $i\alpha : B \to F$.

(4) $\Rightarrow$ (1) Let $g : B \to A$ be a homomorphism and $\bar{\alpha} : B \to D$ a copure monomorphism. By (4), there exists a monic copure map $\alpha : B \to C$ with $C$ copure-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$. So by the copure-injectivity of $C$, there exists a homomorphism $\tilde{h} : D \to C$ such that $\alpha = \tilde{h}\alpha$. Then $\tilde{h}h : D \to A$ and $\tilde{h}h\alpha = h\alpha = g$. Hence, $A$ is $B$-subcopure-injective. \hfill $\Box$

**Proposition 4.** Let $A$ be an $R$-module. The following conditions are equivalent:

1. $A$ is copure-injective.
2. $\CPI^{-1}(A) = \text{Mod} - R$.
3. $A$ is $A$-subcopure-injective.

**Proof.** (1) $\Rightarrow$ (2) For any $R$-module $B$ and any copure-injective module $A$, every copure monomorphism $\alpha : B \to D$ and a homomorphism $g : B \to A$, there exists a homomorphism $h : D \to A$ such that $h\alpha = g$. Hence, $A$ is $B$-subcopure-injective and so $B \in \CPI^{-1}(A)$. Consequently, $\CPI^{-1}(A) = \text{Mod} - R$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Assume that $A$ is $A$-subcopure-injective. For any copure monomorphism $\alpha : A \to B$ with $B$ copure-injective and $1_A : A \to A$, there exists a homomorphism $g : B \to A$ such that $g\alpha = 1_A$. Thus $\alpha$ splits. This means that $A$ is copure-injective. \hfill $\Box$

The next result asserts that subcopure-injectivity domain $\CPI^{-1}(A)$ of $A$ how small can be. It should contain the copure-injective modules at least.

**Proposition 5.** $\bigcap_{A \in \text{Mod} - R} \CPI^{-1}(A) = \{C \in \text{Mod} - R \mid C \text{ is copure-injective}\}$. 
Proof. Suppose that each $R$-module is $B$-subcopure-injective for an $R$-module $B$. Then, by Proposition 4, $B$ is copure-injective. Conversely, let $A$ be any $R$-module and $B$ a copure-injective module. Let $g : B \to A$ be a homomorphism and $\alpha : B \to C$ a copure monomorphism. Since $B$ is copure-injective, the splitting map $\beta : C \to B$ gives the homomorphism $\beta \alpha : C \to B$ such that $\beta \alpha g = 1_B$. So $\beta \alpha g = (\beta \alpha) g = g$. Hence $B \in \mathcal{CPI}^{-1}(A)$ for any $R$-module $A$.

Clearly, $\mathcal{CPI}^{-1}(A)$ contains $2n^{-1}(A)$ for any module $A$. The following example shows that equality need not hold.

Example 6. Let $G = Z(n)$ be a cyclic group of order $n$. Since $G$ is finite it is countably related and so it is copure-injective $\mathbb{Z}$-module [8, Proposition 3]. So $G \in \mathcal{CPI}^{-1}(G)$ by Proposition 4. But $G \notin 2n^{-1}(G)$, otherwise $G$ would be an injective $\mathbb{Z}$-module.

It is natural to investigate conditions to get the coincidence of the injectivity, and subcopure-injectivity domains, either for a certain class of modules or all the modules in $Mod - R$. We start by proving that, for all modules, subcopure-injectivity domains are the same as their subinjectivity domains over a right $V$-ring. Recall that a ring $R$ is a right $V$-ring if and only if all exact sequences in $Mod - R$ are copure if and only if all copure-injective modules are injective (see [8, Proposition 5]).

Corollary 7. Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is a right $V$-ring.
2. $\mathcal{CPI}^{-1}(A) = 2n^{-1}(A)$ for each $R$-module $A$.
3. $\mathcal{CPI}^{-1}(A) \subseteq 2n^{-1}(A)$ for each $R$-module $A$.

Proof. (1) $\Rightarrow$ (2) It is easy since for any module $A$, over a right $V$-ring its extension is copure.
(2) $\Rightarrow$ (3) It is obvious.
(3) $\Rightarrow$ (1) For a copure injective right $R$-module $A$, by Proposition 4, $A \in \mathcal{CPI}^{-1}(A)$. By (3), $A \in 2n^{-1}(A)$. This says that $A$ is injective, and so $R$ is a right $V$-ring by [8, Proposition 5].

Proposition 8. Let $A$ be a module. The following conditions are equivalent:

1. $A$ is copure-injective.
2. $\mathcal{CPI}^{-1}(A)$ is closed under copure submodules.
3. $\mathcal{CPI}^{-1}(A) = \mathcal{CPI}^{-1}(A)$.
4. $\mathcal{CPI}^{-1}(A) \subseteq \mathcal{CPI}^{-1}(A)$.

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear since $\mathcal{CPI}^{-1}(A) = \mathcal{CPI}^{-1}(A) = Mod - R$. 

\[ \square \]
For a copure-injective extension $C$ of $A$, $C \in \mathcal{CPI}^{-1}(A)$, so $A$ is also in $\mathcal{CPI}^{-1}(A)$ by (2). Then by Proposition 4, $A$ is copure-injective.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) For a copure-injective extension $C$ of $A$, $C \in \mathcal{CPI}^{-1}(A)$. This implies that $A$ is $C$-copure-injective i.e. $C = A \oplus B$ for some submodule $B$ of $A$, whence $A$ is copure-injective. □

The rings for which every right $R$-module is copure-injective are called right CDS, [8, Corollary 18]. As a result of Proposition 8, we get the following Corollary.

**Corollary 9.** Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is right CDS.
2. $\mathcal{CPI}^{-1}(A) = \mathcal{In}^{-1}(A)$ for each $R$-module $A$.
3. $\mathcal{CPI}^{-1}(A) \subseteq \mathcal{In}^{-1}(A)$ for each $R$-module $A$.

**Proof.** (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let $A$ be an $R$-module. Since $R$ is a right CDS ring, $A$ is copure-injective. The rest follows from Proposition 8.

(3) $\Rightarrow$ (1) For any right $R$-module $A$, $\mathcal{CPI}^{-1}(A) \subseteq \mathcal{In}^{-1}(A)$ by the hypothesis. Thus every right $R$-module $A$ is copure-injective by Proposition 8 whence $R$ is right CDS. □

**Remark 10.** If $A$ is $R$-subcopure-injective, for a ring $R$ and a module $A$, then $\mathcal{CPI}^{-1}(A)$ and $\mathcal{Mod} - R$ need not be equal. For example if $R$ is copure-injective ring that is not CDS, then for every module $A$, $A$ is $R$-subcopure-injective by Proposition 5. But by the definition of right CDS ring, we can find a module $A$ that is not copure-injective.

**Proposition 11.** Let $A$ be a module. The following conditions are equivalent:

1. $A$ is injective.
2. $\mathcal{CPI}^{-1}(A) = \mathcal{In}^{-1}(A)$.
3. $\mathcal{CPI}^{-1}(A) \subseteq \mathcal{In}^{-1}(A)$.

**Proof.** (1) $\Rightarrow$ (2) It is clear.

(3) $\Rightarrow$ (1) By the copure-injectivity of $E(A)$, $E(A) \in \mathcal{CPI}^{-1}(A)$. By (3), $E(A) \in \mathcal{In}^{-1}(A)$, and hence $A$ is injective. □

**Corollary 12.** Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is semisimple.
2. $\mathcal{CPI}^{-1}(A) = \mathcal{In}^{-1}(A)$ for each $R$-module $A$.
3. $\mathcal{CPI}^{-1}(A) \subseteq \mathcal{In}^{-1}(A)$ for each $R$-module $A$.

**Proof.** (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let $A$ be an $R$-module. Since $R$ is semisimple, $A$ is injective. The rest follows from Proposition 11.
(3) ⇒ (1) For any right $R$-module $A$, $\text{CPI}^{-1}(A) \subseteq \text{In}^{-1}(A)$ by the hypothesis. Thus every right $R$-module $A$ is injective by Proposition 11, whence $R$ is semisimple. 

In general, factors of copure-injective modules need not be copure-injective (see, [8, Remark 24]). But if $R$ is a Dedekind domain, every copure factor of copure-injective module is copure-injective by [8, Corollary 28]. Hence, by the following Proposition, $\text{CPI}^{-1}(A)$ is closed under copure homomorphic images over Dedekind domains for a module $A$.

**Proposition 13.** $\text{CPI}^{-1}(A)$ is closed under copure quotients for any module $A$ if and only if every copure homomorphic image of a copure-injective module is copure-injective.

**Proof.** Let $B$ be a copure submodule of copure-injective module $A$. Since $A \in \text{CPI}^{-1}(\frac{A}{B})$, by the hypothesis $\frac{A}{B} \in \text{CPI}^{-1}(\frac{A}{B})$, and so $\frac{A}{B}$ is copure-injective. Conversely, let $A$ be a module and $C$ a copure submodule of $B$ with $B \in \text{CPI}^{-1}(A)$. By Lemma 2 there exists a copure monomorphism $\alpha : B \to D$ with $D$ copure-injective. Let $f : \frac{B}{C} \to A$ be any homomorphism. Consider the following pushout diagram:

\[
\begin{array}{cccccc}
0 & \to & B & \xrightarrow{\alpha} & D & \xrightarrow{\pi} & \frac{D}{B} & \to & 0 \\
 & \downarrow{\pi} & & \downarrow{\pi'} & & \downarrow{f} & & \downarrow{\alpha'} & & \downarrow{\alpha''} \\
0 & \to & \frac{B}{C} & \xrightarrow{\alpha'} & \frac{D}{B} & \xrightarrow{f} & A & & & \\
\end{array}
\]

where $\pi : B \to \frac{B}{C}$ is the natural epimorphism. By commutativity of the following diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\alpha} & D \\
\downarrow{\pi} & & \downarrow{\pi''} \\
\frac{B}{C} & \xrightarrow{\alpha''} & \frac{D}{C} \\
\end{array}
\]

and the pushout diagram property, there exists a map $\phi : E \to \frac{D}{C}$ such that $\phi\pi' = \pi''$ and $\phi\alpha' = \alpha''$. Since $A$ is $B$-subcopure-injective, there exists a homomorphism $\varphi : D \to A$ such that $\varphi\alpha = f\pi$. Then, $\varphi(C) = \varphi\alpha(C) = f\pi(C) = f(0) = 0$. Hence, $\text{Ker}(\phi\pi') \subseteq \text{Ker}\varphi$, and so there exists $\psi : \frac{D}{C} \to A$ such that $\psi\pi'' = \varphi$. For every $x \in B$, $\psi(x + C) = \psi\pi''(x) = \varphi(x) = f\pi(x) = f(x + C)$. Thus $\psi$ extends $f$. Then by the hypothesis, $\frac{B}{C}$ is copure-injective, so by Lemma 3 $\frac{B}{C} \in \text{CPI}^{-1}(A)$. \qed
Proposition 14. \( \text{CPI}^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \text{CPI}^{-1}(A_i) \) for any set of modules \( \{A_i\}_{i \in I} \).

Proof. Let \( B \in \text{CPI}^{-1}(\bigcap_{i \in I} A_i) \), \( i \in I \) and \( f : B \to A_i \) be a homomorphism. Then there exists a homomorphism \( g : C \to \prod_{i \in I} A_i \) such that \( g\alpha = i_{A_i}f \), where \( \alpha : B \to C \) is the monic map with \( C \) copure-injective and \( i_{A_i} : A_i \to \prod_{i \in I} A_i \) is the inclusion map. Let \( \pi_{A_i} : \prod_{i \in I} A_i \to A_i \) denote the natural projection. Since \( \pi_{A_i}g\alpha = \pi_{A_i}i_{A_i}f = f \), \( f \) is extended to \( \pi_{A_i}g \). Therefore \( B \in \text{CPI}^{-1}(A_i) \) for any \( i \in I \). Conversely, let \( B \in \text{CPI}^{-1}(A_i) \) for all \( i \in I \) and \( f : B \to \prod_{i \in I} A_i \). Hence for each \( i \in I \), there exists \( g_i : C \to A_i \) with \( g_i\alpha = \pi_{A_i}f \). Now define \( g : C \to \prod_{i \in I} A_i \) by \( x \mapsto g_i(x) \). Since \( g\alpha = f \), \( g \) extends \( f \). Thus, \( B \in \text{CPI}^{-1}(\prod_{i \in I} A_i) \). \( \square \)

Corollary 15. Let \( B \) be a module. Then \( B \)-subcopure-injective modules are closed under direct summands and finite direct sums.

Proof. Let \( A \) be a module with decomposition \( A = \bigoplus_{i=1}^n A_i \). By Proposition 14, \( B \in \text{CPI}^{-1}(A) \) if and only if \( B \in \bigcap_{i=1}^n \text{CPI}^{-1}(A_i) \). Now the result follows. \( \square \)

The following shows that Proposition 14 do not hold for infinite direct sums.

Example 16. Let \( K_i = \mathbb{Z}_{p_i} \) and \( G = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_{p_i} \) where \( p_i \) is a prime integer for all \( i \in \mathbb{N} \). Since every \( \mathbb{Z}_{p_i} \) is pure-injective, every \( \mathbb{Z}_{p_i} \) is copure-injective by Proposition 9. So \( G \in \text{CPI}^{-1}(\mathbb{Z}_{p_i}) \) for all \( i \in \mathbb{N} \). But \( G \notin \text{CPI}^{-1}(G) \) since \( G \) is not copure-injective by Examples-(ii).

Proposition 17. If \( B \in \text{CPI}^{-1}(A) \), then every direct summand of \( B \) is in \( \text{CPI}^{-1}(A) \).

Proof. Suppose \( C \) is a direct summand of \( B \), and let \( f : C \to A \) be a homomorphism. By Lemma 2, there exist copure monomorphisms \( i : B \to D \) and \( j : C \to E \) with \( D \) and \( E \) copure-injective. Consider the following diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & C \\
\downarrow j & & \downarrow i \\
E & \longrightarrow & D
\end{array}
\]

where \( i_C : C \to B \) the inclusion map. Since \( D \) is copure-injective, there exists \( h : E \to D \) such that \( hj = i_iC \). Let \( \pi_C : B \to C \) be the projection map. Since \( A \) is \( B \)-subcopure-injective, there exists a homomorphism \( g : D \to A \) such that \( gi = f\pi_C \). Then, \((gh)j = g(hj) = gi_iC = f\pi_Ci_C = f \), and so by Lemma 3, \( A \) is \( C \)-subcopure-injective. \( \square \)
In this section, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules.

A module $C$ is said to be co-absolutely co-pure (c.c. in short) if every exact sequence of modules ending with $C$ is copure, equivalently $\text{Ext}^1_R(C,A) = 0$ for every co-finitely related module $A$. Clearly every projective module is c.c. But the converse need not be true, for instance, the additive group $\mathbb{Q}$ is a c.c. $\mathbb{Z}$-module but $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module (see, [9, Example on page 290]).

**Definition 18.** A right module $A$ is called cc-injective if $\text{Ext}^1_R(B,A) = 0$ for any c.c. module $B$.

Recall that a module $A$ is called cotorsion if $\text{Ext}^1_R(B,A) = 0$ for every flat module $B$. A module $A$ is called linearly compact if any family of cosets having the finite intersection property has a nonempty intersection. A commutative ring is called classical if the injective hull $E(S)$ of all simple modules $S$ are linearly compact (see [17, §3]).

**Example 19.** (1) By definition, any cofinitely related module is cc-injective.

(2) By [9, Remark 15], c.c. modules need not be flat in general. By [9, Corollary 14] c.c. modules are flat over a commutative ring. So, in this case every cotorsion module is cc-injective.

(3) By [9, Remark 12], flat modules need not be c.c. Over a commutative classical ring flat modules are c.c. by [9, Proposition 11]. So, in this case every cc-injective module is cotorsion.

**Remark 20.** Over a commutative ring $R$ every simple $R$-module is cotorsion by [13, Lemma 2.14]. So by Example 19(2), every simple $R$-module is cc-injective.

**Lemma 21.** Every copure-injective module is cc-injective.

**Proof.** Let $A$ be a copure-injective module and $B$ a c.c. module. By [9, Proposition 5], there exists a copure exact sequence $0 \rightarrow D \rightarrow P \rightarrow B \rightarrow 0$ with $P$ projective. If we apply $\text{Hom}(\cdot, A)$ to this sequence, we have $\text{Hom}(P,A) \rightarrow \text{Hom}(D,A) \rightarrow \text{Ext}^1_R(B,A) \rightarrow \text{Ext}^1_R(P,A) = 0$. Since $A$ is copure-injective, $\text{Hom}(P,A) \rightarrow \text{Hom}(D,A)$ is epic, and so $\text{Ext}^1_R(B,A) = 0$ for any c.c. module $B$. Hence $A$ is cc-injective. $\square$

**Proposition 22.** For a ring $R$, the following conditions are equivalent:

(1) $R$ is a right V-ring.

(2) Every copure-injective right $R$-module is injective.

(3) Every cc-injective right $R$-module is injective.

**Proof.** (1) $\Leftrightarrow$ (2) It follows by [8, Proposition 5].

(3) $\Rightarrow$ (2) It immediately from Lemma 21.
(1) ⇒ (3) Let $A$ be a cc-injective $R$-module and $B$ any $R$-module. Since $R$ is right $V$, $B$ is a c.c. module by \[9,\] Proposition 4. Thus $\text{Ext}_R^1(B, A) = 0$ for any $R$-module $B$, and so $A$ is injective. \hfill □

**Proposition 23.** Let $B$ be an $R$-module and $\alpha : B \to C$ a copure monomorphism with $C$ copure-injective. If $C/\text{im}(\alpha)$ is c.c., then every cc-injective module is $B$-subcopure-injective.

*Proof.* Let $A$ be a cc-injective module and $C/\text{im}(\alpha)$ a c.c. module. Applying functor $\text{Hom}(\cdot, A)$ to the exact sequence $0 \to B \to C \to C/\text{im}(\alpha) \to 0$, we have $\text{Hom}(C, A) \to \text{Hom}(B, A) \to \text{Ext}_R^1(C/\text{im}(\alpha), A)$. Since $C/\text{im}(\alpha)$ is c.c., $\text{Ext}_R^1(C/\text{im}(\alpha), A) = 0$ and so $\text{Hom}(C, A) \to \text{Hom}(B, A)$ is epic. Hence $A$ is $B$-subcopure-injective by Lemma \[3\]. \hfill □

**Theorem 24.** Let $A$ and $B$ be two modules. Consider the following conditions:

1. $A$ is $B$-subcopure-injective.
2. For every homomorphism $g : B \to A$, there exist a monomorphism $\alpha : B \to C$ with $C$ copure-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$.
3. For every homomorphism $g : B \to A$, there exist a monomorphism $\alpha : B \to C$ with $C$ cc-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$.
4. For every homomorphism $g : B \to A$ and for any extension $\alpha : B \hookrightarrow C$ with $C/B$ is c.c., there exists $h : C \to A$ such that $h\alpha = g$.

Then (1) ⇔ (2) ⇒ (3) ⇒ (4). Also, if $D/\text{im}(\alpha)$ is c.c. for a copure monomorphism $\alpha : B \to D$ with $D$ copure-injective, then (4) ⇒ (1).

*Proof.* (1) ⇒ (2) Obvious by Lemma \[3\].

(2) ⇒ (3) It follows from Lemma \[21\] since every copure-injective module is cc-injective.

(2) ⇒ (3) Let $\alpha : B \to C$ be a copure-monomorphism and $g : B \to A$ a homomorphism. By (2), exists a monomorphism $\beta : B \to D$ with $D$ copure-injective and a homomorphism $h : D \to A$ such that $h\beta = g$. Since $D$ is copure-injective, there exists a homomorphism $f : C \to D$ such that $f\alpha = \beta$. Hence, $(hf)\alpha = h\beta = g$, and so (1) follows.

(3) ⇒ (4) Let $C$ be an extension of $B$ with $C/B$ is c.c. and $g : B \to A$ a homomorphism. So, $0 \to B \xrightarrow{\alpha} C \to C/B \to 0$ is copure exact. Then consider the exact sequence with $E$ cc-injective:

$0 \to \text{Hom}_R(C/B, E) \to \text{Hom}_R(C, E) \xrightarrow{\alpha^*} \text{Hom}_R(B, E) \to \text{Ext}_R^1(C/B, E) = 0$

Since, $\alpha^*$ is surjective, by (3), there exists a monomorphism $f : B \to E$ and a homomorphism $h : E \to A$ such that $hf = g$. Since $\alpha^*$ is surjective, there exists a homomorphism $\beta : C \to E$ such that $\beta\alpha = f$. Hence, $h(\beta\alpha) = hf = g$, and so (4) follows.
(4) ⇒ (1): Let \( \alpha : B \to D \) be a copure monomorphism with \( D \) copure-injective and \( D/im(\alpha) \) is c.c. So, by (4), for any homomorphism \( g : B \to A \) there exists \( h : D \to A \) such that \( h\alpha = g \). Thus \( A \) is \( B \)-subcopure-injective by Lemma 3.

Now we investigate when the class of \( B \)-subcopure-injective modules is closed under extensions.

**Proposition 25.** Let \( B \) be an \( R \)-module and \( \alpha : B \to C \) a copure monomorphism with \( C \) copure-injective. The class of \( B \)-subcopure-injective modules is closed under extensions if and only if for every exact sequence \( 0 \to A' \to A \to C \to 0 \) with \( A' \) \( B \)-subcopure-injective, \( A \) is \( B \)-subcopure-injective.

**Proof.** Let \( 0 \to A' \to A \to C \to 0 \) be an exact sequence with \( A' \) \( B \)-subcopure-injective. Since \( C \) is copure-injective, it is \( B \)-subcopure-injective. By the hypothesis, \( A \) is \( B \)-subcopure-injective. Conversely, let \( 0 \to A' \to A \xrightarrow{\pi} A'' \to 0 \) be an exact sequence with \( A' \) and \( A'' \) \( B \)-subcopure-injective. Then by Lemma 3 for every map \( g : B \to A \), there exists a map \( h : C \to A'' \) such that \( \pi g = h\alpha \) where \( \alpha : B \to C \) is the copure monomorphism with \( C \) copure-injective. If we consider the pullback diagram:

\[
\begin{array}{ccc}
0 & \to & A' \\
\downarrow & & \downarrow f \\
0 & \to & A' \\
\end{array}
\]

there exists a homomorphism \( \gamma : B \to D \) such that \( f\gamma = g \) and \( \beta\gamma = \alpha \). By hypothesis, \( D \) is \( B \)-subcopure-injective, so by Lemma 3, there exists a homomorphism \( h' : C \to D \) such that \( h'\alpha = \gamma \). Thus, \( fh'\alpha = f\gamma = g \) and so, \( A \) is \( B \)-subcopure-injective by Lemma 3.

A ring \( R \) is said to be right co-noetherian if every homomorphic image of a finitely embedded \( R \)-module is finitely embedded, equivalently for each simple right \( R \)-module \( S \) the injective hull \( E(S) \) is Artinian (see [10, Theorem]). Over a commutative noetherian ring, the injective hull of each simple right \( R \)-module is Artinian by [14, Exercise 4.17]. Thus every commutative Noetherian ring is co-noetherian. In the following, for an ideal \( I \), we deal with an \( R \)-module structure of an \( R/I \)-module.

**Proposition 26.** Let \( R \) be a right co-noetherian ring and \( f : R \to S \) a ring epimorphism. If \( A \) is \( cc \)-injective \( S \)-module, then \( A \) is \( cc \)-injective \( R \)-module.

**Proof.** Let \( A \) be a \( cc \)-injective \( S \)-module. Since \( f : R \to S \) is a ring epimorphism, \( S \cong R/I \) for some ideal \( I \) of \( R \) and so \( A \) can be considered as \( R/I \)-module. Let \( C \) be an extension of \( A \) by a \( cc \) module \( F \) as \( R \)-modules. Since \( F \) is \( cc \), the exact sequence \( 0 \to A \to C \to F \to 0 \) is copure. Then \( A \cap CI = AI \) for each right ideal \( I \) by [7, proposition 16]. Since \( A \) is an \( R/I \)-module, \( A \cap CI = AI = 0 \), and so \( A \cap CI \cong A \). Thus we have the following commutative diagram.
Since \( \frac{C}{A} \otimes_R \overset{\text{c.c.}}{=} \frac{C}{A+CI} \) is c.c. as an \( R/I \)-module, so the second exact sequence splits and so does the first. Hence \( \text{Ext}^1_R(F, A) = 0 \), and \( A \) is cc-injective \( R \)-module.

4. SC-INDIGENT MODULES

Indigent (resp. ps-poor) modules were introduced and some results about them were obtained in \([2]\) (resp. \([11]\)). Proposition 5 says that subcopure-injectivity domain of any module \( A \) contains all copure-injective modules, so studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. It is thus to keep in line with \([2]\), we refer to these modules as subcopure-injectively indigent (sc-indigent for short). In this section, sc-indigent modules investigated over certain rings and compared these modules with indigent modules and ps-poor modules.

**Definition 27.** A module \( A \) is said to be subcopure-injectively indigent (sc-indigent for short), if \( \text{CPI}_1(A) \) consists of only copure-injective modules.

**Remark 28.** Let \( A \) be a module with decomposition \( A = B \oplus C \). If \( B \) is sc-indigent, then so is \( A \), by Proposition \([14]\).

**Proposition 29.** For a ring \( R \), the following conditions are equivalent:

1. \( R \) is right CDS.
2. Every \( R \)-module is sc-indigent.
3. There exists a copure-injective sc-indigent \( R \)-module.
4. \( 0 \) is an sc-indigent \( R \)-module.
5. \( R \) has an sc-indigent module and every sc-indigent \( R \)-module is copure-injective.
6. \( R \) has an sc-indigent module and every factor of an sc-indigent \( R \)-module is sc-indigent.
7. \( R \) has an sc-indigent module and every summand of an sc-indigent \( R \)-module is sc-indigent.

**Proof.** The implications (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (5) are clear since every \( R \)-module is copure-injective.

The implications (2) \( \Rightarrow \) (4) and (2) \( \Rightarrow \) (6) \( \Rightarrow \) (7) are clear.

(4) \( \Rightarrow \) (2) It immediately from Remark \([25]\).

(2) \( \Rightarrow \) (3) The copure-injective extension \( C \) of any module \( A \) is sc-indigent.

(3) \( \Rightarrow \) (1) Let \( C \) be a copure-injective sc-indigent module and \( A \) a module. Since \( C \) is \( A \)-subcopure-injective, \( A \) is copure-injective. Then \( R \) is a right CDS ring.
By (5), there exist an sc-indigent module $B$. Then $A \oplus B$ is also sc-indigent for any module $A$ by Remark 28. So $A$ is copure-injective by (5). Also $A$ is copure-injective. Thus $R$ is a right CDS ring.

(7) \Rightarrow (2) Let $A$ be an $R$-module. Then $A \oplus B$ is an sc-indigent module for some sc-indigent module $B$. Hence, $A$ is sc-indigent by the hypothesis.

Remark 30. Over a commutative uniserial ring $R$, every $R$-module is sc-indigent since such rings are CDS by [4, Theorem 10.4].

Remark 31. An sc-indigent module need not be indigent. Consider the ring $R = \mathbb{Z}/p^2\mathbb{Z}$, for some prime integer $p$. $R$ is an artinian principal ideal ring. Hence it is a CDS-ring by [4, Theorem 10.4]. So every $R$-module is sc-indigent. Since $\mathbb{Z}/p^2\mathbb{Z}$ is injective $\mathbb{Z}/p^2\mathbb{Z}$-module, $2n^{-1}(\mathbb{Z}/p^2\mathbb{Z}) = \text{Mod} - R$. But since $R$ is not a semisimple ring, $\mathbb{Z}/p^2\mathbb{Z}$ is not an indigent $R$-module.

Remark 32. An indigent module need not be sc-indigent. Let $R$ be a commutative Noetherian ring which is not CDS and $\Gamma$ a complete set of representatives of finitely presented right $R$-modules. Set $F := \bigoplus_{S_i \in \Gamma} S_i$. Thus the character module $F^*$ of $F$ is a pure-injective indigent $R$-module by [3, Proposition 3.4]. Since $R$ is commutative, $F^*$ is copure-injective by [8, Proposition 9], and so $\text{CPI}^{-1}(F^*) = \text{Mod} - R$. But since $R$ is not a CDS-ring, $F^*$ is not an sc-indigent $R$-module.

Proposition 33. Indigent modules and sc-indigent modules coincide over a right $V$-ring $R$.

Proof. Let $R$ be a right $V$-ring. Then by Corollary 7, $\text{CPI}^{-1}(A) = 2n^{-1}(A)$ for any $R$-module $A$. Hence $A$ is indigent if and only if $A$ is sc-indigent by [8, Proposition 5].

Proposition 34. A module $A$ is sc-indigent if and only if $\prod_{i \in I} A_i$ is sc-indigent where $A_i = A$ for all $i \in I$.


By Remark 28 and Proposition 34, sc-indigent rings are characterized as follows:

Corollary 35. For a ring $R$, the following are equivalent:

(1) $R_R$ is sc-indigent.

(2) Any direct product of copies of $R$ is sc-indigent.

(3) Every free $R$-module is sc-indigent.

(4) There exists a cyclic projective sc-indigent $R$-module.

Theorem 36. Let $R$ be a ring, $B$ an $R$-module and $A$ an $R/I$-module for any ideal $I$ of $R$. If $B/BI \in \text{CPI}^{-1}(A_{R/I})$, then $B \in \text{CPI}^{-1}(A_R)$.

Proof. Let $B/BI \in \text{CPI}^{-1}(A_{R/I})$, and $C$ be a copure extension of $B$ and $g : B \rightarrow A$ an $R$-homomorphism. Since copure short exact sequences of $R$-modules form a proper class by [7, Proposition 8], $B/BI$ can be embedded in $C/CI$ as
a copure submodule via \( f : B/BI \to C/CI \) defined by \( f(b + BI) = b + CI \) for any \( b \in B \). Since \( BI \subseteq \text{Ker}(g) \), there exists a homomorphism \( h : B/BI \to A \) such that \( h\pi_B = g \) where \( \pi_B : B \to B/BI \). By assumption, there exists an \( R/I \)-homomorphism \( \bar{h} : C/CI \to A \) such that \( \bar{h}f = g \). Since \( h \) is also an \( R \)-homomorphism and \( \bar{h}\pi_Ci_B = g \) where \( \pi_C : C \to C/CI \) and \( i_B : B \to C \) is the inclusion. Thus \( B \in \mathfrak{C}^{-1}(A_R) \).

**Corollary 37.** Let \( I \) be an ideal of a ring \( R \) and \( A \) and \( B \) be \( R/I \)-modules. Then the following statements hold:

1. \( B \in \mathfrak{C}^{-1}(A_R) \) if and only if \( B \in \mathfrak{C}^{-1}(A_{R/I}) \).
2. \( A \) is a copure-injective \( R \)-module if and only if \( A \) is a copure-injective \( R/I \)-module.
3. \( A \) is an sc-indigent \( R \)-module if and only if \( A \) is an sc-indigent \( R/I \)-module.

**Proof.** (1) If \( A_R \) is \( B \)-subcopure-injective, then clearly it is a \( B \)-subcopure-injective \( R/I \)-module. The converse follows by Theorem 36.

(2) By using Proposition 4, (2) follows from (1).

(3) Clear by (1) and (2).

Recall [11] that a module \( A \) is called ps-poor if pure-subinjectivity domain of \( A \) consists of only pure-injective modules. Over a commutative classical ring \( R \), by [8, Corollary 17], pure-injective modules and copure-injective modules coincide. Hence, the following result is immediate.

**Proposition 38.** Let \( R \) be a commutative classical ring. Then an \( R \)-module \( A \) is sc-indigent if and only if \( A \) is ps-poor.

Since by [16, Theorem 2] and [17, Proposition 4.1], every commutative (co-)noetherian ring is classical, we have the following result.

**Corollary 39.** Let \( R \) be a commutative (co-)noetherian ring. Then an \( R \)-module \( A \) is sc-indigent if and only if \( A \) is ps-poor.

**Remark 40.** ps-poor abelian groups and sc-indigent abelian groups coincide by Corollary 39.

**Corollary 41.** Every finitely embedded \( \mathbb{Z} \)-module is copure-injective but not sc-indigent.

**Proof.** Let \( A \) be a finitely embedded \( \mathbb{Z} \)-module. Then \( A \) is cofinitely related by [6, Proposition 17]. So \( A \) is copure-injective by [3, Proposition 3]. Since \( \mathbb{Z} \) is not a CDS ring, by Proposition 29, \( A \) is not an sc-indigent module.

**Proposition 42.** If a ring \( R \) has an sc-indigent cc-injective module \( B \), then every module with its copure injective extension has c.c cokernel is copure-injective.
Proof. Let $A$ be an $R$-module with the exact sequence $0 \to A \to C \to C/A \to 0$, where $A \to C$ is a copure extension of $A$ with $C$ is copure-injective. Consider the sequence $0 \to \text{Hom}(C/A, B) \to \text{Hom}(C, B) \to \text{Hom}(A, B) \to \text{Ext}^1(C/A, B)$. Since $C/A$ is c.c., $\text{Ext}^1(C/A, B) = 0$. So by Lemma 3, $A \in \mathcal{CPI}^{-1}(B)$, that is $A$ is copure-injective.

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References


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