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Pure Elements and Dual Notions of Prime Elements in Lattice Modules

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Abstract: This paper deals with the pure elements and the dual notions of prime elements (that is, second elements). For this, it introduces the definitions of second element and coprime element. Then it is shown that the concepts of the second element and coprime element are equivalent. Moreover, this study gives us a characterization of comultiplication modules. Finally, it defines pure elements and obtains the relation among pure, idempotent and multiplication elements.

Latis Modüllerindeki Pür Elemanlar ve Asal Elemanların Dual Kavramları

Anahtar Kelimeler
Çarpımsal latisler,
Latis modüller,
İkinci elemanlar,
Pür elemenlar

Özet: Bu makale pür elemanlar ve asal elemanların dual kavramları (yani ikinci elemanları) ile ilgilenir. Bunun için, ikinci elemanı ve eşasal elemanını tanıtır. Daha sonra ikinci eleman kavramı ile eşasal eleman kavramlarının denk olduğu elde edilir. Ayrıca, bu çalışma bize eşçarpımsal latis modüllerinin bir karakterizasyonunu verir. Son olarak, pür elemanları tanımlar ve pür, eşgüçlü ve çarpım elemanlar arasındaki ilişkiyi elde eder.

1. Introduction

L is said to be a multiplicative lattice if L is a complete lattice in which there exists a commutative, associative multiplication that distributes over randomly joins and has compact greatest element 1_L (least element 0_L) as a multiplicative identity (zero).

A complete lattice M is called an $L-$lattice module if there exists a multiplication between elements of L and M, denoted by xN for $x \in L$ and $N \in M$, that provides the next conditions:

1. $(xy)N = x(yN)$;
2. $\left(\bigvee_{\alpha} x_\alpha\right)\left(\bigvee_{\beta} N_\beta\right) = \bigvee_{\alpha,\beta} x_\alpha N_\beta$;
3. $1_L N = N$;
4. $0_L N = 0_M$;

for all x, x_α, y in L and for all N, N_β in M.

Throughout this note L denotes a multiplicative lattice and M denotes an $L-$lattice module. Note that 1_M (resp., 0_M) means the greatest element (resp., the least element) of M. $H \in M$ is called proper if $H < 1_M$. Suppose H, K are in M, the join of every $x \in L$ such that $xK \leq H$ is denoted by $(H :_L K)$. Specially, $(0_M :_L 1_M)$ is denoted by $Ann(M)$. On the other hand, for $x \in L$ and $H \in M$, the join of every $N \in M$ such that $xN \leq H$ is denoted by $(H :_M x)$.

An element H in M is said to be meet principal if $(x \wedge (N :_L H))H = xH \wedge N$ for $\forall N \in M$ and $\forall x \in L$. H is said to be join principal if $x \vee (N :_L H) = ((xH \vee N) :_L H)$ for $\forall x \in L$ and for $\forall N \in M$. Then if H is both meet principal and join principal, we say it is principal. In particular, we say H is weak meet principal (weak join principal) if $(N :_L H)H = N \wedge H$ $((xH :_L H) = x \vee (0_M :_L H))$ for all $N \in M$ (for all $x \in L$). Then if H is both weak meet principal and weak join principal, H is called weak principal.

We say $H \in M$ is compact if $H \leq \bigvee_{\alpha} N_\alpha$, then $H \leq N_{\alpha_1} \vee N_{\alpha_2} \vee ... \vee N_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. M is called CG (compactly generated) if each element of M is a join of compact elements in M. Similarly, we say that M is PG (principally generated) if each element of M is a join of principal elements in M.

If $H \in M$ such that $(H :_L 1_M)H = H$, then it is said to be an idempotent element. Specially, if we consider L as an $L-$module over itself, $a \in L$ is called idempotent if $a = a^2$. We say $H < 1_M$ in $L-$module M is prime, if $aX \leq H$ implies $X \leq H$ or $a \leq (H :_L 1_M)$ for all $a \in L$, $X \in M$. Particularly if 0_M is prime, then M is called a prime lattice module. Note that the necessary and sufficient condition for 0_M to be prime in M is $(0_M :_L 1_M) = (0_M :_L H)$ for all $0_M \neq H \in M$. An element

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$H < 1_M$ in M is called a primary element if $aX \leq H$ and $X \nleq H$, then $a^k \leq (H :_L 1_M))$ for all $a \in L, X \in M$. Moreover M is said to be faithful if $Ann(M) = (0_M :_L 1_M) = 0_L$.

Suppose H is an element in M. Then $[H, 1_M]$ is a set of all $K \in M$ which satisfies $H \leq K \leq 1_M$. Moreover the lattice $[H, 1_M]$ is an $L-$lattice module with $a.K = aK \vee H$ for all $a \in L$ and $K \in M$ which satisfies $H \leq K$.

For more information of lattice modules, one is referred to [5]-[7].

Throughout the paper we have examined the concept of multiplication and comultiplication lattice module over L and generalized some important conclusions for multiplication and comultiplication modules over commutative rings, shown by M. M. Ali, H. Ansari-Toroghy, F. Farshadifar, P. F. Smith in [1]-[4] and [8]-[11], to the lattice modules over multiplicative lattice.

In this paper, we introduce second and coprime elements of M in Definition 2 and Definition 3, respectively. Then in Proposition 2, we prove that the notion of the second element and coprime element in M are equivalent. In Section 3, we define the notion of pure element in M in Definition 4. Then in Proposition 4 we characterize a pure element in L. Also, in Proposition 5, we show that $a \in L$ is pure in $L \Leftrightarrow a$ is multiplication and idempotent in L. The main theorem of our study (Theorem 3) states that if L is PG and M is faithful multiplication PG such that 1_M is compact, then $H \in M$ is pure $\Leftrightarrow (H :_L 1_M)$ is a pure element of L. Finally, in Corollary 1, we obtain the relationship among pure, multiplication and idempotent elements in lattice and lattice modules. Consequently, this study contributes to the literature introducing the notion of the second element, coprime element and pure element in a lattice module as well as giving relationships among some special elements in lattice and lattice modules.

2. Dual Notions of Prime Elements

Firstly for the sake of completeness the following definition, lemma and theorem are given.

Definition 1. (See [5] and [7])
(i) M is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$.
(ii) M is said to be a comultiplication lattice module if for every element N of M there exists an element $a \in L$ such that $N = (0_M :_M a)$.

Lemma 1. *(See [5] and [7]) The following statements hold for a lattice module M over a multiplicative lattice L:*

1. *M is multiplication $\Leftrightarrow N = (N :_L 1_M) 1_M$ for every element N in M.*
2. *M is comultiplication $\Leftrightarrow N = (0_M :_M (0_M :_L N))$ for every element N in M.*

Theorem 1. *(See [7]) The following statements are equivalent for a lattice module M over a multiplicative lattice L:*

1. *M is comultiplication.*
2. *For any $N \in M$ and any $c \in L$ with $N < (0_M :_M c)$, there is $b \in L$ such that $c < b$ and $N = (0_M :_M b)$.*
3. *For any $N \in M$ and any $c \in L$ with $N < (0_M :_M c)$, there is $b \in L$ such that $c < b$ and $N \leq (0_M :_M b)$.*

Now, let us define the dual notion of prime element, i.e., second element of a lattice module.

Definition 2. $0_M \neq N \in M$ is called second element in M, if for each $a \in L$, $aN = N$ or $aN = 0_M$. In particular, if 1_M is a second element, M is called second.

Proposition 1. *If N is second, $(0_M :_L N) = p$ is prime. Conversely, if $(0_M :_L N) = p$ is prime for $N \in M$ and M is comultiplication, then N is second.*

Proof. Suppose that $N \in M$ is second and $p = (0_M :_L N)$. Let $ab \leq p$ and $b \nleq p$. Then $bN \neq 0_M$. Since N is a second element, we have $bN = N$. Then $0_M = (ab)N = a(bN) = aN$. Therefore, $a \leq p = (0_M :_L N)$.
Assume M is comultiplication and $p = (0_M :_L N)$ is prime for $N \in M$. Suppose that $aN \neq 0_M$ for $a \in L$. Then $0_M \neq K = aN \leq N$. If $0_M \neq K = aN < N = (0_M :_M (0_M :_L N)) = (0_M :_M p)$, then by Theorem 1(2), there exists an element $b \in L$ such that $p < b$ and $K = aN = (0_M :_M b)$. It follows that $baN = 0_M$, and so $ba \leq p = (0_M :_L N)$. Since p is prime and $b \nleq p$, we have $a \leq p$ and so $aN = 0_M$. This is a contradiction. So $K = aN = N$. Consequently N is a second element in $L-$module M. □

Now, let us introduce the notion of coprime elements in a lattice module.

Definition 3. $0_M \neq N \in M$ is called a coprime element if $(0_M :_L N) = (K :_L N)$ for all $K \in M$, $K < N$. In particular, if 1_M is a coprime element, then we say that M is a coprime module.

One can easily see that 1_M is a coprime element in M necessary and sufficient condition $(N :_L 1_M) = (0_M :_L 1_M)$ for any $N \in M$ such that $N < 1_M$.

Owing to the following proposition, one can see that the concepts of second element and coprime element in a lattice module are equivalent.

Proposition 2. *$K \in M$ is a coprime element $\Leftrightarrow$ K is a second element.*

Proof. Suppose that K is a coprime element in M. Let $aK < K$ for some $a \in L$. Since K is a coprime element we have $a \leq (aK :_L K) = (0_M :_L K)$ and so $aK = 0_M$. Conversly, if K is a second element, then we show that $(N :_L K) = (0_{M:} :_L K)$, for all $N < K$. It is clear that $(0_{M:} :_L K) \leq (N :_L K)$. If $a \leq (N :_L K)$, then we have $aK \leq N < K$. Since K is a second element, then $aK = 0_M$ and $a \leq (0_M :_L K)$. □

Proposition 3. *Suppose M is coprime. Then $N < 1_M$ is primary $\Leftrightarrow$ N is prime.*

Proof. $\Rightarrow$: Assume that M is a coprime $L-$ lattice module. Thus $(0_M : 1_M) = (N : 1_M) = p$ is prime by Proposition 1. Since $(N : 1_M) = p$ is prime, if N is a primary element, we obtain N is prime (See Proposition 1 in [5]).

$\Leftarrow$: Obvious. □

3. Pure Elements in Lattice Modules

Before the definition of the pure element, let us recall some information:

Note that an element $H \in M$ is said to be multiplication if given an element N of M such that $N \le H$, there exists $x \in L$ such that $N = xH$. In particular, if we consider L as a module over itself, an element $x \in L$ is said to be multiplication if given an element y of L such that $y \le x$, there exists $z \in L$ such that $y = zx$. It is clear that if 1_M is multiplication, then $L-$module M is multiplication.

Now, let us define the pure elements in lattice module as the following:

Definition 4. An element $0_M \neq N < 1_M$ in M is called a pure element, if $aN = a1_M \wedge N$, for all $a \in L$. In particular, if we consider L as a module over itself, $c < 1_L$ in L is called a pure element if $ac = a \wedge c$, for $\forall a \in L$.

Theorem 2. *M is coprime with $(0_M :_L 1_M) = p \Leftrightarrow [0_M, N]$ and $[N, 1_M]$ are coprime $L-$lattice modules with $(0_M :_L N) = (N :_L 1_M) = p$, for all pure elements $N \in M$.*

Proof. $\Rightarrow$: Let N be pure. As M is second, $a1_M = 0_M$ or $a1_M = 1_M$ for all $a \in L$. If $a1_M = 0_M$, then $aN = a1_M \wedge N = 0_M$ and if $a1_M = 1_M$, then $aN = a1_M \wedge N = N$. Hence N is second (coprime) and $[0_M, N]$ is a coprime $L-$module.

We shall show that $(0_M :_L N) = (0_M :_L 1_M) = p$. Clearly, $(0_M :_L 1_M) \le (0_M :_L N)$. It is sufficient to show that $(0_M :_L N) \le (0_M :_L 1_M)$. Let $aN = 0_M$ for $a \in L$. If $a1_M \neq 0_M$, then $a1_M = 1_M$, because M is coprime. Hence $aN = 0_M = a1_M \wedge N = N$, as N is pure. But this is a contradiction because N is pure, so $N \neq 0_M$. Therefore $a1_M = 0_M$. It follows that $(0_M :_L N) \le (0_M :_L 1_M)$. Also, as M is coprime, one has $(0_M :_L 1_M) = (N :_L 1_M) = p$ for all $N \in M, N < 1_M$. So we obtain that $(0_M :_L N) = (0_M :_L 1_M) = p$.

We prove that $[N, 1_M]$ is a coprime $L-$module. For all $N \le K$, $a.K$ is defined by $a.K = aK \vee N$ in the module $[N, 1_M]$. Since M is a coprime $L-$module, $a1_M = 1_M$ or $a1_M = 0_M$. Hence $a.1_{[N,1_M]} = a.1_M = a1_M \vee N = 1_M$ or $a.1_{[N,1_M]} = a.1_M = a1_M \vee N = N = 0_{[N,1_M]}$. Therefore $[N, 1_M]$ is coprime.

$\Leftarrow$: Suppose $[0_M, N]$ and $[N, 1_M]$ are coprime $L-$modules for all pure element $N \in M$ with $(0_M :_L N) = (N :_L 1_M) = p$. We show that $(0_M :_L 1_M) = p$. Clearly, $(0_M :_L 1_M) \le (0_M :_L N) = p$. Conversely, if $a \le (0_M :_L N) = (N :_L 1_M) = p$, we have $aN = 0_M$ and $a1_M \le N$. Hence $0_M = aN = a1_M \wedge N = a1_M$, i.e., $a \le (0_M : 1_M)$.

We show M is coprime. If $N \le K < 1_M$, as $[N, 1_M]$ is coprime, one has $(N :_L 1_M) = p = (K :_L 1_M)$. If $N \nleq K$ for $K \wedge N < N$, since $[0_M, N]$ is coprime, one has $(0_M :_L N) = p = (K \wedge N :_L N) = (K :_L N) \ge (K :_L 1_M) \ge (0_M :_L 1_M) = p$. Thus $p = (0_M :_L 1_M) = (K :_L 1_M)$ for all $K < 1_M$. □

Proposition 4. *$a \in L$ is pure in $L \Leftrightarrow b = ab$ for any element $b \in L$ such that $b \le a$.*

Proof. $\Rightarrow$: Let $b \in L$ such that $b \le a$. Since a is a pure element of L, we have $ab = a \wedge b = b$.

$\Leftarrow$: Let $b = ab$ for any element $b \in L$ such that $b \le a$. Since $a \wedge c \le a$ for all $c \in L$, $a \wedge c = a(a \wedge c) \le ac \le a \wedge c$. So, we have $a \wedge c = ac$ for all $c \in L$. □

Proposition 5. *$a \in L$ is pure in $L \Leftrightarrow$ a is multiplication and idempotent in L.*

Proof. $\Rightarrow$: Let a be pure in L. Since a is pure for any element b of L such that $b \le a$, we have $b = ab$. In particular for $a, b \in L$ such that $a = b$, we have $a^2 = a$.

$\Leftarrow$: Let a be a multiplication and idempotent element of L. Let b be an element of L such that $b \le a$. Since a is multiplication element of L then there exists $c \in L$ such that $b = ac$. Because a is an idempotent element, we have $b = ac = a^2c = a(ac) = ab$. □

Theorem 3. *Assume that L is PG and M is faithful multiplication PG such that 1_M is compact. Then $H \in M$ is pure $\Leftrightarrow$ $(H :_L 1_M)$ is a pure element of L.*

Proof. $\Rightarrow$: Suppose that H is a pure element of M. Therefore $xH = x1_M \wedge H$ for every element $x \in L$. And so,

$$\begin{aligned}(xH :_L 1_M) &= (x1_M \wedge H :_L 1_M)\\ &= (x1_M :_L 1_M) \wedge (H :_L 1_M) = x \wedge (H :_L 1_M).\end{aligned}$$

Since M is a multiplication lattice module, there is $y \in L$ such that $H = y1_M$. Then,

$$(xH :_L 1_M) = (xy1_M :_L 1_M) = xy = x(H :_L 1_M).$$

Hence, one has $x \wedge (H :_L 1_M) = x(H :_L 1_M)$, i.e. $(H :_L 1_M)$ is pure in L.

$\Leftarrow$: Assume $(H :_L 1_M)$ is pure. We show that $xH = x1_M \wedge H$ for every $x \in L$. Because 1_M is compact (see [6], Proposition 2ii), then

$$\begin{aligned}(x1_M \wedge H :_L 1_M) &= (x1_M :_L 1_M) \wedge (H :_L 1_M)\\ &= x \wedge (H :_L 1_M) = x(H :_L 1_M).\end{aligned}$$

Since M is a multiplication lattice module, there is $y \in L$ such that $H = y1_M$. Similarly, as 1_M is compact (see [6], Proposition 2ii),

$$\begin{aligned}x(H :_L 1_M) &= x(y1_M :_L 1_M) = xy\\ &= (xy1_M :_L 1_M) = (xH :_L 1_M).\end{aligned}$$

As M is PG, faithful, multiplication such that 1_M is compact, one obtains $x1_M \wedge H = xH$ (see [5], Theorem 5ii). □

Finally, with the next Corollary, the relation among pure, idempotent and multiplication elements is obtained under some conditions.

Corollary 1. *Assume that L is PG and M is faithful multiplication PG such that 1_M is compact. For $x \in L$, the following statements are equivalent:*

1. $N = x1_M$ is a pure element in M.
2. $x = (N :_L 1_M)$ is a pure element in L.
3. $x = (N :_L 1_M)$ is a multiplication and idempotent element in L.
4. $N = x1_M$ is idempotent and multiplication.

Proof. (1) ⇔ (2) is clear by the previous Theorem.

(2) ⇔ (3) is clear by Proposition 5.

(3) ⇒ (4) Suppose $x = (N :_L 1_M)$ is multiplication. Then $x1_M$ is multiplication in M (see [5], Proposition 2iv). Suppose that $x = (N :_L 1_M)$ is an idempotent element in L. We will show that $N = x1_M$ is idempotent, i.e. $(N : 1_M)N = N$. Since M is a multiplication lattice and x is idempotent, one has

$$(N : 1_M)N = (x1_M :_L 1_M)x1_M$$
$$= x(x1_M :_L 1_M)1_M = x^2 1_M = x1_M = N.$$

(4)⇒ (3) Suppose that $N = x1_M$ is multiplication in M. Then $x = (N :_L 1_M)$ is multiplication in L (see [5], Proposition 2iv).
Suppose that $N = x1_M$ is an idempotent element. Hence, $(N : 1_M)N = N$ for $N \in M$. Then,

$$N = x1_M = (x1_M : 1_M)x1_M$$
$$= x(x1_M : 1_M)1_M = x^2 1_M.$$

Since 1_M is compact, we obtain $x^2 = x$ (see [5], Theorem 5ii). □

4. Conclusion

The study gives us the definitions of second, coprime and pure elements in lattices (similarly, in lattice modules). Then it is obtained that in a lattice module the concept of second element is equivalent to the concept of coprime element. Moreover, it is shown that if M is coprime, then N is primary ⇔ N is prime. Under some particular conditions in Theorem 3, it is proved that N is a pure element of M ⇔ $(N :_L 1_M)$ is a pure element of L. Consequently, the relationship among pure, multiplication and idempotent elements in lattice and lattice modules is obtained.

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