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Research Article

On MF-projective modules

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Abstract

In this paper, we study the left orthogonal class of max-flat modules which are the homological objects related to s-pure exact sequences of modules and module homomorphisms. Namely, a right module A is called *MF-projective* if $\mathrm{Ext}^1_R(A,B)=0$ for any max-flat right R-module B, and A is called *strongly MF-projective* if $\mathrm{Ext}^i_R(A,B)=0$ for all max-flat right R-modules B and all $i\geq 1$. Firstly, we give some properties of MF-projective modules and SMF-projective modules. Then we introduce and study MF-projective dimensions for modules and rings. The relations between the introduced dimensions and other (classical) homological dimensions are discussed. We characterize some classes of rings such as perfect rings, QF rings and max-hereditary rings by $(S)MF$-projective modules. We also study the rings whose right ideals are MF-projective. Finally, we characterize the rings whose MF-projective modules are projective.

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1. Introduction

Throughout, R will denote an associative ring with identity, and modules will be unital right R-modules, unless otherwise stated. As usual, we denote by $\mathfrak{M}_R$ $({}_R\mathfrak{M})$ the category of right (left) R-modules. For a module A, $E(A)$, $id(A)$, $pd(A)$ and A^+ denote the injective hull, injective dimension, projective dimension and the character module $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$ of A, respectively.

Let $\mathfrak{C}$ be a class of R-modules and A be an R-module. A homomorphism $f:A\to C$ with $C\in\mathfrak{C}$ is called a $\mathfrak{C}$-*preenvelope* of A if for any homomorphism $g:A\to D$ with $D\in\mathfrak{C}$, there is a homomorphism $h:C\to D$ such that $hf=g$ (see [8]). Moreover, if the only such h are automorphisms of C when $C=D$ and $g=f$, the $\mathfrak{C}$-preenvelope is called a $\mathfrak{C}$-envelope of A. Dually, we have the definitions of a $\mathfrak{C}$-precover and a $\mathfrak{C}$-cover. $\mathfrak{C}$-envelopes ($\mathfrak{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism. We will denote by $\mathfrak{C}^{\perp}=\{X:\mathrm{Ext}^1_R(C,X)=0 \text{ for all } C\in\mathfrak{C}\}$ the right orthogonal class of $\mathfrak{C}$, and by ${}^{\perp}\mathfrak{C}=\{X:\mathrm{Ext}^1_R(X,C)=0 \text{ for all } C\in\mathfrak{C}\}$ the left orthogonal class of $\mathfrak{C}$. A pair $(\mathfrak{F},\mathfrak{C})$ of classes of right R-modules is called a *cotorsion theory* (for the category of R-modules) if $\mathfrak{F}^{\perp}=\mathfrak{C}$ and ${}^{\perp}\mathfrak{C}=\mathfrak{F}$. A cotorsion theory $(\mathfrak{F},\mathfrak{C})$ is called *perfect* (*complete*) if every right R-module has a $\mathfrak{C}$-envelope and an $\mathfrak{F}$-cover (a special $\mathfrak{C}$-preenvelope and a special $\mathfrak{F}$-precover). A cotorsion theory $(\mathfrak{F},\mathfrak{C})$ is said to be

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hereditary if whenever $0 \to L^{'} \to L \to L^{''} \to 0$ is exact with $L, L^{''} \in \mathfrak{F}$, then $L^{'}$ is also in $\mathfrak{F}$ (see [9]). By [9], $(\mathfrak{F}, \mathfrak{C})$ is hereditary if and only if whenever $0 \to C^{'} \to C \to C^{''} \to 0$ is exact with $C, C^{'} \in \mathfrak{C}$, then $C^{''}$ is also in $\mathfrak{C}$.

Since its development, the Cohn purity plays a significant role in module theory and homological algebra. One of the main reason is that, some significant homological objects such as, flat modules, cotorsion modules, absolutely pure modules and pure-injective modules arose from this notion of purity. Recall that, the submodule A of B is called *s-pure submodule of* B [5] if $i \otimes 1_S : A \otimes S \to B \otimes S$ is a monomorphism for each simple left module S. Similarly, the submodule A of B is called *neat submodule of* B if $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A)$ is an epimorphism for each simple right module S. Unlike the generation of pure submodules, the notions of s-pure and neat submodules are not only inequivalent they are also incomparable. The equality of the notions of s-pure and neat submodules is considered in [12], which is hold over the commutative domains whose maximal ideals are invertible, and these domains termed as N-domains. In [6], S. Crivei proved that if the ring is commutative and the maximal ideals are principal, then the notions s-pure and neat submodules coincide. Recently, the commutative rings with this property are completely characterized in [19, Theorem 3.7]. These are exactly the commutative rings whose maximal ideals are finitely generated and locally principal.

A left R-module A is called *max-injective* if for the inclusion map $i : I \to R$ with I maximal left ideal, and any homomorphism $f : I \to A$ there exist a homomorphism $g : R \to A$ such that $gi = f$, or equivalently $\mathrm{Ext}^1_R(R/I, A) = 0$ for any maximal left ideal I. A ring R is said to be left *max-injective* if R is max-injective as a left R-module [26]. As observed by Crivei in [6, Theorem 3.4], a left R-module A is max-injective if and only if A is a neat submodule of every module containing it. A right R-module A is called *max-flat* if $\mathrm{Tor}^R_1(A, R/I) = 0$ for any maximal left ideal I of R (see [25]). A right R-module A is max-flat if and only if A^+ is max-injective by the isomorphism $\mathrm{Ext}^1_R(R/I, A^+) \cong (\mathrm{Tor}^R_1(A, R/I))^+$ for any maximal left ideal I of R. Indeed, we show in Lemma 4.1 that, a right R-module A is max-flat if and only if any short exact sequence ending with A is s-pure.

So far, s-pure and neat submodules and homological objects related to s-pure and neat-exact sequences are studied by many authors (see, [3, 5–7, 12–14, 19, 26, 27]).

The main purpose of this paper is to continue the study and investigation of the homological objects related to s-pure and neat short exact sequences. Namely, we have studied max-flat modules and left orthogonal class of max-flat modules.

Along the way, the concepts of MF-projective and strongly MF-projective modules are first introduced in section 2. Several elementary properties of MF-projective and SMF-projective modules are obtained in this section. We prove that a right R-module A is MF-projective if and only if A is a cokernel of a max-flat preenvelope $f : C \to B$ with B projective. It is shown that a ring R is right perfect if and only if all max-flat right R-modules are $(S)MF$-projective. It is also proven that R is a QF ring if and only if every right R-module is $(S)MF$-projective.

In section 3 of this article, we define and discuss MF-projective dimensions for modules and rings. For a right R-module A, the *MF-projective dimension* $mfpd(A)$ of A is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+i}_R(A, B) = 0$ for any max-flat right R-module B and any integer $i \geq 1$. If no such n exists, set $mfpd(A) = \infty$. Put $rmfpD(R) = sup\{mfpd(A) : A \text{ is a right } R\text{-module}\}$, and call $rmfpD(R)$ the *right MF-projective dimension* of R. It is proven that $rmfD(R) \leq n$ if and only if $id(A) \leq n$ for all max-flat right R-modules A. Certain characterizations of QF rings in terms of MF-projective modules are also obtained. We characterize the rings whose simple right R-modules are MF-projective. We also introduce the notion of right MF-hereditary rings, and then give some characterizations of such rings. It is shown that a ring R is right MF-hereditary if and only if every submodule of an MF-projective right R-module is MF-projective if and only if $rmfD(R) \leq 1$ if and only if $id(A) \leq 1$ for all max-flat right R-modules A.

In section 4, we study max-flat preenvelopes which are epimorphisms. We first consider the commutative rings whose maximal ideals are finitely generated and locally principal over which neat-flat modules and max-flat modules coincide. By using this result, over a commutative ring whose maximal ideals are finitely generated and locally principal it is proven that the following are equivalent: (1) R is max-hereditary; (2) every (simple) R-module has an epic max-flat preenvelope; (3) every simple R-module has an epic projective preenvelope; (4) every (finitely presented) MF-projective module is projective; (5) R is a PS ring.

2. Left orthogonal class of max-flat modules

We begin with the following definition.

Definition 2.1. A right module A is called *MF-projective* if $\mathrm{Ext}^1_R(A, B) = 0$ for any max-flat right R-module B. A is said to be *strongly MF-projective* (SMF-projective for short) if $\mathrm{Ext}^i_R(A, B) = 0$ for all max-flat right R-modules B and all $i \geq 1$.

Recall that a ring R is said to be a *left C-ring* if $\mathrm{Soc}(R/I) \neq 0$ for every proper essential left ideal I of R. Right perfect rings, left semiartinian rings are well known examples of left C-rings ([4, 10.10]).

Remark 2.2. (1) Projective modules are clearly $(S)MF$-projective, but the converse need not to be true in general. For example, let R be a local QF ring $R = k[X]/(X^2)$, where k is a field, and $\overline{X}$ denotes the residue class of X in R. Then every right R-module is $(S)MF$-projective by Proposition 2.11, so is the ideal $\overline{X}$, in particular. However $\overline{X}$ is not projective, because $\overline{X}^2 = 0$ implies that $\overline{X}$ is not a free ideal in the local ring R.

(2) In [11], Fu et al. defined and discussed copure-projective modules. A right module A is called *copure-projective* provided that $\mathrm{Ext}^1_R(A, B) = 0$ for any flat right module B. Since every flat right module is max-flat, every MF-projective right module is copure-projective. For the converse, let R be a left C-ring. It is shown in [24, Lemma 4] that every max-injective left module is injective, so in this case, every max-flat right module is flat. Thus every copure-projective right module is MF-projective.

Recall that the class of max-flat modules is closed under extensions, direct sums, direct summands by [27, Proposition 2.4(2)]. Moreover it is closed under pure submodules and pure quotients by the following lemma.

Lemma 2.3. (1) *The class of max-flat modules is closed under pure submodules and pure quotients.*

(2) *The class of MF-projective modules is closed under extensions, direct sums and direct summands.*

Proof. (1) Consider the pure exact sequence of right R-modules $0 \to B \to A \to A/B \to 0$ with A max-flat. Since $0 \to (A/B)^+ \to A^+ \to B^+ \to 0$ splits and A^+ is max-injective, B^+ and $(A/B)^+$ is max-injective. Hence B and A/B is max-flat.

(2) The class of MF-projective modules is closed under extensions by using the functor $\mathrm{Ext}^1_R(-, F)$ for any max-flat module F. Also, it is closed under direct sums and direct summands by using the isomorphism $\mathrm{Ext}^1_R(\oplus_{i\in I} A_i, F) \cong \prod_{i\in I} \mathrm{Ext}^1_R(A_i, F)$ for any max-flat module F and a family of modules $(A_i)_{i\in I}$ by [23, Theorem 7.13]. □

Recall that a ring R is called left *max-hereditary* if every maximal left ideal is projective (see [1]). This is equivalent to saying that every factor of a max-injective left R-module is max-injective (see [1, Proposition 1.2]). A ring R is called a left *SF-ring* if each simple left R-module is flat (see [22]). The following example shows that a left max-hereditary ring does not need to be left SF-ring.

Example 2.4. Assume that R is a left Noetherian left hereditary ring that is not semisimple. Thus every left ideal of R is projective, and so R is left max-hereditary. But R is not a left SF-ring. Otherwise, since R is left Noetherian, every simple left R-module is finitely presented. If R was a left SF-ring, then every simple left R-module would be projective by [23, Corollary 3.58], whence R would be semisimple, a contradiction.

We shall now give a condition for the converse of Remark 2.2(1).

Proposition 2.5. *Let R be a left max-hereditary ring or a left SF-ring. Then the followings are equivalent for a module A.*

(1) *A is projective.*
(2) *A is SMF-projective.*
(3) *A is MF-projective.*

Proof. We know that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is always true.

(a) First, assume that R is a left max-hereditary ring.

(3) $\Rightarrow$ (1) Let A be an MF-projective right module. Then there is an exact sequence $0 \to C \to B \to A \to 0$ with B projective. Then this exact sequence induces the exactness of $0 \to A^+ \to B^+ \to C^+ \to 0$. Since B^+ is injective, C^+ is max-injective by [1, Proposition 1.2] and so C is max-flat. Thus, $\mathrm{Ext}^1_R(A, C) = 0$, that is, $0 \to C \to B \to A \to 0$ splits. It follows that A is projective.

(b) Now, assume that R is a left SF-ring.

(3) $\Rightarrow$ (1) Let A be an MF-projective right module. Then there is an exact sequence $0 \to C \to B \to A \to 0$ with B projective. Since R is a left SF-ring, $\mathrm{Tor}^R_1(C, R/I) = 0$ for any maximal left ideal I of R, and so C is max-flat. Thus, $\mathrm{Ext}^1_R(A, C) = 0$, that is, $0 \to C \to B \to A \to 0$ splits. It follows that A is projective. □

By definitions, every SMF-projective module is MF-projective. For the converse we have the following condition.

Proposition 2.6. *Let R be a ring and A an MF-projective right R-module. Then A is SMF-projective if and only if for any exact sequence $0 \to C \to B \to A \to 0$ of right R-modules with B projective, C is SMF-projective.*

Proof. Let $0 \to C \to B \to A \to 0$ be an exact sequence of right R-modules with B projective. If A is SMF-projective, then $\mathrm{Ext}^i_R(C, F) \cong \mathrm{Ext}^{i+1}_R(A, F) = 0$ for any max-flat right R-module F and $i \geq 1$. So C is SMF-projective. Conversely, if C is SMF-projective, then $\mathrm{Ext}^i_R(A, F) \cong \mathrm{Ext}^{i-1}_R(C, F) = 0$ for any max-flat right R-module F and $i \geq 2$. But $\mathrm{Ext}^1_R(A, F) = 0$ by hypothesis, and so A is SMF-projective. □

The following proposition gives some characterizations of MF-projective modules in terms of max-flat preenvelopes.

Proposition 2.7. *The following are equivalent for a right R-module A.*

(1) *A is MF-projective.*
(2) *A is projective with respect to every exact sequence $0 \to K \to T \to L \to 0$ with K max-flat.*
(3) *For every exact sequence $0 \to C \to B \to A \to 0$, with B max-flat, $C \to B$ is a max-flat preenvelope of C.*
(4) *A is a cokernel of a max-flat preenvelope $C \to B$ with B projective.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1) Let B be a max-flat right R-module. The exactness of the sequence $0 \to B \to E(B) \to E(B)/B \to 0$ induces the exact sequence $\mathrm{Hom}(A, E(B)) \to \mathrm{Hom}(A, E(B)/B) \to \mathrm{Ext}^1_R(A, B) \to 0$. Since $\mathrm{Hom}(A, E(B)) \to \mathrm{Hom}(A, E(B)/B)$ is epic by (2), $\mathrm{Ext}^1_R(A, B) = 0$. So A is MF-projective.

(3) $\Rightarrow$ (4) Since there is an exact sequence $0 \to C \to P \to A \to 0$ with P projective, (4) follows from (3).

(4) $\Rightarrow$ (1) Let A be a cokernel of a max-flat preenvelope $f : C \to B$ with B projective. Then, there is an exact sequence $0 \to D \to B \to A \to 0$ with $D = Im(f)$. For each max-flat right R-module F, the sequence $\mathrm{Hom}(B, F) \to \mathrm{Hom}(D, F) \to \mathrm{Ext}^1_R(A, F) \to 0$ is exact. Note that $\mathrm{Hom}(B, F) \to \mathrm{Hom}(D, F)$ is epic by (4). Thus $\mathrm{Ext}^1_R(A, F) = 0$, and so A is MF-projective. □

Now we characterize MF-projective modules over a commutative ring.

Proposition 2.8. *The following statements are equivalent for a commutative ring R and an R-module A.*

(1) *A is MF-projective.*
(2) *$P \otimes_R A$ is MF-projective for any projective R-module P.*
(3) *$\mathrm{Hom}(P, A)$ is MF-projective for any finitely generated projective R-module P.*

Proof. (1) $\Rightarrow$ (2) Let P be a projective R-module and consider by [23, Exercise 9.20] the isomorphism $\mathrm{Ext}^1_R(P \otimes_R A, B) \cong \mathrm{Hom}(P, \mathrm{Ext}^1_R(A, B))$. For any max-flat R-module B, we have $\mathrm{Ext}^1_R(A, B) = 0$ since A is MF-projective. This says that $\mathrm{Ext}^1_R(P \otimes_R A, B) = 0$. Thus $P \otimes_R A$ is MF-projective.

(1) $\Rightarrow$ (3) Let P be a finitely generated projective R-module. By using [23, Lemma 3.59] and mimicking the proof of [23, Theorem 9.51], we have the isomorphism $P \otimes_R \mathrm{Ext}^1_R(A, B) \cong \mathrm{Ext}^1_R(\mathrm{Hom}(P, A), B)$. Since A is MF-projective, $\mathrm{Ext}^1_R(A, B) = 0$ for any max-flat R-module B. This says that $\mathrm{Ext}^1_R(\mathrm{Hom}(P, A), B) = 0$, and so $\mathrm{Hom}(P, A)$ is MF-projective.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are clear by letting $P = R$. □

A ring R is called *left max-coherent* if every maximal left ideal is finitely presented. A right R-module A is called *MI-flat* if $\mathrm{Tor}^R_1(A, B) = 0$ for any max-injective left R-module B (see [27]). These modules were discovered when studying max-flat preenvelopes.

Proposition 2.9. *Let R be a left max-coherent ring. Then:*

(1) *Every MF-projective right R-module is MI-flat.*
(2) *Every finitely presented MI-flat right R-module is MF-projective.*

Proof. (1) Let A be an MF-projective right R-module. For any max-injective left R-module E, E^+ is max-flat by [27, Theorem 2.3], and hence $\mathrm{Ext}^1_R(A, E^+) = 0$. Thus from the standard isomorphism $\mathrm{Ext}^1_R(A, E^+) \cong (\mathrm{Tor}^R_1(A, E))^+$ in [8, Theorem 3.2.1], we have $\mathrm{Tor}^R_1(A, E) = 0$. So A is MI-flat.

(2) Let A be a finitely presented MI-flat right R-module. Then A is the cokernel of a max-flat preenvelope $g : C \to B$ with B projective by [27, Proposition 3.7(2)]. Hence, A is MF-projective by Proposition 2.7. □

It is well known that R is a right perfect ring if and only if every flat right R-module is projective. The converse of Proposition 2.9(1) characterizes the right perfect rings over a left max-coherent ring.

Theorem 2.10. *Let R be a ring. Then the followings are equivalent.*

(1) *R is right perfect.*
(2) *All max-flat right R-modules are projective.*
(3) *All max-flat right R-modules are SMF-projective.*
(4) *All max-flat right R-modules are MF-projective.*
(5) *All flat right R-modules are MF-projective.*
Also, if R is a left max-coherent ring, then the above conditions are equivalent to:
(6) *All MI-flat right R-modules are MF-projective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (5) are clear.

(1) $\Rightarrow$ (2) Let A be any max-flat right R-module. Then A^+ is max-injective. Since R is a left C-ring, A^+ is injective by [24, Lemma 4], whence A is flat. By the perfectness of R, A is projective.

(5) $\Rightarrow$ (1) Let A be a flat right R-module. There is an exact sequence $0 \to B \to P \to A \to 0$ with P projective. Note that, by the flatness of A, B is flat. Since A is MF-projective by (5), $\mathrm{Ext}^1_R(A, B) = 0$. So $0 \to B \to P \to A \to 0$ splits, whence A is projective.

(1) $\Rightarrow$ (6) Let A be an MI-flat right R-module and F a max-flat right R-module. Since R is a left C-ring, F^+ is injective by [1, Corollary 1.1], and so F is flat. Also, since R is a left max-coherent ring, R is left coherent by [1, Corollary 1.1]. Thus, right perfectness of R gives from [16, Proposition 1.4] that pure injectivity of F. But F is a pure submodule of F^{++}, so F is a direct summand of a max-flat right R-module F^{++}. Because F^+ is max-injective, $\mathrm{Ext}^1_R(A, F^{++}) \cong (\mathrm{Tor}^R_1(A, F^+))^+ = 0$. Therefore $\mathrm{Ext}^1_R(A, F) = 0$. So, A is MF-projective. □

Recall that R is said to be a *QF-ring* if R is left Noetherian and left self-injective, or equivalently R is right artinian and right self-injective. By a well-known result of Faith and Walker [10], R is QF if and only if every projective right R-module is injective. In the following result, we give a new characterization of a QF ring.

Proposition 2.11. *R is a QF ring if and only if every right R-module is $(S)MF$-projective.*

Proof. Let A be a right R-module and B a max-flat right R-module. Since R is right artinian, R is right perfect, and so B is projective by Theorem 2.10. Thus B is an injective right R-module by the hypoyhesis. This means that $\mathrm{Ext}^{i+1}_R(A, B) = 0$ for any max-flat right R-module B and any $i \geq 0$. Hence A is $(S)MF$-projective. Conversely, let A be a projective right R-module. Since A is max-flat, by the hypoyhesis $\mathrm{Ext}^{i+1}_R(B, A) = 0$ for any right R-module B and any $i \geq 0$. So A is injective, whence R is a QF-ring. □

In the following, we characterize when every simple right module is MF-projective.

Lemma 2.12. *Every simple right R-module is MF-projective if and only if every max-flat right R-module is max-injective.*

Proof. Let A be a max-flat right R-module. Then by the hypothesis, $\mathrm{Ext}^1_R(R/I, A) = 0$ for any maximal right ideal I of R. It follows that A is max-injective. Conversely, let S be a simple right R-module. For any max-flat right R-module A, A is max-injective. Thus $\mathrm{Ext}^1_R(S, A) = 0$, whence S is MF-projective. □

In general, a left SF-ring does not need to be a semisimple ring. The fact that every simple right (left) R-module is projective if and only if R is semisimple together with Proposition 2.5 and Lemma 2.12 gives rise the following corollary.

Corollary 2.13. *Let R be a ring. The followings are equivalent.*

(1) *R is a semisimple ring.*
(2) *R is a left max-coherent left SF-ring.*
(3) *R is a left max-hereditary ring and every simple right R-module is MF-projective.*
(4) *R is a left max-hereditary ring and every max-flat right R-module is max-injective.*

3. MF-projective dimensions

In this section we investigate the MF-projective dimension of modules. We begin with the following definition.

Definition 3.1. Let R be a ring. For a right R-module A, let $mfpd(A)$ denote the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+i}(A, B) = 0$ for any max-flat right R-module B and any integer $i \geq 1$, and call $mfpd(A)$ the *MF-projective dimension* of A. If no such n exists, set $mfpd(A) = \infty$.

Put $rmfpD(R) = sup\{mfpd(A) : A \text{ is a right } R\text{-module}\}$, and call $rmfpD(R)$ the *right MF-projective dimension* of R. Similarly we have $lmfpD(R)$.

The following remark follows from definitions and Proposition 2.11.

Remark 3.2. (1) A module A is SMF-projective if and only if $mfpd(A) = 0$.
(2) A ring R is a QF-ring if and only if $rmfpD(R) = 0$.

The copure projective dimension $cpd(A)$ of an R-module A is defined in [11] as the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+i}(A, B) = 0$ for any flat right R-module B and any $i \geq 1$. The right copure projective dimension of a ring R is defined as $rcpD(R) = sup\{cpd(A) | A \text{ is a right R-module}\}$. By the following proposition, we have the relation with right copure projective dimension of rings.

Proposition 3.3. *Let R be a ring. Then $rmfpD(R) \leq rcpD(R)$. Moreover, if $rcpD(R) < \infty$, then $rmfpD(R) = rcpD(R)$.*

Proof. It is clear that $rmfpD(R) \leq rcpD(R)$, since any flat right R-module is max-flat. Now suppose that $rmfpD(R) = n < \infty$. Let A be a right R-module with $cpd(A) = k < \infty$. Suppose $k > n$. For any flat right R-module B, consider the short exact sequence $0 \to C \to P \to B \to 0$ with P projective. Since B and P are flat, C is flat by [17, Corollary 4.86]. So we get an exact sequence $\mathrm{Ext}_R^k(A, P) \to \mathrm{Ext}_R^k(A, B) \to \mathrm{Ext}_R^{k+1}(A, C)$. Since $rmfpD(R) = n < k$, $\mathrm{Ext}_R^k(A, P) = 0$. Also since $cpd(A) = k$, $\mathrm{Ext}_R^{k+1}(A, C) = 0$. Then $\mathrm{Ext}_R^k(A, B) = 0$, whence $cpd(A) < k$, a contradiction. Thus $k \leq n$, and $rcpD(R) \leq rmfpD(R)$. □

It is clear that $rmfpD(R) \leq rD(R)$, where $rD(R)$ denote the right global dimension of R. In general, $rmfpD(R) \neq rD(R)$. For example, let R be a QF ring with $rD(R) \neq 0$ (e.g. $R = \mathbb{Z}/4\mathbb{Z}$), then $rmfpD(R) = 0$. The next corollary is due to Fu et al. [11, Corollary 4.4].

Corollary 3.4. *Let R be a ring with $rD(R) < \infty$. Then $rmfpD(R) = rcpD(R) = rD(R)$.*

From now on, for the class of SMF-projective right R-modules we write $\mathscr{SMF}$.

Lemma 3.5. *$(\mathscr{SMF}, \mathscr{SMF}^{\perp})$ is a hereditary cotorsion theory.*

Proof. Let $A \in \mathscr{SMF}$ and $B \in \mathscr{SMF}^{\perp}$. Consider the short exact sequence $0 \to C \to P \to A \to 0$ with P projective. Then $\mathrm{Ext}_R^2(A, B) \cong \mathrm{Ext}_R^1(C, B) = 0$ by Proposition 2.6. Let $0 \to B \to E \to D \to 0$ be an exact sequence with E injective. Then $\mathrm{Ext}_R^1(A, D) \cong \mathrm{Ext}_R^2(A, B) = 0$, and so $D \in \mathscr{SMF}^{\perp}$. Now let $G \in {}^{\perp}(\mathscr{SMF}^{\perp})$, then $\mathrm{Ext}_R^2(G, B) \cong \mathrm{Ext}_R^1(G, D) = 0$. Therefore $\mathrm{Ext}_R^i(G, B) = 0$ for any $i \geq 1$ by induction. Since max-flat modules are contained in $\mathscr{SMF}^{\perp}$, $\mathrm{Ext}_R^i(G, F) = 0$ for any max-flat right R-module F and $i \geq 1$, so $G \in \mathscr{SMF}$. Hence $(\mathscr{SMF}, \mathscr{SMF}^{\perp}) = ({}^{\perp}(\mathscr{SMF}^{\perp}), \mathscr{SMF}^{\perp})$ is a cotorsion theory. Let $0 \to K \to L \to M \to 0$ be an exact sequence with $L, M \in \mathscr{SMF}$. Take $N \in \mathscr{SMF}^{\perp}$. Then the sequence $0 = \mathrm{Ext}_R^1(L, N) \to \mathrm{Ext}_R^1(K, N) \to \mathrm{Ext}_R^2(M, N) = 0$ is exact, whence $\mathrm{Ext}_R^1(K, N) = 0$ for any $N \in \mathscr{SMF}^{\perp}$. Thus $K \in \mathscr{SMF}$. □

Now we have the following characterizations of modules with finite MF-projective dimension.

Proposition 3.6. *Let R be a ring, n a nonnegative integer and A a right R-module. The following are equivalent.*

(1) $mfpd(A) \leq n$.
(2) $\mathrm{Ext}_R^{n+i}(A,B)=0$ *for any right R-module* $B \in \mathscr{SMF}^{\perp}$ *and* $i \geq 1$.
(3) $\mathrm{Ext}_R^{n+1}(A,B)=0$ *for any right R-module* $B \in \mathscr{SMF}^{\perp}$.
(4) *If* $0 \to C \to B_{n-1} \to ... \to B_1 \to B_0 \to A \to 0$ *is exact with each B_i projective, then C is SMF-projective.*
(5) *There exists an exact sequence* $0 \to B_n \to B_{n-1} \to ... \to B_1 \to B_0 \to A \to 0$ *with each B_i SMF-projective.*

Proof. (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are trivial.

(1) $\Rightarrow$ (4) Let $0 \to C \to B_{n-1} \to ... \to B_1 \to B_0 \to A \to 0$ be an exact sequence with each B_i projective. Then $\mathrm{Ext}_R^i(C,B) \cong \mathrm{Ext}_R^{n+i}(A,B)=0$ for any max-flat right R-module B and $i \geq 1$ by (1). So C is SMF-projective by definition.

(4) $\Rightarrow$ (3) Let $0 \to C \to B_{n-1} \to ... \to B_1 \to B_0 \to A \to 0$ be an exact sequence with each B_i projective. Then $\mathrm{Ext}_R^{n+1}(A,B) \cong \mathrm{Ext}_R^1(C,B)=0$ for any $B \in \mathscr{SMF}^{\perp}$.

(3) $\Rightarrow$ (2) For any $B \in \mathscr{SMF}^{\perp}$, consider the short exact sequence $0 \to B \to E \to C \to 0$ with E injective. Then the sequence $\mathrm{Ext}_R^{n+1}(A,C) \to \mathrm{Ext}_R^{n+2}(A,B) \to \mathrm{Ext}_R^{n+2}(A,E)=0$ is exact. Since $E \in \mathscr{SMF}^{\perp}$, $C \in \mathscr{SMF}^{\perp}$ by Lemma 3.5, and so $\mathrm{Ext}_R^{n+1}(A,C)=0$ by (3). Therefore $\mathrm{Ext}_R^{n+2}(A,B)=0$, and (2) holds by induction.

(5) $\Rightarrow$ (1) Let B be a max-flat right R-module and $K_1 = ker(B_0 \to A)$, $K_i = ker(B_{i-1} \to B_{i-2})$ for $i \geq 2$. Since each B_i is SMF-projective, we get that $\mathrm{Ext}_R^{n+i}(A,B) \cong \mathrm{Ext}_R^{n+i-1}(K_1,B) \cong ... \cong \mathrm{Ext}_R^i(B_n,B)=0$ for any $i \geq 1$. So, $mfpd(A) \leq n$. □

Now we set out to investigate how MF-projective dimension behave in short exact sequences. It is easy to check the following result.

Proposition 3.7. *Let R be a ring,* $0 \to A \to B \to C \to 0$ *an exact sequence of right R-modules. If two of* $mfpd(A), mfpd(B), mfpd(C)$ *are finite, so is the third. Moreover:*

(1) $mfpd(B) \leq sup\{mfpd(A), mfpd(C)\}$;
(2) $mfpd(A) \leq sup\{mfpd(B), mfpd(C)-1\}$;
(3) $mfpd(C) \leq sup\{mfpd(B), mfpd(A)+1\}$.
(4) *If* $0 < mfpd(A) < \infty$ *and B is SMF-projective, then* $mfpd(C) = mfpd(A)+1$.

Now we are in the position of characterizing the rings with finite MF-projective dimension.

Theorem 3.8. *Let R be a ring, n a nonnegative integer. The following are equivalent.*

(1) $rmfpD(R) \leq n$.
(2) $mfpd(A) \leq n$ *for any cyclic right R-module A.*
(3) $id(A) \leq n$ *for all max-flat right R-modules A.*
(4) $id(A) \leq n$ *for all right R-modules* $A \in \mathscr{SMF}^{\perp}$.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1) Let A be any right R-module and B a max-flat right R-module. Since $id(B) \leq n$, $\mathrm{Ext}_R^{n+i}(A,B)=0$ for any $i \geq 1$. Hence $mfpd(A) \leq n$ by definition.

(2) $\Rightarrow$ (4) Let $A \in \mathscr{SMF}^{\perp}$ and I be a right ideal of R. So $mfpd(R/I) \leq n$, whence by Proposition 3.6, $\mathrm{Ext}_R^{n+1}(R/I,A)=0$ for any $n \geq 0$. Thus $id(A) \leq n$. □

We show in Proposition 2.11 that R is a QF ring if and only if every right R-module is $(S)MF$-projective. The following corollary gives a new characterization of QF rings by using the MF-projective modules.

Corollary 3.9. *Let R be a ring. The following are equivalent.*

(1) *R is a QF-ring.*
(2) *$rmfpD(R) = 0$.*
(3) *Every cyclic right R-module is SMF-projective.*
(4) *Every max-flat right R-module is injective.*
(5) *Every quotient module of an injective right R-module is MF-projective.*
Moreover, if R is a right max-coherent right C-ring, then the above conditions are equivalent to:
(6) *Every simple right R-module is MF-projective.*
(7) *R is a right max-injective ring.*

Proof. By Proposition 2.11 and Theorem 3.8, it is enough to show that (5) $\Rightarrow$ (4) and (6) $\Rightarrow$ (7) $\Rightarrow$ (1).

(5) $\Rightarrow$ (4) For any max-flat right R-module F, there exists an exact sequence $0 \to F \to E \to B \to 0$ with E injective. Then B is MF-projective by (5), and so $\mathrm{Ext}^1_R(B, F) = 0$. Thus the above short exact sequence splits, which implies that F is injective.

(6) $\Rightarrow$ (7) Since every simple right R-module is MF-projective, every max-flat right R-module is max-injective by Lemma 2.12. This means that every flat right R-module is max-injective. Thus R is a right max-injective ring.

(7) $\Rightarrow$ (1) Let A be a projective right R-module. So A is a direct summand of a free module $R^{(I)}$, for some index set I. Since R is a right max-injective ring, $R^{(I)}$ is a max-injective right R-module by [27, Proposition 2.4(2)], and so A is max-injective. Also since R is a right C-ring, A is injective by [24, Lemma 4]. Thus R is a QF ring. □

Next, we introduce and study MF-hereditary rings. But, first, recall that a ring R is called *right hereditary* if every right ideal is projective. It is known that a ring R is right hereditary if and only if every submodule of a projective right R-module is projective (see [23, Theorem 4.23]). We shall say that a ring R is *right MF-hereditary* if every right ideal of R is MF-projective. The next theorem gives some characterizations of such rings.

Corollary 3.10. *Let R be a ring. The following are equivalent.*

(1) *$rmfpD(R) \leq 1$.*
(2) *$id(A) \leq 1$ for all max-flat right R-modules A.*
(3) *R is right MF-hereditary.*
(4) *Every submodule of any MF-projective right R-module is MF-projective.*
(5) *Every submodule of any projective right R-module is MF-projective.*
(6) *Every submodule of any free right R-module is MF-projective.*

Proof. (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (3) are trivial.

(1) $\Leftrightarrow$ (2) follows by Theorem 3.8.

(2) $\Rightarrow$ (4) Let B be a submodule of an MF-projective right R-module A. Consider the short exact sequence $0 \to B \to A \to A/B \to 0$. Then for any max-flat right R-module F, we get an exact sequence $0 = \mathrm{Ext}^1_R(A, F) \to \mathrm{Ext}^1_R(B, F) \to \mathrm{Ext}^2_R(A/B, F)$. Since $id(F) \leq 1$, it follows that $\mathrm{Ext}^2_R(A/B, F) = 0$. So $\mathrm{Ext}^1_R(B, F) = 0$, whence B is MF-projective.

(3) $\Rightarrow$ (2) Let F be a max-flat right R-module and I a right ideal of R. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Since I is MF-projective, we have $0 = \mathrm{Ext}^1_R(I, F) \to \mathrm{Ext}^2_R(R/I, F) \to \mathrm{Ext}^2_R(R, F) = 0$. Thus $\mathrm{Ext}^2_R(R/I, F) = 0$ and so $id(F) \leq 1$. □

It is obvious that every right hereditary ring is right MF-hereditary. The following is an example of a right non-hereditary ring R such that every right ideal is MF-projective.

Example 3.11. Let R be a non-semisimple QF ring. Since by Proposition 2.11, every right R-module is MF-projective over a QF ring R, R is a right MF-hereditary ring. But R is a non-hereditary ring, otherwise it would be semisimple.

Now we discuss the relations between the class of right MF-hereditary rings and the well-known class of right hereditary rings.

Corollary 3.12. *Consider the following statements for a ring R:*

(1) *R is right MF-hereditary and left max-hereditary.*

(2) *R is right MF-hereditary and every MF-projective right R-module is projective.*

(3) *R is right hereditary.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (2) $\Rightarrow$ (3) is clear.

(1) $\Rightarrow$ (2) Let A be an MF-projective right R-module. Since R is left max-hereditary, A is projective by Proposition 2.5.

(3) $\Rightarrow$ (2) Assume R is right hereditary. Let A be an MF-projective right R-module. Consider the exact sequence $0 \to B \to F \to A \to 0$ with F projective. Since R is right hereditary, B is projective and so $\mathrm{Ext}^1_R(A, B) = 0$. This implies that $0 \to B \to F \to A \to 0$ splits, whence A is projective. □

4. Max-flat preenvelopes which are epimorphisms

Recall by [27, Theorem 2.5] that over a left max-coherent ring R, every right R-module has a max-flat preenvelope. It is shown that over a left max-coherent ring R, every right R-module has a monic max-flat preenvelope if and only if R is a left max-injective ring ([27, Theorem 2.11]). It is well known that every right R-module has an epic flat envelope if and only if R is a left semihereditary ring ([21, Corollary 4.3]). In this section, we consider when every R-module has an epic max-flat preenvelope.

The following lemma gives a characterization of max-flat modules in terms of s-purity.

Lemma 4.1. *A right R-module A is max-flat if and only if any short exact sequence ending with A is s-pure.*

Proof. Let $0 \to C \to B \to A \to 0$ be an exact sequence. Since A is max-flat, for any maximal left ideal I of R, we have the exact sequence $0 = \mathrm{Tor}^R_1(A, R/I) \to C \otimes R/I \to B \otimes R/I \to A \otimes R/I \to 0$. So the exact sequence $0 \to C \to B \to A \to 0$ is s-pure. Conversely, let $0 \to B \to F \to A \to 0$ be an s-pure exact sequence with F projective. For any maximal left ideal I of R, we have the exact sequence $0 = \mathrm{Tor}^R_1(F, R/I) \to \mathrm{Tor}^R_1(A, R/I) \to B \otimes R/I \to F \otimes R/I$. Since $B \otimes R/I \to F \otimes R/I$ is monic, $\mathrm{Tor}^R_1(A, R/I) = 0$. Hence, A is max-flat. □

Unlike the generation of pure submodules the notions of s-pure and neat submodules are not only inequivalent they are also incomparable. Recently, the commutative rings for which the notions of s-pure and neat submodules are equivalent are completely characterized in [19, Theorem 3.7]. These are exactly the commutative rings whose maximal ideals are finitely generated and locally principal. A right module A is called *neat-flat* if for any epimorphism $f : B \to A$, the induced map $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, A)$ is epic for any simple right module S, equivalently any short exact sequence ending with A is neat-exact (see [3]). Together with Lemma 4.1 and [3, Lemma 2.3.], we obtain the following.

Corollary 4.2. *Let R be a commutative ring whose maximal ideals are finitely generated and locally principal and let A be an R-module. Then the following are equivalent.*

(1) *A is max-flat.*
(2) *A is neat-flat.*
(3) *A is simple projective, i.e. for any simple R-module S, every homomorphism $f : S \to A$ factors through a finitely generated free R-module F.*

If R is a left max-hereditary ring, then every MF-projective right module is projective by Proposition 2.5. Now for the converse, we have the following characterizations of max-hereditary rings.

Theorem 4.3. *Let R be a commutative ring whose maximal ideals are finitely generated and locally principal. The following are equivalent.*

(1) *R is max-hereditary.*
(2) *Every MF-projective R-module is projective.*
(3) *Every MF-projective R-module is flat.*
(4) *Every finitely presented MF-projective R-module is projective.*
(5) *Every simple R-module has an epic projective preenvelope.*
(6) *Every simple R-module has an epic max-flat preenvelope.*
(7) *Every R-module has an epic max-flat preenvelope.*
(8) *Every submodule of a max-flat R-module is max-flat.*

Proof. (1) $\Rightarrow$ (2) is by Proposition 2.5.

(2) $\Rightarrow$ (3) and (7) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (5) $\Rightarrow$ (8) is by Corollary 4.2 and [18, Theorem 3.7].

(3) $\Rightarrow$ (4) Let A be a finitely presented MF-projective R-module. Then A is flat by (3), and so is projective since A is finitely presented.

(4) $\Rightarrow$ (5) Let S be a simple R-module. Since R is max-coherent, S has a max-flat preenvelope $\psi : S \to F$ with F max-flat. So ψ factors through a finitely generated free module P by Corollary 4.2. This means that there exist homomorphisms $f : S \to P$ and $g : P \to F$ such that $gf = \psi$. Let $B = Im(f)$, $\beta : S \to B$ and $A = P/B$. Now, we claim that the inclusion map $i : B \to P$ is a max-flat preenvelope of B. Let $h : B \to M$ be a homomorphism with M max-flat. Then there exists a homomorphism $\phi : F \to M$ such that $\phi gf = \phi gi\beta = h\beta$. Since β is epic, $h = (\phi g)i$. This proves our claim, whence A is MI-flat by [27, Proposition 3.7(1)]. Since A is finitely presented, A is MF-projective by Proposition 2.9(2), and so is projective by the hypothesis. Thus the splitting of $0 \to B \to P \to A \to 0$ says that B is projective. Hence $S \to B$ is a projective preenvelope which is an epimorphism.

(8) $\Rightarrow$ (1) Let B be a factor of a max-injective R-module A. Then the exact sequence $0 \to C \to A \to B \to 0$ induces the exactness of $0 \to B^+ \to A^+ \to C^+ \to 0$. Since A^+ is max-flat by [27, Theorem 2.3], B^+ is max-flat by (8) and so B is max-injective. Hence by [1, Proposition 1.2], R is max-hereditary.

(8) $\Rightarrow$ (7) For any R-module A, there is a max-flat preenvelope $f : A \to B$. Note that $Im(f)$ is max-flat by (8), so $A \to Im(f)$ is an epic max-flat preenvelope. □

R is called a *right PS* ring [20] if every simple right ideal is projective. It is shown that every submodule of any neat-flat right R-module is neat-flat if and only if R is a right PS ring ([2, Theorem 5.3]). As a consequence of Corollary 4.2 and Theorem 4.3, we obtain a new characterization of max-hereditary rings.

Corollary 4.4. *Let R be a commutative ring whose maximal ideals are finitely generated and locally principal. The following are equivalent.*

(1) *R is a max-hereditary ring.*
(2) *R is a PS ring.*

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