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Research Article

On the symmetric polynomials in the variety of Grassmann algebras

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Abstract

Let K be a field of characteristic zero and L be the associative algebra of rank 2 over K in the variety generated by Grassmann algebras. In this paper, we study the subalgebra L^{S_2} of symmetric polynomials in the algebra L, and give a finite generating set for L^{S_2}.

Keywords: PI-algebra, Grassmann algebras, symmetric polynomial

Grassmann cebirleri sınıfında simetrik polinomlar üzerine

Öz

K karakteristiği sıfır olan bir cisim ve L, Grassmann cebirleri tarafından üretilen varyetede, K cismi üzerinde rankı 2 olan birleşmeli cebir olsun. Bu çalışmada, L cebirinin L^{S_2} simetrik polinomlar alt cebiri incelenmiş ve L^{S_2} için sonlu bir üreteç kümesi verilmiştir.

Anahtar Kelimeler: PI-cebiri, Grassmann cebirleri, simetrik polinom.

1. Introduction

Let A_n be a free unitary associative algebra generated by $x_1, \dots, x_n$ over a field K of characteristic zero. The Grassmann algebra is the factor algebra $G_n = A_n/I$ where I is generated by $x_ix_j + x_jx_i$, $1 \leq i, j \leq n$. The Grassmann algebra is generated by $e_i = x_i + I$, $1 \leq i \leq n$, which implies that $e_ie_j + e_je_i = 0$. As a vector space, the Grassmann algebra has the basis $B = \{e_{i_1} \dots e_{i_k} : i_1 \leq \cdots \leq i_k,\ 1 \leq k \leq n\} \cup \{1\}$.

Let $K\langle X\rangle$ be the free associative algebra generated by X over K where $X = \{x_1, x_2, \dots\}$ is a countable infinite set of variables. We call elements of $K\langle X\rangle$ polynomials. Let A be an algebra over K and $f(x_1, \dots, x_n) \in K\langle X\rangle$. We call $f(x_1, \dots, x_n)$ a polynomial identity of A, if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. The algebra A is called a *PI-algebra* if it has a nontrivial

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polynomial identity. We denote by $T(A)$ the set of all polynomial identities of A. Since $T(A)$ is an ideal of $K\langle X\rangle$ which is invariant under all endomorphisms of $K\langle X\rangle$, it is a T-*ideal* of A.

One of the objectives of the theory of PI-algebras is finding the generating sets for T-ideal of an algebra. Given a commutative unitary algebra, the T-ideal of the algebra is generated by the commutator $[x,y]=xy-yx$. Since $\big[[x,y],z\big]=0$ for all $x,y,z\in G_n$, the Grassmann algebra G_n is a PI-algebra. It is shown that the T-ideal of the Grassmann algebras is generated by $\big[[x,y],z\big]$. It is shown by Latyshev (1976) and by Krakovski and Regev (1973).

The variety defined by the polynomial identity $\big[[x,y],z\big]=0$ from $T(G_n)$ is called the variety generated by the Grassmann algebra. Let us denote by L the free associative algebra of rank 2 generated by $\{x,y\}$ in the variety generated by the Grassmann algebra.

In this paper, we investigate the subalgebra of symmetric polynomials in the algebra L. We give a generating set for the algebra of symmetric polynomials as an algebra and obtain the presentation of the commutator ideal of the algebra of symmetric polynomials.

2. Preliminaries

Let K be a field of characteristic zero, $K[x_1,\dots,x_n]$ be the commutative algebra of polynomials. A polynomial $f(x_1,\dots,x_n)\in K[x_1,\dots,x_n]$ is symmetric if it is invariant under every permutation of the variables $x_1,\dots,x_n$.

The polynomials $\sigma_1,\dots,\sigma_n\in K[x_1,\dots,x_n]$ are called the elementary symmetric polynomials, where

$$\sigma_1=x_1+\cdots+x_n$$

$$\sigma_2=x_1x_2+x_1x_3+\cdots+x_2x_3+\cdots+x_{n-1}x_n$$

$$\sigma_3=x_1x_2x_3+x_1x_2x_4+\cdots+x_{n-2}x_{n-1}x_n$$

$$\vdots$$

$$\sigma_n=x_1x_2x_3\dots x_n.$$

The elementary symmetric polynomials are generators of the algebra of symmetric polynomials. Every symmetric polynomial can be written uniquely as a polynomial in the elementary symmetric polynomials. The generating set is not unique. The polynomials

$p_1, \dots, p_n \in K[x_1, \dots, x_n]$ form another generating set for the symmetric polynomials, where $p_k = x_1^k + \cdots + x_n^k$. (see Cox et al., 2015; Strumfels, 2008; van der Waerden, 1949.)

We refer the readers to the work by Fındık and Öğüşlü (2019) for a generating set of symmetric polynomials in the free metabelian Lie algebra of rank 2 as one of the generalizations in a Lie algebraic setting.

Now let L be the algebra of rank 2 freely generated by elements x, y, in the variety generated by the Grassmann algebra consisting of unitary associative algebras over the ground field K. The ideal I of L generated by all commutators $[r, s] = rs - sr$, where $r, s \in L$, is called the commutator ideal of L. The elements of I are of the form $\sum \alpha p[r, s]t$ where $p, t \in L$, $\alpha \in K$. It is well known (see e.g. Drensky (1996)) that the basis of the commutator ideal I as a vector space consists of elements $x^a y^b [x, y]$, $a, b \geq 0$. The commutative polynomial algebra $K[x, y]$ acts on I by the following action.

$$p \sum \alpha_{ab} x^a y^b [x, y] = \sum \alpha_{ab} p x^a y^b [x, y]$$

where $p \in K[x, y]$ and $\sum \alpha_{ab} x^a y^b [x, y] \in I$. Therefore the commutator ideal I is a free $K[x, y]$-module generated by the commutator $[x, y]$ of x and y.

The polynomial identity $\big[[x, y], z\big] = 0$ implies the identity

$$[x, y][z, t] = -[x, z][y, t],$$

and

$$x^a y^b [x, y] = y^b x^a [x, y] = [x, y] x^a y^b = [x, y] y^b x^a$$

is satisfied in I.

We define the sets of symmetric polynomials of L and I by

$$L^{S_2} = \{p(x, y) \in L : p(x, y) = p(y, x)\}$$

and

$$I^{S_2} = \{p(x, y) \in I : p(x, y) = p(y, x)\}$$

respectively. These sets are subalgebras of invariants of the symmetric group S_2.

3. Results and Discussion

Lemma 3.1. Let $p(x,y)$ be an element in I^{S_2}. Then $p(x,y)$ is of the form

$$p(x,y) = \sum_{0\le a<b} \alpha_{ab}(x^a y^b - x^b y^a)[x,y]$$

for some $\alpha_{ab} \in K$.

Proof. The element $p(x,y) \in I^{S_2} \subseteq I$ can be expressed as

$$p(x,y) = \sum_{0\le a,b} \alpha_{ab} x^a y^b [x,y] = \sum_{a\ne b} \alpha_{ab} x^a y^b [x,y] + \sum_{0\le a} \alpha_{aa} x^a y^a [x,y]$$

Since $p(x,y) \in I^{S_2}$, $p(x,y) = p(y,x)$ holds. Hence

$$p(x,y) = \sum_{a\ne b} \alpha_{ab} y^a x^b [y,x] + \sum_{0\le a} \alpha_{aa} y^a x^a [y,x]$$

and

$$\sum_{0\le a} 2\alpha_{aa} x^a y^a [x,y] + \sum_{a\ne b} \alpha_{ab} x^a y^b [x,y] - \sum_{a\ne b} \alpha_{ab} y^a x^b [y,x] = 0.$$

Since $\sum_{0\le a} 2\alpha_{aa} x^a y^a [x,y] = 0$ by the suggested basis, we have $\alpha_{aa} = 0$ for $0 \le a$. Therefore we have

$$\sum_{a<b} (\alpha_{ab} + \alpha_{ba}) x^a y^b [x,y] + \sum_{b<a} (\alpha_{ab} + \alpha_{ba}) x^a y^b [x,y] = 0$$

where each sum equals zero by linear independence. So that $\alpha_{ab} = -\alpha_{ba}$ for all $a \ne b$. Thus we have the following computations which provide the desired form of the element $p(x,y)$.

$$p(x,y) = \sum_{a<b} \alpha_{ab} x^a y^b [x,y] + \sum_{b<a} \alpha_{ab} x^a y^b [x,y]$$

$$= \sum_{a<b} \alpha_{ab} x^a y^b [x,y] + \sum_{a<b} \alpha_{ba} x^b y^a [x,y]$$

$$= \sum_{0\le a<b} \alpha_{ab} (x^a y^b - x^b y^a)[x,y].$$

Corollary 3.2. The set

$$\{(x^a y^b - x^b y^a)[x,y]: 0 \leq a < b\}$$

is the basis of I^{S_2}.

Proof. The given set spans I^{S_2} as a vector space by Lemma 3.1. It is sufficient to show that the set is linearly independent. Let

$$\sum_{a<b} \alpha_{ab}(x^a y^b - x^b y^a)[x,y] = 0.$$

We can fix $a+b$ to n since L^{S_2} is a graded vector space.

$$\sum_{a+b=n, a<b} \alpha_{ab}(x^a y^b - x^b y^a)[x,y] = 0$$

As I is the $K[x,y]$-module generated by $[x,y]$, we have

$$\sum_{a+b=n, 0\leq a<b} \alpha_{ab} x^a y^b - \sum_{a+b=n, 0\leq a<b} \alpha_{ab} x^b y^a = 0.$$

So $\alpha_{ab} = 0$ where $0 \leq a < b$ since each sum equals zero.

Theorem 3.3. The presentation of I^{S_2} is

$$I^{S_2} = \langle m_{\boldsymbol{ab}} \mid 0 \leq a < b, m_{\boldsymbol{ab}} m_{\boldsymbol{a'b'}} = 0 \rangle$$

where $m_{\boldsymbol{ab}} = (x^a y^b - x^b y^a)[x,y]$.

Proof. Let $m_{\boldsymbol{a'b'}} = \left(x^{\boldsymbol{a'}} y^{\boldsymbol{b'}} - x^{\boldsymbol{b'}} y^{\boldsymbol{a'}}\right)[x,y]$.

$$m_{\boldsymbol{ab}} m_{\boldsymbol{a'b'}} = (x^a y^b - x^b y^a)[x,y]\left(x^{\boldsymbol{a'}} y^{\boldsymbol{b'}} - x^{\boldsymbol{b'}} y^{\boldsymbol{a'}}\right)[x,y]$$

$$= (x^a y^b - x^b y^a)\left(x^{\boldsymbol{a'}} y^{\boldsymbol{b'}} - x^{\boldsymbol{b'}} y^{\boldsymbol{a'}}\right)[x,y][x,y]$$

$$= (x^a y^b - x^b y^a)\left(x^{\boldsymbol{a'}} y^{\boldsymbol{b'}} - x^{\boldsymbol{b'}} y^{\boldsymbol{a'}}\right)(xy - yx)[x,y]$$

$$= (x^a y^b - x^b y^a)\left(x^{\boldsymbol{a'}} y^{\boldsymbol{b'}} - x^{\boldsymbol{b'}} y^{\boldsymbol{a'}}\right)(xy)[x,y] - (x^a y^b - x^b y^a)\left(x^{\boldsymbol{a'}} y^{\boldsymbol{b'}} - x^{\boldsymbol{b'}} y^{\boldsymbol{a'}}\right)(xy)[x,y]$$

$$= 0.$$

Theorem 3.4. The algebra L^{S_2} is generated by the set $\{x+y, x^2+ y^2, (y-x)[x,y]\}$, and the algebra I^{S_2} is a left $K[x,y]^{S_2}$-module generated by the element $(y-x)[x,y]$.

Proof. The algebra $(L/I)^{S_2} \cong K[x,y]^{S_2}$ is generated by $x+y, x^2+ y^2$. Thus it is sufficient to show that the algebra I^{S_2} is contained in the algebra generated by $x+y, x^2+ y^2, (y-x)[x,y]$. Corollary 3.2 gives that I^{S_2} has the basis

$$\{(x^a y^b - x^b y^a)[x,y]: 0 \leq a < b\},$$

which can be also considered as generating set. Direct computations give that

$$(x^a y^b - x^b y^a)[x,y] = q(x,y)r(x,y)(y-x)[x,y]$$

where

$$q(x,y) = \left(\frac{(x+y)^2 - (x^2+y^2)}{2}\right)^a$$

and

$$r(x,y) = \sum_{i=1}^{b-a} x^{b-a-i} y^{i-1}.$$

The polynomial $r(x,y)$ can be written as $r(x,y) = \frac{x^{b-a} - y^{b-a}}{x-y}$. It is clear that $r(x,y) = r(y,x)$ and $q(x,y) = q(y,x)$. Hence $q(x,y), r(x,y) \in K[x+y, x^2+y^2]$, and $(x^a y^b - x^b y^a)[x,y]$ is included in $\langle x+y, x^2+ y^2, (y-x)[x,y]\rangle$. This also shows that I^{S_2} is a left $K[x+y, x^2+y^2] = K[x,y]^{S_2}$-module.

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