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BEST PROXIMITY PROBLEMS FOR NEW TYPES OF $\mathcal{Z}$-PROXIMAL CONTRACTIONS WITH AN APPLICATION

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Abstract. In this study, we establish existence and uniqueness theorems of best proximity points for new types of $\mathcal{Z}$-proximal contractions defined on a complete metric space. The presented results improve and generalize some recent results in the literature. Several examples are constructed to demonstrate the generality of our results. As applications of the obtained results, we discuss sufficient conditions to ensure the existence of a unique solution for a variational inequality problem.

1. Introduction

Khojasteh et al. [14] presented the notion of $\mathcal{Z}$-contraction involving a new class of mappings namely simulation functions and proved new fixed point theorems by using different methods than others in literature.

Definition 1.1 ([14]). *A simulation function is a mapping* $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ *satisfying the following conditions:*

- (ζ_1) $\zeta(0,0) = 0$,
- (ζ_2) $\zeta(a,b) < b - a$ *for all* $a, b > 0$,
- (ζ_3) *If* $(a_n), (b_n)$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(a_n, b_n) < 0. \tag{1.1}$$

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Theorem 1.2 ([14]). *Let (M,d) be a complete metric space and $\mathcal{T}: M \to M$ be a $\mathcal{Z}$-contraction with respect to ζ satisfying the conditions (ζ_1)-(ζ_3) in Definition 1.1, that is,*

$$\zeta(d(\mathcal{T}u,\mathcal{T}v),d(u,v)) \geq 0, \quad \text{for all } u,v \in M.$$

Then $\mathcal{T}$ has a unique fixed point and, for every initial point $u_0 \in M$, the Picard sequence $\{\mathcal{T}^n u_0\}$ converges to this fixed point.

Afterwards, Argoubi et al. [3] partly modified Definition 1.1, by removing the condition (ζ_1), because of the fact that the condition (ζ_1) was not used in the proof of Theorem 1.2. On the other hand, Roldan-Lopez-de-Hierro et al. [17] extended the family of all simulation functions by replacing the condition (ζ_3) in Definition 1.1 with the following proviso.

(ζ_4) If $(a_n),(b_n)$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0$ and $a_n < b_n$ for all $n \in \mathbb{N}$, then the inequality (1.1) is satisfied.

In this study, we will consider simulation functions satisfying the conditions (ζ_2) and (ζ_4). For the sake of openness, we identify the following families of function.

$$\begin{aligned}
\mathcal{Z}_1 &= \{\zeta : \ \zeta \text{ satisfies conditions } (\zeta_1),(\zeta_2) \text{ and } (\zeta_3)\},\\
\mathcal{Z}_2 &= \{\zeta : \ \zeta \text{ satisfies conditions } (\zeta_2) \text{ and } (\zeta_3)\},\\
\mathcal{Z}_3 &= \{\zeta : \ \zeta \text{ satisfies conditions } (\zeta_1),(\zeta_2) \text{ and } (\zeta_4)\},\\
\mathcal{Z}_4 &= \{\zeta : \ \zeta \text{ satisfies conditions } (\zeta_2) \text{ and } (\zeta_4)\}.
\end{aligned}$$

Remark 1.3. *It is obvious that $\mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \mathcal{Z}_4$ and also $\mathcal{Z}_3 \subset \mathcal{Z}_4$.*

Example 1.4. *Let $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ be a function defined by*

$$\zeta(t,s) = \begin{cases} 1 & \text{if } (s,t) = (0,0),\\ 2(s-t) & \text{if } s < t,\\ \lambda s - t & \text{otherwise,} \end{cases}$$

where $\lambda \in (0,1)$. Then it is easy to see that $\zeta \in \mathcal{Z}_4$, but $\zeta \notin \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$.

The main concern of the paper is to establish existence and uniqueness theorems of best proximity points for new types of $\mathcal{Z}$-proximal contractions in complete metric spaces. The obtained results extend and complement some known results from the literature. Several examples are constructed to demonstrate the new concepts and the generality of our results. Also, sufficient conditions to guarantee the existence of a unique solution to the problem of variational inequality are discussed.

2. Preliminaries

A best proximity point generates to a fixed point if the mapping under consideration is a self-mapping. For more details on this research subject, we refer the reader to [1, 2, 4–7, 9–13, 16, 18–22].

Let P and Q two nonempty subsets of a metric space (M,d). We will use the following notations:

$$\begin{aligned} d(P,Q) &:= \inf\{d(p,q) : p \in P, q \in Q\}; \\ P_0 &:= \{p \in P : d(p,q) = d(P,Q) \text{ for some } q \in Q\}; \\ Q_0 &:= \{q \in Q : d(p,q) = d(P,Q) \text{ for some } p \in P\}. \end{aligned}$$

Throughout this study, the set of all best proximity points of a non-self-mapping $\mathcal{T} : P \to Q$ will be denoted by

$$B_{est}(\mathcal{T}) = \{u \in P : d(u, \mathcal{T}u) = d(P,Q)\}.$$

Jleli and Samet [12] introduced the concepts of α-ψ-proximal contractive and α-proximal admissible mappings and established best proximity point theorems for such mappings defined on complete metric spaces. Subsequently, Hussain et al. [9] modified the aforesaid notions and substantiated certain best proximity point theorems.

Definition 2.1 ([12]). *Let $\mathcal{T} : P \to Q$ and $\alpha : P \times P \to [0,\infty)$ be given mappings. Then $\mathcal{T}$ is said to be α-proximal admissible, if*

$$\left.\begin{array}{l} \alpha(u_1,u_2) \geq 1 \\ d(p_1, \mathcal{T}u_1) = d(P,Q) \\ d(p_2, \mathcal{T}u_2) = d(P,Q) \end{array}\right\} \Longrightarrow \alpha(p_1,p_2) \geq 1,$$

for all $u_1, u_2, p_1, p_2 \in P$.

Definition 2.2 ([9]). *Let $\mathcal{T} : P \to Q$ and $\alpha, \eta : P \times P \to [0,\infty)$ be given mappings. Then $\mathcal{T}$ is said to be (α,η)-proximal admissible, if*

$$\left.\begin{array}{l} \alpha(u_1,u_2) \geq \eta(u_1,u_2) \\ d(p_1, \mathcal{T}u_1) = d(P,Q) \\ d(p_2, \mathcal{T}u_2) = d(P,Q) \end{array}\right\} \Longrightarrow \alpha(p_1,p_2) \geq \eta(p_1,p_2),$$

for all $u_1, u_2, p_1, p_2 \in P$.

Note that if we take $\eta(u,v) = 1$ for all $u, v \in P$, then the previous definition reduces to Definition 2.1.

Very recently, Tchier et al. [22] introduced the concept of $\mathcal{Z}$-proximal contractions as follows.

Definition 2.3 ([22]). *Let P and Q be two nonempty subsets of a metric space (M,d). A non-self-mapping $\mathcal{T} : P \to Q$ is said to be a $\mathcal{Z}$-proximal contraction, if there exists a simulation function $\zeta \in \mathcal{Z}_2$ such that*

$$\left.\begin{array}{l} d(p, \mathcal{T}u) = d(P,Q) \\ d(q, \mathcal{T}v) = d(P,Q) \end{array}\right\} \Longrightarrow \zeta(d(p,q), d(u,v)) \geq 0, \tag{2.1}$$

for all $p, q, u, v \in P$.

Let us introduce the following notions which will be used in our main results.

Definition 2.4. *Let $\mathcal{T} : P \to Q$ and $\alpha, \eta : P \times P \to [0, \infty)$ be given mappings. Then $\mathcal{T}$ is said to be triangular (α, η)-proximal admissible, if*

(1) *$\mathcal{T}$ is (α, η)-proximal admissible;*
(2) *$\alpha(u, v) \geq \eta(u, v)$ and $\alpha(v, z) \geq \eta(v, z)$ implies that $\alpha(u, z) \geq \eta(u, z)$, for all $u, v, z \in P$.*

Definition 2.5. *Let P and Q be two nonempty subsets of a metric space (M, d), $\zeta \in \mathcal{Z}_4$ and $\alpha, \eta : P \times P \to [0, \infty)$ be mappings. A non-self mapping $\mathcal{T} : P \to Q$ is said to be (α, η)-$\mathcal{Z}$-proximal contraction, if*

$$\left.\begin{array}{l} \alpha(u, v) \geq \eta(u, v) \\ d(p, \mathcal{T}u) = d(P, Q) \\ d(q, \mathcal{T}v) = d(P, Q) \end{array}\right\} \Longrightarrow \zeta(d(p, q), d(u, v)) \geq 0, \tag{2.2}$$

for all $p, q, u, v \in P$.

We provide the following examples illustrating Definition 2.5 where Definition 2.3 is not applicable.

Example 2.6. *Let $M = \mathbb{R}$ be endowed with the usual metric d, $P = [0, \frac{1}{2}] \cup \{1, 10\}$ and $Q = [0, \frac{1}{6}] \cup \{1, 10\}$. Define a mapping $\mathcal{T} : P \to Q$ by*

$$\mathcal{T}u = \begin{cases} 10, & \text{if } u = 1, \\ 1, & \text{if } u = 10, \\ \frac{u}{6}, & \text{if } u \in \left[0, \frac{1}{2}\right]. \end{cases}$$

It is obvious that $d(P, Q) = 0$ and $P_0 = Q_0 = Q$. Now, define $\alpha, \eta : P \times P \to [0, \infty)$ by

$$\alpha(u, v) = \begin{cases} 4, & \text{if } u, v \in \left[0, \frac{1}{2}\right], \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \eta(u, v) = 2.$$

Then $\mathcal{T}$ is (α, η)-$\mathcal{Z}$-proximal contraction, but not a $\mathcal{Z}$-proximal contraction where $\zeta(t, s) = \frac{1}{2}s - t$ for all $t, s \in [0, \infty)$. Indeed, let us consider

$$\begin{array}{l} \alpha(u, v) \geq \eta(u, v), \\ d(p, \mathcal{T}u) = d(q, \mathcal{T}v) = d(P, Q). \end{array} \tag{2.3}$$

Taking into account (2.3), we get that $u, v \in \left[0, \frac{1}{2}\right]$, and so $p = \mathcal{T}u = \frac{u}{6}$ and $q = \mathcal{T}v = \frac{v}{6}$. Then

$$\begin{aligned} \zeta\left(d(p, q), d(u, v)\right) &= \frac{1}{2} d(u, v) - d(p, q) \\ &= \frac{1}{2} |u - v| - \frac{1}{6} |u - v| \geq 0. \end{aligned}$$

It means that $\mathcal{T}$ is (α, η)-$\mathcal{Z}$-proximal contraction. On the other hand, let

$$\begin{array}{l} d(0, \mathcal{T}0) = d(P, Q) = 0, \\ d(10, \mathcal{T}1) = d(P, Q) = 0. \end{array}$$

Then

$$\begin{aligned}\zeta\left(d(0,10),d(0,1)\right) &= \frac{1}{2}d(0,1) - d(0,10)\\ &= \frac{1}{2} - 10 \not\geq 0,\end{aligned}$$

and hence $\mathcal{T}$ is not a $\mathcal{Z}$-proximal contraction.

Example 2.7. *Let $M = \{(0,1),(1,0),(-1,0),(0,-1)\}$ be endowed with the Euclidian metric d. Consider $P = \{(0,1),(1,0)\}$ and $Q = \{(0,-1),(-1,0)\}$. We have $d(P,Q) = \sqrt{2}$. Let $\mathcal{T} : P \to Q$ be given as $\mathcal{T}(u,v) = (-v,-u)$. Choose $\zeta(t,s) = ks - t$ for $s,t \geq 0$, with $k \in (0,1)$. Take $\alpha,\eta : P \times P \to [0,\infty)$ as*

$$\alpha(u,v) = \begin{cases}1, & \text{if } u = v,\\ 0, & \text{otherwise,}\end{cases} \quad \text{and} \quad \eta(u,v) = \begin{cases}\frac{1}{4}, & \text{if } u = v,\\ 3, & \text{otherwise.}\end{cases}$$

Let $u,v,p,q \in P$ such that

$$\alpha(u,v) \geq \eta(u,v) \quad \text{and} \quad d(p,\mathcal{T}u) = d(q,\mathcal{T}v) = d(P,Q) = \sqrt{2}.$$

We should have $u = v = p = q = (0,1)$ or $u = v = p = q = (1,0)$. Then, $\zeta(d(p,q),d(u,v)) = \zeta(0,0) = 0$, that is, $\mathcal{T}$ is (α,η)-$\mathcal{Z}$-proximal contraction.

On the other hand, by taking $u = p = (0,1)$ and $q = v = (1,0)$, we have

$$d(p,\mathcal{T}u) = d(q,\mathcal{T}v) = d(P,Q),$$

but $\zeta(d(p,q),d(u,v)) = \zeta(\sqrt{2},\sqrt{2}) = (k-1)\sqrt{2} < 0$, that is, $\mathcal{T}$ is not a $\mathcal{Z}$-proximal contraction.

3. Main Results

The first result of this study is the following.

Theorem 3.1. *Let (P,Q) be a pair of nonempty subsets of a complete metric space (M,d) such that P_0 is nonempty, $\mathcal{T} : P \to Q$ and $\alpha,\eta : P \times P \to [0,\infty)$ be given mappings. Suppose the following conditions are satisfied:*

- *(i) P_0 is closed and $\mathcal{T}(P_0) \subseteq Q_0$;*
- *(ii) $\mathcal{T}$ is triangular (α,η)-proximal admissible;*
- *(iii) there exist $u_0,u_1 \in P_0$ such that $d(u_1,\mathcal{T}u_0) = d(P,Q)$ and $\alpha\left(u_0,u_1\right) \geq \eta\left(u_0,u_1\right)$;*
- *(iv) $\mathcal{T}$ is a continuous (α,η)-$\mathcal{Z}$-proximal contraction.*

Then $\mathcal{T}$ has a best proximity point in P. If $\alpha(u,v) \geq \eta(u,v)$ for all $u,v \in B_{est}(\mathcal{T})$, then $\mathcal{T}$ has a unique best proximity point $u^* \in P$. Moreover, for each $u \in M$, we have $\lim_{n\to\infty} \mathcal{T}^n u = u^*$.

Proof. By virtue of the assertion *(iii)*, there exist $u_0,u_1 \in P_0$ such that

$$d(u_1,\mathcal{T}u_0) = d(P,Q) \quad \text{and} \quad \alpha\left(u_0,u_1\right) \geq \eta\left(u_0,u_1\right).$$

Since $\mathcal{T}(P_0) \subseteq Q_0$, there exists $u_2 \in P_0$ such that

$$d(u_2, \mathcal{T}u_1) = d(P, Q).$$

Thus, we get

$$\begin{aligned} &\alpha(u_0, u_1) \geq \eta(u_0, u_1), \\ &d(u_1, \mathcal{T}u_0) = d(u_2, \mathcal{T}u_1) = d(P, Q). \end{aligned}$$

Since $\mathcal{T}$ is an (α, η)-proximal admissible, we conclude that $\alpha(u_1, u_2) \geq \eta(u_1, u_2)$. Now, we have

$$d(u_2, \mathcal{T}u_1) = d(P, Q) \quad \text{and} \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2).$$

Again, since $\mathcal{T}(P_0) \subseteq Q_0$, there exists $u_3 \in P_0$ such that

$$d(u_3, \mathcal{T}u_2) = d(P, Q),$$

and thus

$$\begin{aligned} &\alpha(u_1, u_2) \geq \eta(u_1, u_2), \\ &d(u_2, \mathcal{T}u_1) = d(u_3, \mathcal{T}u_2) = d(P, Q). \end{aligned}$$

Since $\mathcal{T}$ is (α, η)-proximal admissible, this implies that $\alpha(u_2, u_3) \geq \eta(u_2, u_3)$. Thereby, we have

$$d(u_3, \mathcal{T}u_2) = d(P, Q) \quad \text{and} \quad \alpha(u_2, u_3) \geq \eta(u_2, u_3).$$

By repeating this process, a sequence $\{u_n\}$ in P_0 can be constituted by the following way:

$$d(u_{n+1}, \mathcal{T}u_n) = d(P, Q) \quad \text{and} \quad \alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1}), \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$. If there exists n_0 such that $u_{n_0} = u_{n_0+1}$, then

$$d(u_{n_0}, \mathcal{T}u_{n_0}) = d(u_{n_0+1}, \mathcal{T}u_{n_0}) = d(P, Q).$$

This means that u_{n_0} is a best proximity point of $\mathcal{T}$ and the proof is finalized. Due to this reason, we suppose that $u_n \neq u_{n+1}$ for all n. Using (3.1), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} &\alpha(u_{n-1}, u_n) \geq \eta(u_{n-1}, u_n), \\ &d(u_n, \mathcal{T}u_{n-1}) = d(u_{n+1}, \mathcal{T}u_n) = d(P, Q). \end{aligned}$$

Since $\mathcal{T}$ is an (α, η)-$\mathcal{Z}$-proximal contraction, for all $n \in \mathbb{N}$, we obtain

$$0 \leq \zeta(d(u_n, u_{n+1}), d(u_{n-1}, u_n)) < d(u_{n-1}, u_n) - d(u_n, u_{n+1}). \tag{3.2}$$

It follows from the above inequality that

$$0 < d(u_n, u_{n+1}) < d(u_{n-1}, u_n), \quad \text{for all } n \in \mathbb{N}.$$

Therefore the sequence $\{d(u_n, u_{n+1})\}$ is decreasing and so there exists $r \geq 0$ such that $\lim_{n\to\infty} d(u_n, u_{n+1}) = r$. Now, our purpose is to show that $r = 0$. On the contrary, assume that $r > 0$. Set the sequences $\{a_n\}$ and $\{b_n\}$ as $a_n = d(u_n, u_{n+1})$ and $b_n = d(u_{n-1}, u_n)$. Then since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r$ and $a_n < b_n$ for all n, by the axiom (ζ_4), we deduce

$$0 \leq \limsup_{n\to\infty} \zeta(d(u_n, u_{n+1}), d(u_{n-1}, u_n)) < 0,$$

which is a contradiction. That's why $r = 0$, that is,

$$\lim_{n\to\infty} d\left(u_n, u_{n+1}\right) = 0. \tag{3.3}$$

Let us prove now that $\{u_n\}$ is a Cauchy sequence in P_0. Suppose, to the contrary, that $\{u_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and

$$d\left(u_{m_k}, u_{n_k}\right) \geq \varepsilon \quad \text{and} \quad d\left(u_{m_k}, u_{n_k-1}\right) < \varepsilon. \tag{3.4}$$

Using the triangular inequality and (3.4), we have

$$\begin{aligned} \varepsilon \leq d\left(u_{m_k}, u_{n_k}\right) &\leq d\left(u_{m_k}, u_{n_k-1}\right) + d\left(u_{n_k-1}, u_{n_k}\right) \\ &< \varepsilon + d\left(u_{n_k-1}, u_{n_k}\right). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.3), we obtain

$$\lim_{k\to\infty} d\left(u_{m_k}, u_{n_k}\right) = \varepsilon. \tag{3.5}$$

Again, using the triangular inequality,

$$\left|d\left(u_{m_k+1}, u_{n_k+1}\right) - d\left(u_{m_k}, u_{n_k}\right)\right| \leq d\left(u_{m_k+1}, u_{m_k}\right) + d\left(u_{n_k}, u_{n_k+1}\right),$$

which yields that

$$\lim_{k\to\infty} d\left(u_{m_k+1}, u_{n_k+1}\right) = \varepsilon. \tag{3.6}$$

Since $\mathcal{T}$ is triangular (α, η)-proximal admissible, by using (3.1), we infer that

$$\alpha(u_m, u_n) \geq \eta(u_m, u_n), \text{ for all } n, m \in \mathbb{N} \cup \{0\} \text{ with } m < n. \tag{3.7}$$

By combining (3.1) and (3.7), for all $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} &\alpha(u_{m_k}, u_{n_k}) \geq \eta(u_{m_k}, u_{n_k}), \\ &d(u_{m_k+1}, \mathcal{T}u_{m_k}) = d(u_{n_k+1}, \mathcal{T}u_{n_k}) = d(P, Q). \end{aligned}$$

Since $\mathcal{T}$ is an (α, η)-$\mathcal{Z}$-proximal contraction, the last equation gives us that, for all $k \in \mathbb{N} \cup \{0\}$

$$0 \leq \zeta(d(u_{m_k+1}, u_{n_k+1}), d(u_{m_k}, u_{n_k})) < d(u_{m_k}, u_{n_k}) - d(u_{m_k+1}, u_{n_k+1}). \tag{3.8}$$

Choose the sequences $\{a_k = d(u_{m_k+1}, u_{n_k+1})\}$ and $\{b_k = d(u_{m_k}, u_{n_k})\}$. Then, from (3.5), (3.6) and (3.8), we conclude that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \varepsilon$ and $a_k < b_k$ for all k. Taking lim sup of (3.8) and considering (ζ_4), we get

$$0 \leq \limsup_{n\to\infty} \zeta\left((d(u_{m_k+1}, u_{n_k+1}), d(u_{m_k}, u_{n_k})\right) < 0,$$

which is a contradiction. Accordingly, $\{u_n\}$ is a Cauchy sequence in P_0. Since P_0 is a closed subset of the complete metric space (M, d), there exists $u \in P_0$ such that

$$\lim_{n\to\infty} d(u_n, u) = 0.$$

In view of the fact that $\mathcal{T}$ is continuous, we deduce that

$$\lim_{n\to\infty} d(\mathcal{T}u_n, \mathcal{T}u) = 0.$$

Thus, using the last two equations and (3.1), we have

$$d(P,Q) = \lim_{n\to\infty} d(u_{n+1}, \mathcal{T}u_n) = d(u, \mathcal{T}u),$$

which means that $u \in P_0 \subseteq P$ is a best proximity point of $\mathcal{T}$. As the final step, we shall show that the set $B_{est}(\mathcal{T})$ is a singleton. Assume that v is another best proximity point of $\mathcal{T}$. Then, by hypothesis, we have $\alpha(u,v) \geq \eta(u,v)$, and thus

$$\begin{aligned} &\alpha(u,v) \geq \eta(u,v), \\ &d(u, \mathcal{T}u) = d(v, \mathcal{T}v) = d(P,Q). \end{aligned}$$

Then, by the argument (iv), we infer that

$$0 \leq \zeta\left(d(u,v), d(u,v)\right) < d(u,v) - d(u,v) = 0,$$

which is a contradiction. Thus, the best proximity point of $\mathcal{T}$ is unique. □

The following example illustrates Theorem 3.1.

Example 3.2. *Let $M = [0,\infty)\times[0,\infty)$ be endowed with the metric $d((u_1,u_2),(v_1,v_2)) = |u_1 - v_1| + |u_2 - v_2|$. Take $P = \{1\} \times [0,\infty)$ and $Q = \{0\} \times [0,\infty)$. We mention that $d(P,Q) = 1$, $P_0 = P$ and $Q_0 = Q$. Consider the mapping $\mathcal{T} : P \to Q$ as*

$$\mathcal{T}(1,u) = \begin{cases} (0, \frac{u^2+1}{4}) & \text{if } 0 \leq u \leq 1 \\ (0, u - \frac{1}{2}) & \text{if } u > 1. \end{cases}$$

Note that $\mathcal{T}$ is continuous at $u_0 = 1$ and $\mathcal{T}(P_0) \subseteq Q_0$. Consider $\zeta(a,b) = kb - a$ with $k \in (\frac{1}{2}, 1)$, for all $a, b \geq 0$. Define $\alpha, \eta : P \times P \to [0,\infty)$ as follows

$$\begin{cases} \alpha((1,u),(1,v)) = 1 & \text{if} \quad u,v \in [0,1] \\ \alpha((1,u),(1,v)) = 0 & \text{if not,} \end{cases} \quad \text{and} \quad \begin{cases} \eta((1,u),(1,v)) = \frac{1}{3} & \text{if} \quad u,v \in [0,1] \\ \eta((1,u),(1,v)) = 2 & \text{if not.} \end{cases}$$

Let $(1,u),(1,v),(1,p)$ and $(1,q)$ in P such that

$$\begin{cases} \alpha((1,u),(1,v)) \geq \eta((1,u),(1,v)) \\ d((1,p), \mathcal{T}(1,u)) = d(P,Q) = 1, \\ d((1,q), \mathcal{T}(1,v)) = d(P,Q) = 1. \end{cases}$$

Then, necessarily, $(u,v) \in [0,1] \times [0,1]$. Also, $p = \frac{1+u^2}{4}$ and $q = \frac{1+v^2}{4}$. Here, we have that $\alpha((1,p),(1,q)) \geq \eta((1,p),(1,q))$, that is, $\mathcal{T}$ is (α,η)-proximal admissible.

Moreover,

$$\begin{aligned}\zeta(d((1,p),(1,q)),d((1,u),(1,v))) &= \zeta(d((1,\frac{1+u^2}{4}),(1,\frac{1+v^2}{4})),d((1,u),(1,v)))\\ &= \zeta(|\frac{u^2}{4}-\frac{v^2}{4}|,|u-v|)\\ &= k|u-v|-|\frac{u^2}{4}-\frac{v^2}{4}|\\ &= k|u-v|-\frac{1}{4}(u+v)|u-v|\\ &\geq (k-\frac{1}{2})|u-v|\geq 0.\end{aligned}$$

Then $\mathcal{T}$ is an (α,η)-$\mathcal{Z}$-proximal contraction. Also, for $u_0=(1,1)$ and $u_1=(1,\frac{1}{2})$, we have

$$d(u_1,\mathcal{T}u_0)=d((1,\frac{1}{2}),(0,\frac{1}{2}))=1=d(P,Q)\quad \textit{and}\quad \alpha(u_0,u_1)\geq\eta(u_0,u_1),$$

that is, condition (iii) holds. Moreover, it is obvious that $\mathcal{T}$ is triangular (α,η)-proximal admissible. All hypotheses of Theorem 3.1 are verified, so $\mathcal{T}$ admits a best proximity point, which is $u=(1,2-\sqrt{3})$.

In the subsequent result, we replace the continuity assertion in the previous theorem with the following condition in P:

(C) If a sequence $\{u_n\}$ in P converges to $u\in P$ such that $\alpha(u_n,u_{n+1})\geq\eta(u_n,u_{n+1})$, then $\alpha(u_n,u)\geq\eta(u_n,u)$ for all $n\in\mathbb{N}$.

Theorem 3.3. *Let (P,Q) be a pair of nonempty subsets of a complete metric space (M,d) such that P_0 is nonempty, $\mathcal{T}:P\to Q$ and $\alpha,\eta:P\times P\to[0,\infty)$ be given mappings. Suppose the following conditions are satisfied:*

(i) P_0 is closed and $\mathcal{T}(P_0)\subseteq Q_0$;
(ii) $\mathcal{T}$ is triangular (α,η)-proximal admissible;
(iii) there exist $u_0,u_1\in P_0$ such that $d(u_1,\mathcal{T}u_0)=d(P,Q)$ and $\alpha(u_0,u_1)\geq\eta(u_0,u_1)$;
(iv) (C) holds and $\mathcal{T}$ is an (α,η)-$\mathcal{Z}$-proximal contraction.

Then $\mathcal{T}$ has a best proximity point in P. If $\alpha(u,v)\geq\eta(u,v)$ for all $u,v\in B_{est}(\mathcal{T})$, then $\mathcal{T}$ has a unique best proximity point $u^*\in P$. Moreover, for each $u\in M$, we have $\lim_{n\to\infty}\mathcal{T}^nu=u^*$.

Proof. By pursuing on the lines of the proof of Theorem 3.1, there exists a Cauchy sequence $\{u_n\}\subset P_0$ satisfying the expression (3.1) and $u_n\to p$. In view of (i), P_0 is closed and so $p\in P_0$. Also, since $\mathcal{T}(P_0)\subseteq Q_0$, there exists $z\in P_0$ such that

$$d(z,\mathcal{T}p)=d(P,Q). \tag{3.9}$$

On the other hand, by (C), we get

$$\alpha(u_n,p)\geq\eta(u_n,p),\quad \text{for all } n\in\mathbb{N}.$$

Thus, from (3.1), we have

$$\alpha(u_n, p) \geq \eta(u_n, p),$$
$$d(u_{n+1}, \mathcal{T}u_n) = d(z, \mathcal{T}p) = d(P, Q).$$

Therefore, from the assertion (iv), we conclude

$$0 \leq \zeta(d(u_{n+1}, z), d(u_n, p)) < d(u_n, p) - d(u_{n+1}, z), \tag{3.10}$$

and so

$$\lim_{n\to\infty} d(u_{n+1}, z) \leq 0.$$

By the uniqueness of limit, we obtain $z = p$. Consequently, from (3.9), we have $d(p, \mathcal{T}p) = d(P, Q)$. Uniqueness of the best proximity point follows from the proof of Theorem 3.1. □

Example 3.4. *Let $X = \mathbb{R}^2$ be endowed with the Euclidian metric, $P = \{(0, u) : u \geq 0\}$ and $Q = \{(1, u) : u \geq 0\}$. Note that $d(P, Q) = 1$, $P_0 = P$ and $Q_0 = Q$. Define $\mathcal{T} : P \to Q$ and $\alpha : P \times P \to [0, \infty)$ by*

$$\mathcal{T}(0, u) = \begin{cases} (1, \frac{u}{9}), & \text{if } 0 \leq u \leq 1, \\ (1, \frac{1}{2}), & \text{if } u > 1, \end{cases}$$

and

$$\alpha((0, u), (0, v)) = \begin{cases} 2\eta((0, u), (0, v)), & \text{if } \ u, v \in [0, 1] \text{ or } u = v, \\ 0 = \eta((0, u), (0, v)), & \text{otherwise.} \end{cases}$$

Choose $\zeta(a, b) = \frac{2}{3}b - a$ for all $a, b \in [0, \infty)$. Let $u, v, p, q \geq 0$ such that

$$\begin{cases} \alpha((0, u), (0, v)) \geq \eta((0, u), (0, v)) \\ d((0, p), \mathcal{T}(0, u)) = d(P, Q) = 1 \\ d((0, q), \mathcal{T}(0, v)) = d(P, Q) = 1. \end{cases}$$

Then $u, v \in [0, 1]$ or $u = v$. We distinguish the following cases.
Case 1: *$u, v \in [0, 1]$. Here, $\mathcal{T}(0, u) = (1, \frac{u}{9})$ and $\mathcal{T}(0, v) = (1, \frac{v}{9})$. Also,*

$$\sqrt{1 + (p - \frac{u}{9})^2} = \sqrt{1 + (q - \frac{v}{9})^2} = 1,$$

that is, $p = \frac{u}{9}$ and $q = \frac{v}{9}$. So, $\alpha((0, p), (0, q)) \geq \eta((0, p), (0, q))$. Moreover,

$$\begin{aligned} \zeta(d((0, p), (0, q)), d((0, u), (0, v))) &= \frac{2}{3}d((0, u), (0, v)) - d((0, \frac{u}{9}), (0, \frac{v}{9})) \\ &= \frac{2}{3}|u - v| - \frac{|u - v|}{9} \geq 0. \end{aligned}$$

Case 2: *$u = v > 1$. Here, $\mathcal{T}(0, u) = (1, \frac{1}{2})$ and $\mathcal{T}(0, v) = (1, \frac{1}{2})$. Similarly, we get that $p = q = \frac{1}{2}$. So, $\alpha((0, p), (0, q)) \geq \eta((0, p), (0, q))$. Also, $\zeta(d((0, p), (0, q)), d((0, u), (0, v))) \geq 0$.*
Case 3: *$u, v > 1$ with $u \neq v$. Then, the proof is similar to that in Case 2.*

In each case, we get that $\mathcal{T}$ is (α,η)-proximal admissible. It is also easy to see that $\mathcal{T}$ is triangular (α,η)-proximal admissible. Also, $\mathcal{T}$ is (α,η)-$\mathcal{Z}$-proximal contraction. Moreover, if $\{u_n=(0,p_n)\}$ is a sequence in P such that $\alpha(u_n,u_{n+1})\geq\eta(u_n,u_{n+1})$ for all n and $u_n=(0,p_n)\to u=(0,p)$ as $n\to\infty$, then $p_n\to p$. We have $p_n,p_{n+1}\in[0,1]$ or $p_n=p_{n+1}$. We get that $p\in[0,1]$ or $p_n=p$. This implies that $\alpha(u_n,u)\geq\eta(u_n,u)$ for all n.

Also, there exists $(u_0,u_1)=((0,1),(0,\frac{1}{9}))\in P_0\times P_0$ such that

$$d(u_1,\mathcal{T}u_0)=1=d(P,Q)\quad\text{and}\quad\alpha(u_0,u_1)\geq\eta(u_0,u_1).$$

Consequently, all conditions of Theorem 3.3 are satisfied. Therefore, $\mathcal{T}$ has a unique best proximity point in P which is $(0,0)$.

Corollary 3.5. *Let (P,Q) be a pair of nonempty subsets of a complete metric space (M,d). Suppose that $\mathcal{T}:P\to Q$ is a $\mathcal{Z}$-proximal contraction and P_0 is nonempty closed subset of M with $\mathcal{T}(P_0)\subseteq Q_0$. Then $\mathcal{T}$ has a unique best proximity point $u^*\in P$. Moreover, for each $u\in M$, we have $\lim_{n\to\infty}\mathcal{T}^nu=u^*$.*

Proof. The proof follows from Theorem 3.1 (Theorem 3.3), if we take $\alpha(u,v)=\eta(u,v)=1$. □

Remark 3.6. *Theorem 3.1 (Theorem 3.3) extend and improve various best proximity point and fixed point results in complete metric spaces. Furthermore, some best proximity point and fixed point results in metric spaces endowed with a graph or a binary relation can be derived from our results under some suitable α-admissible mappings.*

4. A Variational Inequality Problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H, with inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. A variational inequality problem can be stated as follows:

$$\text{Find } u\in C \text{ such that } \langle Su,v-u\rangle\geq 0 \text{ for all } v\in C, \tag{4.1}$$

where $S:H\to H$ is a given operator. This problem has been a classical subject in economics, operations research and mathematical physics, particularly in the calculus of variations associated with the minimization of infinite-dimensional functionals; see, for instance, [15] and the references therein. It is closely related to many problems of nonlinear analysis, such as optimization, complementarity and equilibrium problems and finding fixed points; see, for instance, [8,15,23]. To solve problem (4.1), we define the metric projection operator $P_C:H\to C$. Here, we recall that for each $u\in H$, there exists a unique nearest point $P_Cu\in C$ satisfying the inequality

$$\|u-P_Cu\|\leq\|u-v\|,\quad\text{for all } v\in C.$$

The following lemmas correlate the solvability of a variational inequality problem to the solvability of a special fixed point problem.

Lemma 4.1. *Let $z \in H$. Then $u \in C$ satisfies the inequality $\langle u - z, y - u\rangle \geq 0$, for all $y \in C$ if and only if $u = P_C z$.*

Lemma 4.2. *Let $S : H \to H$. Then $u \in C$ is a solution of $\langle Su, v - u\rangle \geq 0$, for all $v \in C$, if and only if $u = P_C(u - \lambda Su)$, with $\lambda > 0$.*

Theorem 4.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $S : H \to H$ is such that $P_C(I - \lambda S) : C \to C$ is a $\mathcal{Z}$-proximal contraction. Then there exists a unique element $u^* \in C$ such that $\langle Su^*, v - u^*\rangle \geq 0$ for all $v \in C$. Moreover, for any arbitrary element $u_0 \in C$, the sequence $\{u_n\}$ defined by $u_{n+1} = P_C(u_n - \lambda Su_n)$ where $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, converges to u^*.*

Proof. We consider the operator $F : C \to C$ defined by $Fx = P_C(x - \lambda Sx)$ for all $x \in C$. By Lemma 4.2, $u \in C$ is a solution of $\langle Su, v - u\rangle \geq 0$ for all $v \in C$, if and only if $u = Fu$. Now, F satisfies all the hypotheses of Corollary 3.5 with $P = Q = C$. It now follows from Corollary 3.5 that the fixed point problem $u = Fu$ admits a unique solution $u^* \in C$. □

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