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TRACE REGULARIZATION PROBLEM FOR HIGHER ORDER DIFFERENTIAL OPERATOR

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Abstract. We establish a regularized trace formula for higher order self-adjoint differential operator with unbounded operator coefficient.

1. Introduction and History

The first study on the regularized trace of scalar differential operators was performed by Gelfand and Levitan [10]. They studied the boundary value problem

$$y'' + q\,(x)\,y = \lambda y, \quad y'\,(0) = y'\,(\pi) = 0 \quad \text{with } q(x)\;\epsilon\;C^1[0,\pi]$$

and they found the formula

$$\sum_{n=0}^{\infty} (\lambda_n - \mu_n) = \frac{1}{4}\left(q\,(0) + q(\pi)\right)\ ,$$

under the assumption $\int_0^\pi q(x)dx = 0$. Where the μ_n are the eigenvalues of this problem. $\lambda_n = n^2$ are the eigenvalues of the same problem with $q\,(x) = 0$.
After that original work by Gelfand-Levitan, there was a huge interest and many scientists used the same method to obtain the regularized traces of ordinary differential operators. Later, Dikii [5] gave another proof of Gelfand-Levitan's formula from a different point of view. Afterward, Dikii [6] and Gelfand [9] made significant progress in literature by computing regularized sums of powers of eigenvalues. Later on, Levitan [17] calculated the regularized traces of Sturm Liouville Problem with a new method. This research led to Faddeev [7], who connected the trace theory with singular differential operators. Gasimov [8] made the first study combining singular operators with discrete spectrum.

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Thereafter, many scientists such as Halberg and Kramer [13], Jafaev [15], Makin [19], Yang [23] investigated the regularized traces of various scalar differential operators. The list of these works is given in Levitan and Sargsyan [18] and Sadovnichii and Podolskii [21].
Among the studies, only a few of them are focused on the regularized trace of operator-differential equation with operator coefficient. Halilova [14] obtained the regularized trace of the Sturm-Liouville equation with bounded operator coefficient. Adıgüzelov [1] found a formulation of the subtracting eigenvalues of two self-adjoint operators in [0,∞) with bounded operator coefficient. Bayramoğlu and Adıgüzelov [4] examined the regularized trace of singular second order differential operator with bounded operator coefficient. Adıgüzelov and Baksi [2], Sen, Bayramov and Oruçoğlu [22] and Adıgüzelov, Avcı and Gül [12] obtained the equalities for the regularized traces of differential operators with bounded operator coefficient. Aslanova [3] calculated the trace formula of Bessel equation with spectral parameter-dependent boundary condition.
Maksudov, Bayramoğlu and Adıgüzelov [20] investigated the regularized trace formulation of the Sturm Liouville equation with unbounded operator coefficient.
In the present paper, we compute the regularized trace formula for higher order Sturm-Liouville problem

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left(\alpha_q-\beta_q-\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right)=\frac{1}{4}\left(trQ(0)-trQ(\pi)\right) .$$

2. Notation and Preliminaries

Let H be an infinite dimensional separable Hilbert space with inner product $(.,.)$ and corresponding norm $\|.\|$. Let $H_1 = L_2(0,\pi;\ H)$ be the set of all strongly measurable functions f defined on $[0,\pi]$ and taking the values in the space H. The following conditions hold for every $f \in H_1$:
1. The scalar function $(f(x),g)$ is Lebesgue measurable on $[0,\pi]$, for every $g\in H$,
2. $\int_0^{\pi}\|f(x)\|^2dx<\infty$.
H_1 is a normed linear space. We will denote the inner product and norm by $(.,.)_{H_1}$ and $\|.\|_{H_1}$ in H_1. If the inner product is defined as $(f_1,f_2)_{H_1} = \int_0^{\pi}(f_1(x),f_2(x))dx$, for any arbitrary elements f_1, f_2 of H_1, then H_1 becomes a separable Hilbert space, [16]. Let $\{\Phi_q(x)\}_1^{\infty}$ be an orthonormal basis of H_1.
Consider the following differential expressions

$$\ell_0(v)=(-1)^m v^{(2m)}(x)+Av(x), \qquad (m\in\ \mathbb{Z}^+)$$

$$\ell(v)=(-1)^m v^{(2m)}(x)+Av(x)+Q(x)v(x). \tag{2.1}$$

with boundary conditions

$$v^{(2i+1)}(0)=v^{(2i)}(\pi)=0, \qquad (i=0,1,\ldots,m-1)$$

in H_1. Here, A is a densely defined operator in H. This operator takes its values in H and satisfies the conditions $A=A^*\geq I$, $A^{-1}\in\sigma_\infty(H)$, where I is the identity operator of H. $\sigma_\infty(H)$ denotes the set of all completely continuous operators from H to H.

Let $\{\gamma_i\}_{i=1}^{\infty}$ be the increasing sequence of eigenvalues of the operator A counted with respect to their multiplicities and a corresponding orthonormal sequence $\{\varphi_i\}_{i=1}^{\infty}$ of eigenvectors.

Denote by $D(L_0^{'})$ the set of the functions $v(x)$ in the space H_1, and the following conditions are satisfied:

(v1) $v(x)$ has continuous $2m^{\ th}$ order derivative on $[0,\pi]$ with respect to the norm in the space H,

(v2) $v(x) \in D(A)$ for every $x \in [0,\pi]$, and $Av(x)$ is continuous on $[0,\pi]$ with respect to the norm in H,

(v3) $v^{(2i+1)}(0) = v^{(2i)}(\pi) = 0, \qquad (i = 0,1,2,\cdots,m-1)$.

Here, $D(L_0^{'})$ is dense in H_1. Define a linear operator $L_0^{'} : D(L_0^{'}) \to H_1$ as $L_0^{'}v = \ell_0(v)$.

The construction above shows that $L_0^{'}$ is symmetric. Considering the linearity of $L_0^{'}$, its eigenvalues can be calculated by mathematical induction. Therefore, the eigenvalues of $L_0^{'}$ are the form $(k+\frac{1}{2})^{2m}+\gamma_j, \qquad (k = 0,1,2,\cdots; \quad j = 1,2,\cdots)$ and the orthonormal eigenvectors corresponding to these eigenvalues are the form $\sqrt{\frac{2}{\pi}}\varphi_j cos\left(k+\frac{1}{2}\right)x$. We can see that the orthonormal eigenvector sequence of the symmetric operator $L_0^{'}$ is a complete orthonormal system in H_1. Since $L_0^{'}$ is symmetric, then it is closable. Thus, we can define L_0 as $L_0 = \overline{L_0^{'}}$

Assume that the operator function $Q(x)$ in 2.1 verifies the conditions:

(*Q*1) $Q(x) : H \to H$ is a self-adjoint operator for every $x \in [0,\pi]$,

(*Q*2) $Q(x)$ is weak measurable on $[0,\pi]$, that is the scalar function $(Q(x)f,g)$ is measurable on $[0,\pi]$ for every $f,g \in H$,

(*Q*3) The function $\|Q(x)\|$ is bounded on $[0,\pi]$.

In the present paper, we establish a regularized trace formula for the operator $L = L_0 + Q$.

Now, we search some inequalities for the eigenvalues and resolvent operators of L_0 and L.

Consider the closed symmetric operator $L_0 : D(L_0) \to H_1$.

Since the eigenvector system $\{\varphi_j \cos\ (\ k+\frac{1}{2})x\}_{k=0,j=1}^{\infty\quad\infty}$ of L_0 is complete, L_0 is self-adjoint, [2]. Moreover, since the bounded operator $Q : H_1 \to H_1$ is self-adjoint, the operator, $L = L_0 + Q$ is also self-adjoint. Therefore, L_0 and L have purely-discrete spectrum, [2]. Let $\{\beta_i\}_{i=1}^{\infty}$ and $\{\alpha_i\}_{i=1}^{\infty}$ be increasing sequences of eigenvalues of L_0 and L. Denote by $\rho(L_0)$ and $\rho(L)$ the resolvent sets of L_0 and L.

We can prove the Theorem 2.1, by using [2].

Theorem 2.1. *Let the operator function* $Q(x)$ *satisfy the conditions* $(Q1)$ *to* $(Q3)$. *If* $\gamma_j \sim aj^{\ell} \qquad (0 < a, \ell < \infty)$ *as* $j \to \infty$, *then* $\alpha_n, \beta_n \sim dn^{\frac{2m\ell}{2m+\ell}}$ *as* $n \to \infty$,

where $d = \left(\frac{\ell a^{\frac{1}{\ell}}}{2b}\right)^{\frac{2m\ell}{2m+\ell}}$ *and* $b = \int_0^{\frac{\pi}{2}}(sint)^{\frac{2}{\ell}-1}(cost)^{1+\frac{1}{m}}dt.$

From Theorem 2.1, one can see that the sequence $\{\beta_n\}$ has a subsequence $\beta_{n_1} < \beta_{n_2} < ... < \beta_{n_p} < ...$ such that

$$\beta_q - \beta_{n_p} > d_0\left(q^{\frac{2m\ell}{2m+\ell}} - n_p^{\frac{2m\ell}{2m+\ell}}\right), \ (q = n_p+1, n_p+2,\cdots). \tag{2.2}$$

Here, d_0 is a positive constant.
Let $R_\alpha^0 = (L_0 - \alpha I)^{-1}$, $R_\alpha = (L - \alpha I)^{-1}$ be the resolvent operators of L_0 and L .
If $\ell > \frac{2m}{2m-1}$, then by Theorem 2.1, R_α^0 and R_α are nuclear operators for $\alpha \neq \alpha_q, \beta_q \quad (q = 1, 2, ...)$. In this case, we have the formula

$$tr\left(R_\alpha - R_\alpha^0\right) = trR_\alpha - trR_\alpha^0 = \sum_{q=1}^{\infty}\left(\frac{1}{\alpha_q - \alpha} - \frac{1}{\beta_q - \alpha}\right), \tag{2.3}$$

[11]. Let $|\alpha| = b_p = 2^{-1}(\beta_{n_p+1} + \beta_{n_p})$. This says that for the large value of p , the inequalities
$\beta_{n_p} < b_p < \beta_{n_p+1}$ and $\alpha_{n_p} < b_p < \alpha_{n_p+1}$ are satisfied. By using the last inequalities, one can prove that the series $\sum_{q=1}^{\infty} \frac{\alpha}{\alpha_q - \alpha}$ *and* $\sum_{q=1}^{\infty} \frac{\alpha}{\beta_q - \alpha}$ are uniform convergent on the circle $|\alpha| = b_p$. Hence by 2.3

$$\sum_{q=1}^{n_p}(\alpha_q - \beta_q) = -\frac{1}{2\pi i}\int_{|\alpha|=b_p} \alpha tr(R_\alpha - R_\alpha^0)d\alpha. \tag{2.4}$$

We have two lemmas by using [2]:

Lemma 2.2. *If* $\gamma_j \sim aj^\ell$ *as* $j \to \infty$ *for* $a > 0, \quad \ell > \frac{2m}{2m-1}$ *, then*

$$\|R_\alpha^0\|_{\sigma_1(H_1)} < const.n_p^{1-\delta}, \qquad (\delta = \frac{2m\ell}{2m+\ell} - 1),$$

on the circle $|\alpha| = b_p$.

Lemma 2.3. *If the operator function* $Q(x)$ *satisfies conditions (Q1) to (Q3), and*
$\gamma_j \sim aj^\ell$ *as* $j \to \infty$ *, then for the large values of* p

$$\|R_\alpha\|_{H_1} < const.n_p^{-\delta}$$

on the circle $|\alpha| = b_p$ *, where* $a > 0, \quad \ell > \frac{2m}{2m-1}$.

3. Main Results

In this section, we will compute regularized trace formula for the operator L.
With the well-known formula $R_\alpha = R_\alpha^0 - R_\alpha Q R_\alpha^0 \quad (\alpha \in \rho(L_0) \cap \rho(L))$ and by 2.4, we obtain:

$$\sum_{q=1}^{n_p}(\alpha_q - \beta_q) = \sum_{j=1}^{s} E_{pj} + E_p^{(s)} \quad . \tag{3.1}$$

Here,

$$E_{pj} = \frac{(-1)^j}{2\pi ij}\int_{|\alpha|=b_p} tr\left[(QR_\alpha^0)^j\right]d\alpha, \qquad (j = 1, 2, ..), \tag{3.2}$$

$$E_p^{(s)} = \frac{(-1)^s}{2\pi i}\int_{|\alpha|=b_p} \alpha tr\left[R_\alpha (QR_\alpha^0)^{s+1}\right]d\alpha. \tag{3.3}$$

Theorem 3.1. *If the operator function* $Q(x)$ *satisfies conditions (Q1) to (Q3) and* $\gamma_j \sim a j^{\ell}$ *as* $j \to \infty$ *then*

$$\lim_{p\to\infty} E_{pj} = 0, \qquad (j = 2, 3, 4, ...),$$

where $a > 0$ *and* $\ell > \frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1}$.

Proof: Substituting p= 2 into 3.2 , we obtain the equality

$$E_{p2} = \frac{1}{2\pi i}\sum_{j=1}^{n_p}\sum_{k=n_p+1}^{\infty}\left(\int_{\alpha=b_p}\frac{d\alpha}{(\alpha-\beta_j)(\alpha-\beta_k)}\right)(Q\Phi_j,\Phi_k)_{H_1}(Q\Phi_k,\Phi_j)_{H_1}. \quad (3.4)$$

It readily follows that

$$|E_{p2}| \le \|Q\|^2_{H_1}\Lambda_p. \quad (3.5)$$

Here, $\Lambda_p = \sum_{k=n_p+1}^{\infty}(\beta_k - \beta_{n_p})^{-1}$, $(p = 1, 2, \cdots)$.
Using 3.5, we obtain

$$\lim_{p\to\infty} E_{p2} = 0, \qquad \left(\ell > \frac{2m}{2m-1}\right) . \quad (3.6)$$

Now, we wish to see that

$$\lim_{p\to\infty} E_{p3} = 0. \quad (3.7)$$

By 3.2, we get:

$$\begin{aligned} E_{p3} &= \sum_{j=1}^{n_p}\sum_{k=1}^{n_p}\sum_{s=n_p+1}^{\infty}[F(j,k,s)+F(s,k,j)+F(j,s,k)] \\ &+ \sum_{j=1}^{n_p}\sum_{k=n_p+1}^{\infty}\sum_{s=n_p+1}^{\infty}[F(j,k,s)+F(s,k,j)+F(k,j,s)], \end{aligned} \quad (3.8)$$

where,

$$F(j,k,s) = g(j,k,s)(Q\Phi_j,\Phi_k)_{H_1}(Q\Phi_k,\Phi_s)_{H_1}(Q\Phi_s,\Phi_j)_{H_1},$$

$$g(j,k,s) = \frac{1}{6\pi i}\int_{|\alpha|=b_p}\frac{1}{(\alpha-\beta_j)(\alpha-\beta_k)(\alpha-\beta_s)}d\alpha$$

If we consider $g(j,k,s) = \overline{g(j,k,s)}$ and $Q = Q^*$, then

$$F(s,k,j) = \overline{F(j,k,s)}, \qquad F(k,j,s) = \overline{F(j,k,s)}, \qquad F(j,s,k) = \overline{F(j,k,s)}. \quad (3.9)$$

Using 3.8 and 3.9, we obtain

$$E_{p3} = I_1 + I_2 \ , \qquad , \quad (3.10)$$

$$I_1 = \sum_{j=1}^{n_p}\sum_{k=1}^{n_p}\sum_{s=n_p+1}^{\infty}[F(j,k,s) + 2\overline{F(j,k,s)}],$$

$$I_2 = \sum_{j=1}^{n_p} \sum_{k=n_p+1}^{\infty} \sum_{s=n_p+1}^{\infty} [F(j,k,s) + 2\overline{F(j,k,s)}].$$

$$I_1 = I_{11} + 2\overline{I_{11}}, \qquad I_2 = I_{21} + 2\overline{I_{21}} \qquad , \tag{3.11}$$

$$I_{11} = \sum_{j=1}^{n_p} \sum_{k=1}^{n_p} \sum_{s=n_p+1}^{\infty} F(j,k,s),$$

$$I_{21} = \sum_{j=1}^{n_p} \sum_{k=n_p+1}^{\infty} \sum_{s=n_p+1}^{\infty} F(j,k,s).$$

Hence we get:

$$|I_{11}| \le \frac{1+\delta}{d_0^2\delta} \|Q\|_{H_1}^3 n_p^{\frac{1-2\delta^2}{1+\delta}} \qquad , \tag{3.12}$$

$$|I_{21}| \le (\frac{1+\delta}{d_0\delta})^2 \|Q\|_{H_1}^3 n_p^{-\frac{2\delta^2}{1+\delta}} \qquad , \left(\ell > \frac{2m}{2m-1}\right). \tag{3.13}$$

By 3.10, 3.11, 3.12 and 3.13, we find

$$\lim_{p\to\infty} E_{p3} \ = 0 \qquad\qquad (\ell > \frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1}). \tag{3.14}$$

Evaluate the limit $\lim_{p\to\infty} E_{pj}$ $\quad (j = 4,5,...)$ to complete the proof:
According to 3.2

$$\begin{aligned} |E_{pj}| &\le \frac{1}{2\pi j} \int_{|\alpha|=b_p} |tr(QR_\alpha^0)^j| d\alpha \\ &\le \int_{|\alpha|=b_p} \|(QR_\alpha^0)^j\|_{\sigma_1(H_1)} d\alpha \\ &\le \int_{|\alpha|=b_p} \|QR_\alpha^0\|_{\sigma_1(H_1)} \|(QR_\alpha^0)^{j-1}\|_{H_1} d\alpha \\ &\le \|Q\|_{H_1} \int_{|\alpha|=b_p} \|R_\alpha^0\|_{\sigma_1(H_1)} \|QR_\alpha^0\|_{H_1}^{j-1} d\alpha \\ &\le const. \int_{|\alpha|=b_p} \|R_\alpha^0\|_{\sigma_1(H_1)} \|R_\alpha^0\|_{H_1}^{j-1} d\alpha. \end{aligned} \tag{3.15}$$

Since $R_\alpha = R_\alpha^0$ for $Q(x) \equiv 0$, then according to Lemma 2.3

$$\|R_\alpha^0\|_{(H_1)} < \frac{4}{d_0} n_p^{-\delta}, \qquad \left(|\alpha| = b_p; \qquad \delta = \frac{2m\ell}{2m+\ell} - 1\right). \tag{3.16}$$

By 3.15, 3.16, and Lemma 2.2, we obtain:

$$|E_{pj}| < const. \int_{|\alpha|=b_p} n_p^{1-\delta} n_p^{-\delta(j-1)} d\alpha < const.b_p n_p^{1-\delta j}.$$

For the large values of p , since $b_p = \frac{1}{2}(\beta_{n_p+1} + \beta_{n_p}) \le const.n_p^{1+\delta}$, we arrive at the inequality $|E_{pj}| < const.n_p^{2-\delta(j-1)}$.
If $\delta > \frac{2}{3}$ or $\ell > \frac{10m}{6m-5}$, then we have:

$$\lim_{p\to\infty} E_{pj} \ = 0 \qquad (j = 4, 5, ...). \tag{3.17}$$

On the other hand, if $\frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1} > \frac{10m}{6m-5}$, then by 3.6 and 3.14 with $\ell > \frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1}$ give:

$$\lim_{p\to\infty} E_{pj} \ = 0 \quad (j = 2, 3, ...). \tag{3.18}$$

Since the eigenvalues of L_0 are the form $(k+\frac{1}{2})^{2m}+\gamma_j, \quad (k = 0, 1, 2, ...; j = 1, 2, ...)$, we have

$$\beta_q = (k_q + \frac{1}{2})^{2m} + \gamma_{j_q}, \qquad (q = 1, 2, ...). \tag{3.19}$$

Assume that the operator function $Q(x)$ holds the additional conditions:
(Q4) $Q(x)$ has weak H derivatives of the second order on $[0, \pi]$ and the function $(Q(x)^{''} f, g)$ is continuous for every $f, g \in H$,
(Q5) $Q^{(i)}(x) : H \to H \qquad (i = 0, 1, 2)$ are self-adjoint nuclear operators and the functions $\|Q^{(i)}(x)\|_{\sigma_1(H)} \qquad (i = 0, 1, 2)$ are bounded and measurable on $[0, \pi]$.
Our main result is the following:

Theorem 3.2. *If the operator function $Q(x)$ satisfies the conditions (Q4) to (Q5) and $\gamma_j \sim aj^\ell$ as $j \to \infty$, then we have*

$$\lim_{p\to\infty} \sum_{q=1}^{n_p} \left(\alpha_q - \beta_q - \frac{1}{\pi} \int_0^\pi (Q(x)\varphi_{j_q}, \varphi_{j_q})dx \right) \ = \frac{1}{4} \left(trQ(0) - trQ(\pi) \right) \ , \tag{3.20}$$

where $a > 0, \qquad \ell > \frac{2m+2\sqrt{2}m}{2\sqrt{2}m-\sqrt{2}-1} \qquad j_1, j_2, ...$ are natural numbers satisfying the equality 3.19.
The limit on the left side is called regularized trace of L

Proof: According to the formula given by 3.2

$$E_{p1} = -\frac{1}{2\pi i} \int_{|\alpha|=b_p} tr\left(QR_\alpha^0\right) d\alpha. \tag{3.21}$$

Since QR_α^0 is a nuclear operator for every $\alpha \in \rho(L_0)$ and $\{\Phi_q(x)\}_1^\infty$ is an orthonormal basis of H_1 , we have:

$$tr\left(QR_\alpha^0\right) = \sum_{q=1}^{\infty} (QR_\alpha^0 \Phi_q, \Phi_q)_{H_1},$$

[11]. Replacing $tr(QR_\alpha^0)$ into the equality 3.21 and considering

$$R_\alpha^0 \Phi_q = (L_0 - \alpha I)^{-1} \Phi_q = (\beta_q - \alpha)^{-1} \Phi_q \ ,$$

then we obtain

$$\begin{aligned} E_{p1} &= -\frac{1}{2\pi i}\int_{|\alpha|=b_p}\left(\sum_{q=1}^{\infty}(QR_\alpha^0\Phi_q,\Phi_q)_{H_1}\right)d\alpha \\ &= -\frac{1}{2\pi i}\int_{|\alpha|=b_p}[\sum_{q=1}^{\infty}\frac{1}{\beta_q-\alpha}(Q\Phi_q,\Phi_q)_{H_1}]\,d\alpha \\ &= \sum_{q=1}^{\infty}(Q\Phi_q,\Phi_q)_{H_1}\frac{1}{2\pi i}\int_{|\alpha|=b_p}\frac{d\alpha}{\alpha-\beta_q} \end{aligned} \tag{3.22}$$

Since the orthonormal eigenvectors corresponding to the eigenvalues $(k+\frac{1}{2})^{2m}+\gamma_j$ of L_0 are $\sqrt{\frac{2}{\pi}}\varphi_j cos(k+\frac{1}{2})x \qquad (j=1,2,...)$, we have:

$$\Phi_q(x)=\sqrt{\frac{2}{\pi}}\varphi_{j_q}cos(k_q+\frac{1}{2})x \qquad (q=1,2,...). \tag{3.23}$$

According to the Cauchy's integral formula:

$$\frac{1}{2\pi i}\int_{|\alpha|=b_p}\frac{d\alpha}{\alpha-\beta_q}=\begin{cases}1 & ,q\le n_p\\ 0 & ,q>n_p\end{cases} \tag{3.24}$$

Substituting 3.23 and 3.24 in 3.22, we obtain

$$\begin{aligned} E_{p1} &= \sum_{q=1}^{n_p}(Q\Phi_q,\Phi_q)_{H_1} \\ &= \sum_{q=1}^{n_p}\int_0^\pi (Q(x)\Phi_q(x),\Phi_q(x))\,dx \\ &= \sum_{q=1}^{n_p}\int_0^\pi\left(Q(x)\sqrt{\frac{2}{\pi}}\varphi_{j_q}cos(k_q+\frac{1}{2})x,\sqrt{\frac{2}{\pi}}\varphi_{j_q}cos(k_q+\frac{1}{2})x\right)dx \\ &= \frac{2}{\pi}\sum_{q=1}^{n_p}\int_0^\pi cos^2(k_q+\frac{1}{2})x\left(Q(x)\varphi_{j_q},\varphi_{j_q}\right)dx \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi\left(1+cos2(k_q+\frac{1}{2})x\right)(Q(x)\varphi_{j_q},\varphi_{j_q})dx \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi\left(Q(x)\varphi_{j_q},\varphi_{j_q}\right)dx+\frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi cos(2k_q+1)x(Q(x)\varphi_{j_q},\varphi_{j_q})dx \end{aligned}$$

and substituting the last equality in 3.1 , we have

$$\begin{aligned} &\sum_{q=1}^{n_p}\left(\alpha_q-\beta_q-\frac{1}{\pi}\int_0^\pi(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right) \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi cos(2k_q+1)x(Q(x)\varphi_{j_q},\varphi_{j_q})dx+\sum_{j=2}^{s}E_{pj}+E_p^{(s)} \end{aligned} \tag{3.25}$$

If the operator function $Q(x)$ holds the conditions (Q4) and (Q5), the double series

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)cos2kxdx$$

is absolutely convergent. Therefore

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})cos(2k_q+1)xdx$$

$$=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)cos(2k+1)xdx\ . \qquad (3.26)$$

Now, let us arrange the expression on the right side of 3.26 as follows:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)cos(2k+1)xdx\\
=\ &\frac{1}{2\pi}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx-(-1)^k\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx\right)\\
=\ &\frac{1}{4}\sum_{j=1}^{\infty}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx\right)cos(k0)\right.\\
-\ &\left.\sum_{k=0}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)coskxdx\right)cos(k\pi)\right\} \qquad (3.27)
\end{aligned}$$

The difference of sums according to k on the right side of 3.27 is the difference of the values at 0 and at π of the Fourier series of the function $(Q(x)\varphi_j,\varphi_j)$ having second order derivative according to the functions $\{coskx\}_{k=0}^{\infty}$ on $[0,\pi]$. Hence by 3.26 and 3.27 we find:

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})cos2k_qxdx=\frac{1}{2}\sum_{j=1}^{\infty}((Q(0)\varphi_j,\varphi_j)+(Q(\pi)\varphi_j,\varphi_j))$$

or

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})cos2k_qxdx=\frac{1}{2}\left(trQ(0)+trQ(\pi)\right). \qquad (3.28)$$

By using Lemma 2.2 and Lemma 2.3, we get:

$$\lim_{p\to\infty}E_p^{(s)}=0 \qquad (s>3\delta^{-1}) \qquad (3.29)$$

By 3.25, 3.28, 3.29 and Theorem 3.1, we have the main result for regularized trace as

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left(\alpha_q-\beta_q-\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right)=\frac{1}{4}\left(trQ\left(0\right)-trQ\left(\pi\right)\right)\quad .$$

The proof is completed.

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