Çankaya University Journal of Science and Engineering
Volume 17, No. 1 (2020) 001-010
Date Received: February 9, 2020
Date Accepted: March 24, 2020

Infinite Taylor Series Method for Solving Lane-Emden Type Equations

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Abstract: In this research article, a numerical approach to the solutions of different forms of Lane-Emden type of singular initial value problems is presented, The Taylor Series Method has been applied. Application was on singular initial value problems. Comparison with exact solution shows considerable acceleration in convergence. The method is effective and easy to implement.

Keywords: Lane-Emden, Taylor Series, Singular Initial Value Problem.

1. Introduction

Singular initial value problems (IVPs) are very useful in various ranges of mathematical problems which are challenging in nature because of the singularity. They are applicable in pure science, modeling and other aspect of human existence. The numerical solution is however challenging because of the singularity at the origin. Researchers have applied different types of methods to solve the Lane-Emden type equations formulated as:

$$y'' + \frac{2}{x} y' + f(y) = 0, \qquad 0 < x \leq 1$$

$$y(0) = A, y'(0) = B \tag{1}$$

Also, studies have also been carried out on other classes of singular initial value problems (IVPs) of the form.

$$y'' + \frac{2}{x} y' + f(x, y) = g(x), \qquad 0 < x \leq 1$$
$$y(0) = A, y'(0) = B \tag{2}$$

Where A and B are said to be constants $f(x,y)$ is a continuous real value function while $g(x)$ $\in [0.1]$ Yahya and Liu (2007). Mathematicians, Ramos 2008 and Olayiwola 2019 respectively obtained a series approach to the Lane-Emden equations and comparisons with He's Homotopy perturbation method and variational Iteration method for solution of Emden-Fowler type of singular initial value problems (IVPs).

In other research, Tsirivas (2012) it was investigated what can be said of the sequence $(\beta_n S_n(f, Z_0))$ when theorem of Seleznev concerning the case where the radius of convergence of the power series is zero. A new explicit formula proposed by Jos´e Juan and Enrique (2012) for the remainder that generalizes classic ones, namely, Scholomilch, Lebesgue, Cauchy, and Euler's remainders. Inspired by the explicit expression for an arbitrary polynomial x −→ px, ∀x ∈ R.

In view of the great importance of Taylor series in analysis, it may be regarded as extremely surprising that so few attempts at generalization have been made. Widde (1890) researched on the problem of the representation of an arbitrary function by means of linear combinations of prescribed functions has received no small amount of attention. It is well known that one phase of this problem leads directly to Taylor series, the prescribed functions in this case being polynomials.

Approximate solutions of the Lane-Emden equations were obtained by Homotopy Perturbation method by Chen and Chen (2004) and El-Mistikawy (2009). The method was proposed by He (1999) and has been successfully applied to solve many types of linear and nonlinear functional equation. He (2000) apply Homotopy Perturbation method with Laplace Transform, method for obtaining solutions of linear and nonlinear Lane-Emden type differential equation, which gives more accurate solutions when compared with exact solutions.

In this study, Taylor series method is applied to solve the general Lane-Emden type of differential equations used in modeling and application to physical and astrophysics problems He (2003). Many benefits are derived from this type of equation mostly in the area of science and engineering. Momoniat and Harley (2006) obtained an approximate implicit solution by reducing the Lane-Emden equation of first order differential equation using the Lie group analysis and determining a power series solution of the reduced equation.

The Taylor Series Method is spell out. Some examples and result analysis and conclusion were also presented.

2. The Method

Illustrating the basic idea of this method [], also [] we considered the following general nonlinear differential equation:

$$y'' + \frac{u}{x} y' + f(x, y) = g(x) \tag{3}$$

$$n \in \Re$$

$$y(0) = A, \qquad y'(0) = B$$

Multiply (3) through by x and making y' the subject, we obtained the following.

$$x\left(y'' + \frac{n}{x} y' + f(x, y) \right) = xg(x) \tag{4}$$

$$xy'' + ny' + xf(x, y) = xg(x) \tag{5}$$

$$y' = \frac{1}{n}(xg(x)) - \frac{1}{n} f(x, y) - \frac{1}{n}(xy'') \tag{6}$$

Similarly, for higher derivatives of y , such as;

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{n}\left(xg(x) - \frac{1}{n}(xf(x, y)) - \frac{1}{n}(xy'') \right) \right) \tag{7}$$

$$y'' = \frac{1}{n+1}(xg(x))' - \frac{1}{n+1}(xf(x, y))' - \frac{1}{n+1}(xy''') \tag{8}$$

$$y''' = \frac{d^2}{dx^2}(y') = \frac{d^2}{dx^2}\left(\frac{1}{n}\left(xg(x) - \frac{1}{n}(xf(x, y)) - \frac{1}{n}(xy'') \right) \right) \tag{9}$$

$$y''' = \frac{1}{n+2}(xg(x))'' - \frac{1}{n+2}(xf(x, y))'' - \frac{1}{n+1}(xy^{iv}) \tag{10}$$

$$y''' = \frac{1}{n+2}(xg(x))'' - \frac{1}{n+2}(xf(x, y))'' - \frac{1}{n+1}(xy^{iv}) \tag{11}$$

$$y^{iv} = \frac{d^3}{dx^3}(y') = \frac{d^3}{dx^3}\left(\frac{1}{n}\left(xg(x) - \frac{1}{n}(xf(x, y)) - \frac{1}{n}(xy'') \right) \right) \tag{12}$$

$$y^{iv} = \frac{1}{n+3}(xg(x))^{iii} - \frac{1}{n+3}(xf(x, y))^{iii} - \frac{1}{n+3}(xy^{v})$$

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We finally obtained

$$y^{(k)} = \frac{1}{n+(k-1)}(xg(x))^{k-1} - \frac{1}{n+(k-1)}(xf(x,y))^{k-1} - \frac{1}{n+(k-1)}xy^{(k-1)} \quad (13)$$

Where $k = 1,2,3,\ldots$

For $x = 0$, $y(0) = A,$ $y'(0) = B$ then

$y''(0),\ y'''(0),\ y^{iv}(0), \ldots, y^{(k)}(0)$

The series;

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \ldots + \frac{y^k(0)}{k!}x^k \quad (14)$$

3. Numerical Examples

Example 1: We consider the following:

$$y'' + \frac{2}{x}y' + \left(y - x^4 + x^3\right) = 21x^2 - 12x + 6 \quad (15)$$

$y(0) = 0,$ $y'(0) = 0$

From Taylor Series Algorithm we have;

$$y' = \frac{1}{2}(xg(x)) - \frac{1}{2}(xf(x,y)) - \frac{1}{2}xy'' \quad (16)$$

$$y' = \frac{1}{2}\left(x\left(21x^2 - 12x + 6\right)\right) - \frac{1}{2}\left(x\left(y - x^4 + x^3\right)\right) - \frac{1}{2}xy'' \quad (17)$$

$$y' = \frac{1}{2}\left(21x^3 - 12x^2 + 6x\right) - \frac{1}{2}\left(xy - x^5 + x^4\right) - \frac{1}{2}xy'' \quad (18)$$

At $x = 0$, $y' = 0$

$$y'' = \frac{1}{3}\left(63x^2 - 24x + 6\right) - \frac{1}{3}\left(xy' + y - 5x^4 + 4x^3\right) - \frac{1}{3}xy''' \tag{19}$$

At $x = 0$, $y'' = 2$

$$y''' = \frac{1}{4}(126x - 24) - \frac{1}{4}\left(xy'' + 2y' - 20x^3 + 12x^2\right) - \frac{1}{4}xy^{iv} \tag{20}$$

At $x = 0$, $y''' = -6$

$$y^{iv} = \frac{1}{5}(126) - \frac{1}{5}\left(xy''' + y'' + 2y'' - 60x^2 + 24x\right) - \frac{1}{5}xy^{iv} \tag{21}$$

At $x = 0$, $y^{iv} = 24$

By Taylor Series we obtained;

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) \tag{22}$$

$$y(x) = 0 + \frac{0}{1} + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(-6) + \frac{x^4}{4!}(24) \tag{23}$$

$$y(x) = x^2 - x^3 + x^4 \tag{24}$$

Eqn. (24) is the exact solution of Eqn. (15)

Example 2 we consider the following:

$$y'' + \frac{2}{x}y' + y = 2e^x\left(x^2 + 3x + 3\right), \quad y(0) = 0, \quad y'(0) = 0 \tag{25}$$

Expanding the R.H.S. and implementing the algorithm, we have:

$$y'' + \frac{2}{x}y' + y = 6 + 12x + 11x^2 + 6x^3 + \frac{9}{4}x^4 + \frac{19}{30}x^5 + \frac{17}{120}x^6 + \frac{11}{420}x^7 \tag{26}$$

Using the Algorithm we have,

$$y'(x) = \frac{1}{2}(xg(x)) - \frac{1}{2}(xf(x,y) - \frac{1}{2}xy'' \tag{27}$$

$$y' = \frac{1}{2}\left(6x + 12x^2 + 11x^3 + 6x^4 + \frac{9}{4}x^5 + \frac{19}{30}x^6 + \frac{17}{120}x^7 + \frac{11}{420}x^8\right) - \frac{1}{2}xy - \frac{1}{2}xy'' \tag{28}$$

Let $x=0$ it implies $y'=0$

$$y''=\frac{1}{3}\left(6+24x+33x^2+24x^3+\frac{45}{4}x^4+\frac{114}{30}x^5+\frac{119}{120}x^6+\frac{11}{420}x^8\right)-\frac{1}{3}(xy'+y)-\frac{1}{3}xy''' \quad (29)$$

Let $x=0$ it implies $y''=2$

Similarly,

$$y'''=\frac{1}{4}\left(24+66x+72x^2+\frac{4\times45}{4}x^3+\frac{5\times114}{30}x^4+\frac{6\times119}{120}x^5+\frac{8\times88}{420}x^6\right)-\frac{1}{4}(xy''+2y')-\frac{1}{4}(xy^{\prime v}) \quad (30)$$

Let $x=0$, Then, $y'''=6$

$$y^{\prime v}=\frac{1}{5}\left(66+144x+135x^2+76x^3+\frac{595}{20}x^4+\frac{4224}{420}x^5\right)-\frac{1}{5}(xy'''+y''+2y'')-\frac{1}{5}(xy^{v}) \quad (31)$$

Let $x=0$, Then, $y^{\prime v}=12$ (32)

$$y^{v}=\frac{1}{6}\left(144+270x+228x^2+\frac{2380}{20}x^3+\frac{21120}{420}x^4\right)-\frac{1}{6}(xy^{\prime v}+y'''+3y''')-\frac{1}{6}(xy^{v}') \quad (33)$$

Let $x=0$, Then, $y^{v}=20$ (34)

$$y^{v}'=\frac{1}{7}\left(270+456x+357x^2+\frac{844480}{420}x^3\right)-\frac{1}{7}(xy^{v}+y^{\prime v}+4y^{\prime v})-\frac{1}{7}(xy^{v}') \quad (35)$$

Let $x=0$, Then, $y^{v}'=30$ (36)

$$y^{v}''=\frac{1}{8}\left(456+714x+\frac{253440}{420}x^2\right)-\frac{1}{8}(xy^{v}'+y^{v}+5y^{v})-\frac{1}{8}(xy^{v}'') \quad (37)$$

Let $x=0$, Then, $y^{v}''=42$ (38)

$$y^{v}'''=\frac{1}{9}\left(714+\frac{506880}{420}x\right)-\frac{1}{9}(xy^{v}''+y^{v}'+6y^{v}')-\frac{1}{9}(xy^{v}''') \quad (40)$$

Let x=0, Then, $y^{v}'''=56$ (41)

$$y^{\prime x}=\frac{1}{10}\left(\frac{506880}{420}\right)-\frac{1}{10}(8(42))=\frac{506880}{4200}-\frac{336}{10}=\frac{506880-141120}{4200} \quad (42)$$

Let x=0, Then, $y^{\prime x}=87.08$ (43)

$x = 0,\ \ y(0),\ \ y'(0) = 0$

By Taylor series method, x=0 we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \frac{x^5}{5!}y^{v}(0) + \frac{x^6}{6!}y^{v}{}'(0) + \frac{x^7}{7!}y^{v}{}''(0)$$
$$\frac{x^8}{8!}y^{v}{}'''(0) + \cdots + \frac{x^n}{n!}y^n(0) \tag{44}$$

$$y(x) = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \frac{x^7}{5!} + \frac{x^8}{6!} \tag{45}$$

$$y(x) = x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots\right) \approx x^2e^x \tag{46}$$

Example 3 we consider the following:

$$y'' + \frac{2}{x}(y + y') - y = 0 \tag{47}$$

$y(0) = 1,\ \ y'(0) = -1$

From Taylor series we have:

$$y'(x) = \frac{1}{2}(x(g(x))) - \frac{1}{2}(xf(x, y)) - \frac{1}{2}xy'' \tag{48}$$

This implies that:

$$y' = -\frac{1}{2}((2 - x)y) - \frac{1}{2}xy'' \tag{49}$$

Let $x = 0$

Then $y' = -1$ (50)

Also,

$$y'' = -\frac{1}{3}\left((2-x)y' + y(-1)\right) - \frac{1}{3}xy''' \tag{51}$$

When $x = 0$,

$$y'' = 1 \tag{52}$$

$$y''' = -\frac{1}{4}\left((2-x)y'' + y'(-1) - y'\right) - \frac{1}{4}xy^{iv} \tag{53}$$

When $x = 0$,

$$y''' = -1 \tag{54}$$

$$y^{iv} = -\frac{1}{5}\left((2-x)y''' + y''(-1) - 2y''\right) - \frac{1}{5}xy^{v} \tag{55}$$

When $x = 0$,

$$y^{iv} = 1 \tag{56}$$

Therefore,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iy}(0) + \frac{x^5}{5!}y^{v}(0) + \ldots + \frac{x^n}{n!}y^{n}(0) \tag{57}$$

$$y(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} \tag{58}$$

As $n \to \infty$, $\quad y(x) = e^{-x}$ (59)

4. Conclusions

In this article, we applied the Taylor series method in solving some examples of singular Initial value problems, homogeneous and inhomogeneous. Analytical solutions were obtained. The peculiarity of this method is on the type of equations, due to the existence of the singularity at the origin i.e. at which is clearly eliminated.

The method was applied to generate results for higher derivatives of and it shows rapid convergence to series of solution when compared with the exact solutions.

Authorship contribution statement

M. O. Olayiwola: Supervision, Conceptualization, Methodology, Reviewing and Editing, **A. Adegoke:** Investigation, Visualization, Typing and Software.

Declaration of Competing Interest

The authors declare that there is no competing financial interests or personal relationships that influence the work in this paper.

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