Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (5) (2020), 1804 – 1824
DOI : 10.15672/hujms.599132

Research Article

Almost h-conformal semi-invariant submersions from almost quaternionic Hermitian manifolds

Kwang Soon Park

Division of General Mathematics, Room 4-107, Changgong Hall, University of Seoul, Seoul 02504, Republic of Korea

Abstract

As a generalization of Riemannian submersions, horizontally conformal submersions, semi-invariant submersions, h-semi-invariant submersions, almost h-semi-invariant submersions, conformal semi-invariant submersions, we introduce h-conformal semi-invariant submersions and almost h-conformal semi-invariant submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. We study their properties: the geometry of foliations, the conditions for total manifolds to be locally product manifolds, the conditions for such maps to be totally geodesic. Finally, we give some examples of such maps.

Mathematics Subject Classification (2010). 53C15, 53C26, 53C43

Keywords. horizontally conformal submersion, quaternionic manifold, totally geodesic

1. Introduction

Riemannian submersions were independently introduced by B. O'Neill [20] and A. Gray [11] in 1960s. Using the notion of almost Hermitian submersions, B. Watson [29] obtained some differential geometric properties among fibers, base manifolds, and total manifolds. After that, many geometers study this area and there are a lot of results on this topic.

As a generalization of Riemannian submersions, a horizontally conformal submersion was introduced independently by B. Fuglede [14] and T. Ishihara [18] in 1970s and it is a particular type of conformal maps.

Given a C^∞-submersion F from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N), according to the conditions on the map $F : (M, g_M) \mapsto (N, g_N)$, we have the following types of submersions: a Riemannian submersion ([11, 13, 20]), an almost Hermitian submersion [29], an invariant submersion [27], an anti-invariant submersion [24], a slant submersion ([9, 25]), a semi-invariant submersion [26], a semi-slant submersion [23], a quaternionic submersion [15], an h-anti-invariant submersion and an almost h-anti-invariant submersion [22], an h-semi-invariant submersion and an almost h-semi-invariant submersion [21], a horizontally conformal submersion ([4, 12]), a conformal anti-invariant submersion [1], a conformal semi-invariant submersion [2], etc.

It is well-known that Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([7, 30]), Kaluza-Klein theory ([8, 16]), Supergravity and superstring theories ([17, 19]), etc. And the quaternionic Kähler manifolds have

Email address: parkksn@gmail.com
Received: 31.07.2019; Accepted: 24.01.2020

applications in physics as the target spaces for nonlinear $\sigma-$models with supersymmetry [10].

The paper is organized as follows. In Section 2 we remind some notions, which are needed in the following sections. In Section 3 we give the definitions of h-conformal semi-invariant submersions and almost h-conformal semi-invariant submersions and obtain some properties on them: the characterizations of such maps, the harmonicity of such maps, the conditions for such maps to be totally geodesic, the integrability of distributions, the geometry of foliations, etc. In Section 4 we give some examples of h-conformal semi-invariant submersions and almost h-conformal semi-invariant submersions.

2. Preliminaries

Let (M, g_M) and (N, g_N) be Riemannian manifolds, where g_M and g_N are Riemannian metrics on C^∞-manifolds M and N, respectively.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a C^∞-map.

We call the map F a *C^∞-submersion* if F is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$.

Then the map F is said to be a *Riemannian submersion* ([13, 20]) if F is a C^∞-submersion and

$$(F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \mapsto (T_{F(p)}N, (g_N)_{F(p)})$$

is a linear isometry for any $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space T_pM to M at p.

The map F is called *horizontally weakly conformal* at $p \in M$ if it satisfies either (i) $(F_*)_p = 0$ or (ii) $(F_*)_p$ is surjective and there exists a positive number $\lambda(p) > 0$ such that

$$g_N((F_*)_pX, (F_*)_pY) = \lambda^2 g_M(X, Y) \quad \text{for } X, Y \in (\ker(F_*)_p)^\perp. \tag{2.1}$$

We call the point p a *critical point* if it satisfies the type (i) and call the point p a *regular point* if it satisfies the type (ii). And the positive number $\lambda(p)$ is said to be *dilation* of F at p. The map F is called *horizontally weakly conformal* if it is horizontally weakly conformal at any point of M. If the map F is horizontally weakly conformal and it has no critical points, then we call the map F a *horizontally conformal submersion*. The horizontally conformal submersion F is said to be *horizontally homothetic* if $X(\lambda) = 0$ for $X \in \Gamma((\ker F_*)^\perp)$.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion.

Given any vector field $U \in \Gamma(TM)$, we write

$$U = \mathcal{V}U + \mathcal{H}U, \tag{2.2}$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Define the (O'Neill) tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{2.3}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.4}$$

for vector fields $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of g_M ([13, 20]). Then it is well-known that

$$g_M(\mathcal{T}_U V, W) = -g_M(V, \mathcal{T}_U W) \tag{2.5}$$

$$g_M(\mathcal{A}_U V, W) = -g_M(V, \mathcal{A}_U W) \tag{2.6}$$

for $U, V, W \in \Gamma(TM)$.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a C^∞-map.

Then the *second fundamental form* of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_*Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N, [4].

Recall that F is said to be *harmonic* if the tension field $\tau(F) = trace(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$, [4].

Lemma 2.1 ([28])**.** *Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : (M, g_M) \mapsto (N, g_N)$ a C^∞-map. Then we have*

$$\nabla^F_X F_*Y - \nabla^F_Y F_*X - F_*([X, Y]) = 0 \tag{2.7}$$

for $X, Y \in \Gamma(TM)$.

Remark 2.2. (1) From (2.7), we see that the second fundamental form ∇F_* is symmetric.

(2) By (2.7), we obtain

$$[V, X] \in \Gamma(\ker F_*) \tag{2.8}$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation λ.

We call a vector field $X \in \Gamma(TM)$ *basic* if (i) $X \in \Gamma((\ker F_*)^\perp)$ and (ii) X is F-related with some vector field $\overline{X} \in \Gamma(TN)$. (i.e., $(F_*)_p X(p) = \overline{X}(F(p))$ for any $p \in M$.)

Given any fiber $F^{-1}(y)$, $y \in N$, and any basic vector fields $X, Y \in \Gamma((\ker F_*)^\perp)$, we have

$$\lambda(x)^2 g_M(X, Y)(x) = g_N(F_*X, F_*Y)(y) = constant$$

for any $x \in F^{-1}(y)$ so that

$$V(\lambda^2 g_M(X, Y)) = V(g_N(F_*X, F_*Y)) = 0 \quad \text{for } V \in \Gamma(\ker F_*). \tag{2.9}$$

Then we get

Proposition 2.3 ([12])**.** *Let $F : (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation λ. Then we obtain*

$$\mathcal{A}_X Y = \frac{1}{2}\{\mathcal{V}[X, Y] - \lambda^2 g_M(X, Y)\nabla_{\mathcal{V}}(\frac{1}{\lambda^2})\} \tag{2.10}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Here, $\nabla_{\mathcal{V}}$ denotes the gradient vector field in the distribution $\ker F_* \subset TM$. (i.e., $\nabla_{\mathcal{V}} f = \sum_{i=1}^{m} V_i(f)V_i$ for $f \in C^\infty(M)$ and a local orthonormal frame $\{V_1, \cdots, V_m\}$ of $\ker F_*$.)

Lemma 2.4 ([4])**.** *Let $F : (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation λ. Then we have*

$$(\nabla F_*)(X, Y) = X(\ln \lambda)F_*Y + Y(\ln \lambda)F_*X - g_M(X, Y)F_*(\nabla \ln \lambda), \tag{2.11}$$

$$(\nabla F_*)(V, W) = -F_*(\mathcal{T}_V W), \tag{2.12}$$

$$(\nabla F_*)(X, V) = -F_*(\nabla_X V) = -F_*(\mathcal{A}_X V) \tag{2.13}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$.

Let (M, g_M, J) be an almost Hermitian manifold, where J is a compatible almost complex structure on M (i.e., $J^2 = -id$, $g_M(JX, JY) = g_M(X, Y)$ for $X, Y \in \Gamma(TM)$).

We call a horizontally conformal submersion $F : (M, g_M, J) \mapsto (N, g_N)$ a *conformal anti-invariant submersion* [1] if $J(\ker F_*) \subset (\ker F_*)^\perp$.

A horizontally conformal submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *conformal semi-invariant submersion* [2] if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1, \ J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$

Let M be a $4m-$dimensional C^∞-manifold and let E be a rank 3 subbundle of $\mathrm{End}(TM)$ such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1,2,3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1,2,3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on M and (M, E) an *almost quaternionic manifold* [3].

Moreover, let g be a Riemannian metric on M such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1,2,3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{2.14}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{2.15}$$

for all vector fields $X, Y \in \Gamma(TM)$, where the indices are taken from $\{1,2,3\}$ modulo 3.

Then we call (M, E, g) an *almost quaternionic Hermitian manifold* [15].

For convenience, the above basis $\{J_1, J_2, J_3\}$ satisfying (2.14) and (2.15) is said to be a *quaternionic Hermitian basis*.

Let (M, E, g) be an almost quaternionic Hermitian manifold.

We call (M, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1,2,3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for any vector field $X \in \Gamma(TM)$, where the indices are taken from $\{1,2,3\}$ modulo 3 [15].

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1,2,3\}$, where ∇ is the Levi-Civita connection of the metric g), then (M, E, g) is said to be a *hyperkähler manifold*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric* [6].

Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds.

A map $F : M \mapsto N$ is called a $(E_M, E_N)-$*holomorphic map* if given a point $x \in M$, for any $J \in (E_M)_x$ there exists $J' \in (E_N)_{F(x)}$ such that

$$F_* \circ J = J' \circ F_*.$$

A Riemannian submersion $F : M \mapsto N$ which is a $(E_M, E_N)-$holomorphic map is called a *quaternionic submersion* [15].

Moreover, if (M, E_M, g_M) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that F is a *quaternionic Kähler submersion* (or a *hyperkähler submersion*) [15].

Then we know that any quaternionic Kähler submersion is a harmonic map [15].

Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ R(\mathcal{D}_1) = \mathcal{D}_1, \ R(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [21].

We call such a basis $\{I, J, K\}$ an *h-semi-invariant basis*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R, \ R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$ [21].

We call such a basis $\{I, J, K\}$ an *almost h-semi-invariant basis.*

Throughout this paper, we will use the above notations.

3. Almost h-conformal semi-invariant submersions

In this section, we define h-conformal semi-invariant submersions and almost h-conformal semi-invariant submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. And we study their properties: the integrability of distributions, the geometry of foliations, the conditions for such maps to be totally geodesic, etc.

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A horizontally conformal submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *h-conformal semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ R(\mathcal{D}_1) = \mathcal{D}_1,\ R(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ an *h-conformal semi-invariant basis.*

Definition 3.2. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A horizontally conformal submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-conformal semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R,\ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,\ R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ an *almost h-conformal semi-invariant basis.*

Remark 3.3. (1) Let F be an h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal semi-invariant basis. Then the fibers of the map F are quaternionic CR-submanifolds [5].

(2) Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an h-conformal semi-invariant submersion. Then the map F is also an almost h-conformal semi-invariant submersion.

Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-conformal semi-invariant submersion with an almost h-conformal semi-invariant basis $\{I, J, K\}$.

Denote the orthogonal complement of $R\mathcal{D}_2^R$ in $(\ker F_*)^\perp$ by μ^R for $R \in \{I, J, K\}$. We easily see that μ^R is R-invariant for $R \in \{I, J, K\}$.

Then given $X \in \Gamma(\ker F_*)$, we write

$$RX = \phi_R X + \omega_R X, \tag{3.1}$$

where $\phi_R X \in \Gamma(\mathcal{D}_1^R)$ and $\omega_R X \in \Gamma(R\mathcal{D}_2^R)$ for $R \in \{I, J, K\}$.

Given $Z \in \Gamma((\ker F_*)^\perp)$, we get

$$RZ = B_R Z + C_R Z, \tag{3.2}$$

where $B_R Z \in \Gamma(\mathcal{D}_2^R)$ and $C_R Z \in \Gamma(\mu^R)$ for $R \in \{I, J, K\}$.

We see that

$$(\ker F_*)^\perp = R\mathcal{D}_2^R \oplus \mu^R \quad \text{for } R \in \{I, J, K\} \tag{3.3}$$

and

$$g_M(C_R X, RV) = 0 \tag{3.4}$$

for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\mathcal{D}_2^R)$.

Define $\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y$ for $X, Y \in \Gamma(\ker F_*)$.

We also define

$$(\nabla_X \phi_R)Y := \widehat{\nabla}_X \phi_R Y - \phi_R \widehat{\nabla}_X Y \tag{3.5}$$

and

$$(\nabla_X \omega_R)Y := \mathcal{H}\nabla_X \omega_R Y - \omega_R \widehat{\nabla}_X Y \tag{3.6}$$

for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.

Then we easily obtain

Lemma 3.4. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then we get*

(1)

$$\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y = \phi_R \widehat{\nabla}_X Y + B_R \mathcal{T}_X Y$$
$$\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y = \omega_R \widehat{\nabla}_X Y + C_R \mathcal{T}_X Y$$

for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.

(2)

$$\mathcal{V}\nabla_Z B_R W + \mathcal{A}_Z C_R W = \phi_R \mathcal{A}_Z W + B_R \mathcal{H}\nabla_Z W$$
$$\mathcal{A}_Z B_R W + \mathcal{H}\nabla_Z C_R W = \omega_R \mathcal{A}_Z W + C_R \mathcal{H}\nabla_Z W$$

for $Z, W \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$.

(3)

$$\widehat{\nabla}_X B_R Z + \mathcal{T}_X C_R Z = \phi_R \mathcal{T}_X Z + B_R \mathcal{H}\nabla_X Z$$
$$\mathcal{T}_X B_R Z + \mathcal{H}\nabla_X C_R Z = \omega_R \mathcal{T}_X Z + C_R \mathcal{H}\nabla_X Z$$

for $X \in \Gamma(\ker F_*)$, $Z \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$.

Remark 3.5. By (3.5), (3.6), and Lemma 3.4 (1), we have

$$(\nabla_X \omega_R)Y = B_R \mathcal{T}_X Y - \mathcal{T}_X \omega_R Y \tag{3.7}$$

$$(\nabla_X \omega_R)Y = C_R \mathcal{T}_X Y - \mathcal{T}_X \phi_R Y \tag{3.8}$$

for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.

Now, we investigate the integrability of some distributions.

Lemma 3.6. *Let F be an h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal semi-invariant basis. Then we have the followings:*

(i) *The distribution $\mathcal{D}_2$ is always integrable.*

(ii) *The following conditions are equivalent:*

  (a) *The distribution $\mathcal{D}_1$ is integrable.*

  (b) $(\nabla F_*)(W, IV) - (\nabla F_*)(V, IW) \in \Gamma(F_*\mu^I)$ *for* $V, W \in \Gamma(\mathcal{D}_1)$.

  (c) $(\nabla F_*)(W, JV) - (\nabla F_*)(V, JW) \in \Gamma(F_*\mu^J)$ *for* $V, W \in \Gamma(\mathcal{D}_1)$.

  (d) $(\nabla F_*)(W, KV) - (\nabla F_*)(V, KW) \in \Gamma(F_*\mu^K)$ *for* $V, W \in \Gamma(\mathcal{D}_1)$.

Proof. By (2.7), we have $[V, W] \in \Gamma(\ker F_*)$ for $V, W \in \Gamma(\ker F_*)$.

We claim that $\mathcal{T}_V RW = \mathcal{T}_W RV$ for $V, W \in \Gamma(\mathcal{D}_2)$ and $R \in \{I, J, K\}$.

Given $X \in \Gamma(\ker F_*)$, we get

$$\begin{aligned} g_M(\mathcal{T}_V RW, X) &= -g_M(RW, \nabla_V X) = -g_M(RW, \nabla_X V) = g_M(\nabla_X RW, V) \\ &= -g_M(\nabla_X W, RV) = -g_M(\nabla_W X, RV) = g_M(X, \nabla_W RV) \\ &= g_M(X, \mathcal{T}_W RV), \end{aligned}$$

which means our claim.

Given $V, W \in \Gamma(\mathcal{D}_2)$ and $Z \in \Gamma(\mathcal{D}_1)$, we obtain

$$g_M([V,W],Z) = g_M(\nabla_V W - \nabla_W V, Z) = g_M(\mathcal{T}_V RW - \mathcal{T}_W RV, RZ) = 0,$$

which implies (i).

For (ii), given $V, W \in \Gamma(\mathcal{D}_1)$, $Z \in \Gamma(\mathcal{D}_2)$, and $R \in \{I, J, K\}$, we have

$$\begin{aligned} g_M([V,W],Z) &= \frac{1}{\lambda^2} g_N(F_*\nabla_V RW - F_*\nabla_W RV, F_*RZ) \\ &= \frac{1}{\lambda^2} g_N((\nabla F_*)(W, RV) - (\nabla F_*)(V, RW), F_*RZ) \end{aligned}$$

so that we get $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

Therefore, the result follows. □

Theorem 3.7. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ is integrable.*

(b) $\mathcal{A}_Y\omega_I B_I X - \mathcal{A}_X \omega_I B_I Y + \phi_I(\mathcal{A}_Y C_I X - \mathcal{A}_X C_I Y) \in \Gamma(\mathcal{D}_2^I)$ *and*

$$\begin{aligned} &\frac{1}{\lambda^2} g_N(\nabla_Y^F F_* C_I X - \nabla_X^F F_* C_I Y, F_* IV) \\ &= g_M(\mathcal{A}_Y B_I X - \mathcal{A}_X B_I Y - C_I Y(\ln\lambda)X + C_I X(\ln\lambda)Y \\ &+ 2g_M(X, C_I Y)\nabla(\ln\lambda), IV) \end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\mathcal{D}_2^I)$.

(c) $\mathcal{A}_Y\omega_J B_J X - \mathcal{A}_X \omega_J B_J Y + \phi_J(\mathcal{A}_Y C_J X - \mathcal{A}_X C_J Y) \in \Gamma(\mathcal{D}_2^J)$ *and*

$$\begin{aligned} &\frac{1}{\lambda^2} g_N(\nabla_Y^F F_* C_J X - \nabla_X^F F_* C_J Y, F_* JV) \\ &= g_M(\mathcal{A}_Y B_J X - \mathcal{A}_X B_J Y - C_J Y(\ln\lambda)X + C_J X(\ln\lambda)Y \\ &+ 2g_M(X, C_J Y)\nabla(\ln\lambda), JV) \end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\mathcal{D}_2^J)$.

(d) $\mathcal{A}_Y\omega_K B_K X - \mathcal{A}_X \omega_K B_K Y + \phi_K(\mathcal{A}_Y C_K X - \mathcal{A}_X C_K Y) \in \Gamma(\mathcal{D}_2^K)$ *and*

$$\begin{aligned} &\frac{1}{\lambda^2} g_N(\nabla_Y^F F_* C_K X - \nabla_X^F F_* C_K Y, F_* KV) \\ &= g_M(\mathcal{A}_Y B_K X - \mathcal{A}_X B_K Y - C_K Y(\ln\lambda)X + C_K X(\ln\lambda)Y \\ &+ 2g_M(X, C_K Y)\nabla(\ln\lambda), KV) \end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\mathcal{D}_2^K)$.

Proof. Given $X, Y \in \Gamma((\ker F_*)^\perp)$, $W \in \Gamma(\mathcal{D}_1^R)$, and $R \in \{I, J, K\}$, we have

$$\begin{aligned} g_M([X,Y],W) &= g_M(\nabla_X B_R Y, RW) + g_M(\nabla_X C_R Y, RW) \\ &- g_M(\nabla_Y B_R X, RW) - g_M(\nabla_Y C_R X, RW) \\ &= -g_M(\nabla_X R B_R Y, W) + g_M(\mathcal{A}_X C_R Y, RW) \\ &+ g_M(\nabla_Y R B_R X, W) - g_M(\mathcal{A}_Y C_R X, RW) \\ &= -g_M(\nabla_X \omega_R B_R Y, W) - g_M(\phi_R \mathcal{A}_X C_R Y, W) \\ &+ g_M(\nabla_Y \omega_R B_R X, W) + g_M(\phi_R \mathcal{A}_Y C_R X, W) \ (\text{since } \phi_R B_R = 0) \\ &= g_M(\mathcal{A}_Y \omega_R B_R X - \mathcal{A}_X \omega_R B_R Y + \phi_R \mathcal{A}_Y C_R X - \phi_R \mathcal{A}_X C_R Y, W) \end{aligned}$$

so that

$$g_M([X,Y],W) = 0 \quad \text{for } W \in \Gamma(\mathcal{D}_1^R) \tag{3.9}$$
$$\Leftrightarrow \mathcal{A}_Y\omega_R B_R X - \mathcal{A}_X\omega_R B_R Y + \phi_R\mathcal{A}_Y C_R X - \phi_R\mathcal{A}_X C_R Y \in \Gamma(\mathcal{D}_2^R).$$

Given $V \in \Gamma(\mathcal{D}_2^R)$, by using (2.11) and (3.4), we get

$$\begin{aligned} g_M([X,Y],V) &= g_M(\nabla_X B_R Y, RV) + g_M(\nabla_X C_R Y, RV) \\ &\quad - g_M(\nabla_Y B_R X, RV) - g_M(\nabla_Y C_R X, RV) \\ &= g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X, RV) \\ &\quad + \frac{1}{\lambda^2} g_N(-X(\ln\lambda)F_*C_R Y - C_R Y(\ln\lambda)F_*X \\ &\quad + g_M(X, C_R Y)F_*\nabla(\ln\lambda) + \nabla_X^F F_*C_R Y \\ &\quad + Y(\ln\lambda)F_*C_R X + C_R X(\ln\lambda)F_*Y - g_M(Y, C_R X)F_*\nabla(\ln\lambda) \\ &\quad - \nabla_Y^F F_*C_R X, F_*RV) \\ &= g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X + C_R X(\ln\lambda)Y - C_R Y(\ln\lambda)X \\ &\quad + 2g_M(X, C_R Y)\nabla(\ln\lambda), RV) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla_Y^F F_*C_R X - \nabla_X^F F_*C_R Y, F_*RV) \end{aligned}$$

so that

$$g_M([X,Y],V) = 0 \quad \text{for } V \in \Gamma(\mathcal{D}_2^R) \tag{3.10}$$
$$\Leftrightarrow \frac{1}{\lambda^2} g_N(\nabla_Y^F F_*C_R X - \nabla_X^F F_*C_R Y, F_*RV)$$
$$= g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X + C_R X(\ln\lambda)Y - C_R Y(\ln\lambda)X$$
$$+2g_M(X, C_R Y)\nabla(\ln\lambda), RV).$$

Using (3.9) and (3.10), we obtain $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

Therefore, we have the result. □

Theorem 3.8. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Assume that the distribution $(\ker F_*)^\perp$ is integrable. Then the following conditions are equivalent:*

(a) *The map F is horizontally homothetic.*
(b) $\lambda^2 g_M(\mathcal{A}_Y B_I X - \mathcal{A}_X B_I Y, IV) = g_N(\nabla_Y^F F_*C_I X - \nabla_X^F F_*C_I Y, F_*IV)$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^I)$.
(c) $\lambda^2 g_M(\mathcal{A}_Y B_J X - \mathcal{A}_X B_J Y, JV) = g_N(\nabla_Y^F F_*C_J X - \nabla_X^F F_*C_J Y, F_*JV)$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^J)$.
(d) $\lambda^2 g_M(\mathcal{A}_Y B_K X - \mathcal{A}_X B_K Y, KV) = g_N(\nabla_Y^F F_*C_K X - \nabla_X^F F_*C_K Y, F_*KV)$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^K)$.

Proof. Given $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(\mathcal{D}_2^R)$, and $R \in \{I, J, K\}$, from the proof of Theorem 3.7, we have

$$\begin{aligned} g_M([X,Y],V) \quad = \quad & g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X + C_R X(\ln\lambda)Y \\ & - C_R Y(\ln\lambda)X + 2g_M(X, C_R Y)\nabla(\ln\lambda), RV) \\ & - \frac{1}{\lambda^2} g_N(\nabla_Y^F F_*C_R X - \nabla_X^F F_*C_R Y, F_*RV). \end{aligned} \tag{3.11}$$

Using (3.11), it is easy to see $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$, $(a) \Rightarrow (d)$.

Conversely, from (3.11), we get

$$g_M(C_R X(\ln\lambda)Y - C_R Y(\ln\lambda)X + 2g_M(X, C_R Y)\nabla(\ln\lambda), RV) = 0 \tag{3.12}$$

Applying $Y = RV$ at (3.12), we obtain

$$g_M(\nabla(\ln\lambda), C_R X) g_M(RV, RV) = 0,$$

which implies

$$g_M(\nabla(\lambda), X) = 0 \quad \text{for } X \in \Gamma(\mu^R). \tag{3.13}$$

Applying $Y = C_R X$, $X \in \Gamma(\mu^R)$, at (3.12), we have

$$2g_M(X, C_R^2 X) g_M(\nabla(\ln\lambda), RV) = -2g_M(X, X) g_M(\nabla(\ln\lambda), RV) = 0,$$

which implies

$$g_M(\nabla(\lambda), RV) = 0 \quad \text{for } V \in \Gamma(\mathcal{D}_2^R). \tag{3.14}$$

By (3.13) and (3.14), we get $(b) \Rightarrow (a)$, $(c) \Rightarrow (a)$, $(d) \Rightarrow (a)$.

Therefore, the result follows. □

We deal with some particular type of conformal submersions.

Definition 3.9. Let F be an almost h-conformal semi-invariant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). If $R(\mathcal{D}_2^R) = (\ker F_*)^\perp$ for $R \in \{I, K\}$ and $J(\ker F_*) = \ker F_*$ (i.e., $\mathcal{D}_2^J = \{0\}$), then we call the map F an *almost h-conformal anti-holomorphic semi-invariant submersion.*

We call such a basis $\{I, J, K\}$ an *almost h-conformal anti-holomorphic semi-invariant basis.*

Remark 3.10. (1) We easily see that $J(\ker F_*) = \ker F_*$ implies $J((\ker F_*)^\perp) = (\ker F_*)^\perp$.

(2) Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an h-conformal semi-invariant submersion. Then it is not possible to get $R(\mathcal{D}_2) = (\ker F_*)^\perp$ for $R \in \{I, J, K\}$. If not, then $K(\mathcal{D}_2) = (\ker F_*)^\perp$ and $K(\mathcal{D}_2) = IJ(\mathcal{D}_2) = I((\ker F_*)^\perp) = \mathcal{D}_2$, contradiction!

So, our definition makes sense for this case. See Example 4.7.

Corollary 3.11. *Let F be an almost h-conformal anti-holomorphic semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal anti-holomorphic semi-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ is integrable.*
(b) $\mathcal{A}_{IV_1} IV_2 = \mathcal{A}_{IV_2} IV_1$ *for* $V_1, V_2 \in \Gamma(\mathcal{D}_2^I)$.
(c) $\mathcal{A}_{KV_1} KV_2 = \mathcal{A}_{KV_2} KV_1$ *for* $V_1, V_2 \in \Gamma(\mathcal{D}_2^K)$.

Proof. We see that $C_R = 0$, $B_R = R$ on $(\ker F_*)^\perp$ and $\omega_R = R$ on $\mathcal{D}_2^R$ for $R \in \{I, K\}$.

Applying $X = RV_1$ and $Y = RV_2$, $V_1, V_2 \in \Gamma(\mathcal{D}_2^R)$, at Theorem 3.7, we have

$$\mathcal{A}_{RV_1} RV_2 - \mathcal{A}_{RV_2} RV_1 \in \Gamma(\mathcal{D}_2^R)$$

and

$$0 = g_M(\mathcal{A}_{RV_2} RV_1 - \mathcal{A}_{RV_1} RV_2, V) \quad \text{for } V \in \Gamma(\mathcal{D}_2^R),$$

which are equivalent to

$$\mathcal{A}_{RV_1} RV_2 = \mathcal{A}_{RV_2} RV_1 \quad \text{for } V_1, V_2 \in \Gamma(\mathcal{D}_2^R).$$

Hence, we get $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$.

Therefore, we obtain the result. □

We consider the geometry of foliations and the condition for such maps to be horizontally homothetic throughout this section.

Theorem 3.12. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M.*

(b) $\mathcal{A}_X C_I Y + \mathcal{V}\nabla_X B_I Y \in \Gamma(\mathcal{D}_2^I)$ *and*

$$g_N(\nabla^F_X F_* IV, F_* C_I V) = \lambda^2 g_M(\mathcal{A}_X B_I Y - C_I Y(\ln\lambda)X + g_M(X, C_I Y)\nabla(\ln\lambda), IV)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^I)$.

(c) $\mathcal{A}_X C_J Y + \mathcal{V}\nabla_X B_J Y \in \Gamma(\mathcal{D}_2^J)$ *and*

$$g_N(\nabla^F_X F_* JV, F_* C_J V) = \lambda^2 g_M(\mathcal{A}_X B_J Y - C_J Y(\ln\lambda)X + g_M(X, C_J Y)\nabla(\ln\lambda), JV)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^J)$.

(d) $\mathcal{A}_X C_K Y + \mathcal{V}\nabla_X B_K Y \in \Gamma(\mathcal{D}_2^K)$ *and*

$$g_N(\nabla^F_X F_* KV, F_* C_K V) = \lambda^2 g_M(\mathcal{A}_X B_K Y - C_K Y(\ln\lambda)X + g_M(X, C_K Y)\nabla(\ln\lambda), KV)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^K)$.

Proof. Given $X, Y \in \Gamma((\ker F_*)^\perp)$, $W \in \Gamma(\mathcal{D}_1^R)$, and $R \in \{I, J, K\}$, we obtain

$$g_M(\nabla_X Y, W) = -g_M(\phi(\mathcal{A}_X C_R Y + \mathcal{V}\nabla_X B_R Y), W)$$

so that

$$g_M(\nabla_X Y, W) = 0 \Leftrightarrow \mathcal{A}_X C_R Y + \mathcal{V}\nabla_X B_R Y \in \Gamma(\mathcal{D}_2^R). \tag{3.15}$$

Given $V \in \Gamma(\mathcal{D}_2^R)$, by using (2.11) and (3.4), we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\mathcal{A}_X B_R Y, RV) - g_M(C_R Y, \nabla_X RV) \\ &= g_M(\mathcal{A}_X B_R Y, RV) + \frac{1}{\lambda^2} g_N(F_* C_R Y, RV(\ln\lambda)F_* X \\ &- g_M(X, RV)F_*\nabla(\ln\lambda) - \nabla^F_X F_* RV) \\ &= g_M(\mathcal{A}_X B_R Y + g_M(C_R Y, X)\nabla(\ln\lambda) - C_R Y(\ln\lambda)X, RV) \\ &- \frac{1}{\lambda^2} g_N(F_* C_R Y, \nabla^F_X F_* RV) \end{aligned}$$

so that

$$\begin{aligned} & g_M(\nabla_X Y, V) = 0 \\ & \Leftrightarrow g_N(F_* C_R Y, \nabla^F_X F_* RV) = \lambda^2 g_M(\mathcal{A}_X B_R Y \\ & + g_M(C_R Y, X)\nabla(\ln\lambda) - C_R Y(\ln\lambda)X, RV). \end{aligned} \tag{3.16}$$

By (3.15) and (3.16), we get $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

Therefore, the result follows. □

We introduce another notion on distributions and investigate it.

Definition 3.13. Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Given $R \in \{I, J, K\}$, we call the distribution $\mathcal{D}_2^R$ *parallel along* $(\ker F_*)^\perp$ if $\nabla_X V \in \Gamma(\mathcal{D}_2^R)$ for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\mathcal{D}_2^R)$.

Lemma 3.14. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Assume that the distribution $\mathcal{D}_2^R$ is parallel along $(\ker F_*)^\perp$ for $R \in \{I, J, K\}$. Then the following conditions are equivalent:*

(a) *The map F is horizontally homothetic.*

(b)

$$\lambda^2 g_M(\mathcal{A}_X B_I Y, IV) = g_N(\nabla^F_X F_* IV, F_* C_I Y)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^I)$.

(c)
$$\lambda^2 g_M(\mathcal{A}_X B_J Y, JV) = g_N(\nabla^F_X F_* JV, F_* C_J Y)$$
for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}^J_2)$.

(d)
$$\lambda^2 g_M(\mathcal{A}_X B_K Y, KV) = g_N(\nabla^F_X F_* KV, F_* C_K Y)$$
for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}^K_2)$.

Proof. Given $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(\mathcal{D}^R_2)$, and $R \in \{I, J, K\}$, by the proof of Theorem 3.12, we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\mathcal{A}_X B_R Y + g_M(C_R Y, X)\nabla(\ln\lambda) \\ &\quad - C_R Y(\ln\lambda)X, RV) - \frac{1}{\lambda^2} g_N(F_* C_R Y, \nabla^F_X F_* RV). \end{aligned} \tag{3.17}$$

Since $g_M(\nabla_X Y, V) = -g_M(Y, \nabla_X V) = 0$, from (3.17), we get $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$, $(a) \Rightarrow (d)$.

Conversely, from (3.17), we obtain

$$-g_M(C_R Y, \nabla(\ln\lambda)) g_M(X, RV) + g_M(X, C_R Y) g_M(\nabla(\ln\lambda), RV) = 0. \tag{3.18}$$

Applying $X = RV$ at (3.18), we have

$$-g_M(C_R Y, \nabla(\ln\lambda)) g_M(RV, RV) = 0,$$

which implies

$$g_M(X, \nabla(\lambda)) = 0 \quad \text{for } X \in \Gamma(\mu^R). \tag{3.19}$$

Applying $X = C_R Y$ at (3.18), we get

$$g_M(C_R Y, C_R Y) g_M(\nabla(\ln\lambda), RV) = 0,$$

which implies

$$g_M(\nabla(\lambda), RV) = 0 \quad \text{for } V \in \Gamma(\mathcal{D}^R_2). \tag{3.20}$$

Using (3.19) and (3.20), we obtain $(b) \Rightarrow (a)$, $(c) \Rightarrow (a)$, $(d) \Rightarrow (a)$.

Therefore, the result follows. □

Lemma 3.15. *Let F be an almost h-conformal anti-holomorphic semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal anti-holomorphic semi-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution* $(\ker F_*)^\perp$ *defines a totally geodesic foliation on* M.
(b) *The distribution* $\mathcal{D}^I_2$ *is parallel along* $(\ker F_*)^\perp$.
(c) *The distribution* $\mathcal{D}^K_2$ *is parallel along* $(\ker F_*)^\perp$.

Proof. We see that $B_R = R$ and $C_R = 0$ on $(\ker F_*)^\perp$ for $R \in \{I, K\}$.

Given $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\mathcal{D}^R_2)$, from Theorem 3.12, we have

$$\begin{aligned} (a) &\Leftrightarrow \mathcal{V}\nabla_X RY \in \Gamma(\mathcal{D}^R_2) \text{ and } g_M(\mathcal{A}_X RY, RV) = 0 \\ &\Leftrightarrow \nabla_X RY \in \Gamma(\mathcal{D}^R_2). \end{aligned}$$

Hence, we get $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$.

Therefore, we obtain the result. □

Theorem 3.16. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution* $\ker F_*$ *defines a totally geodesic foliation on* M.

(b) $\mathcal{T}_V \omega_I U + \widehat{\nabla}_V \phi_I U \in \Gamma(\mathcal{D}_1^I)$ *and*

$$g_N(\nabla^F_{\omega_I V} F_* X, F_* \omega_I U) = \lambda^2 g_M(C_I \mathcal{T}_U \phi_I V + \mathcal{A}_{\omega_I V} \phi_I U + g_M(\omega_I V, \omega_I U)\nabla(\ln\lambda), X)$$

for $U, V \in \Gamma(\ker F_*)$ *and* $X \in \Gamma(\mu^I)$.

(c) $\mathcal{T}_V \omega_J U + \widehat{\nabla}_V \phi_J U \in \Gamma(\mathcal{D}_1^J)$ *and*

$$g_N(\nabla^F_{\omega_J V} F_* X, F_* \omega_J U) = \lambda^2 g_M(C_J \mathcal{T}_U \phi_J V + \mathcal{A}_{\omega_J V} \phi_J U + g_M(\omega_J V, \omega_J U)\nabla(\ln\lambda), X)$$

for $U, V \in \Gamma(\ker F_*)$ *and* $X \in \Gamma(\mu^J)$.

(d) $\mathcal{T}_V \omega_K U + \widehat{\nabla}_V \phi_K U \in \Gamma(\mathcal{D}_1^K)$ *and*

$$g_N(\nabla^F_{\omega_K V} F_* X, F_* \omega_K U) = \lambda^2 g_M(C_K \mathcal{T}_U \phi_K V + \mathcal{A}_{\omega_K V} \phi_K U + g_M(\omega_K V, \omega_K U)\nabla(\ln\lambda), X)$$

for $U, V \in \Gamma(\ker F_*)$ *and* $X \in \Gamma(\mu^K)$.

Proof. Given $U, V \in \Gamma(\ker F_*)$, $W \in \Gamma(\mathcal{D}_2^R)$, and $R \in \{I, J, K\}$, by using (3.4), we have

$$g_M(\nabla_V U, RW) = -g_M(\omega_R(\widehat{\nabla}_V \phi_R U + \mathcal{T}_V \omega_R U), RW)$$

so that

$$g_M(\nabla_V U, RW) = 0 \Leftrightarrow \widehat{\nabla}_V \phi_R U + \mathcal{T}_V \omega_R U \in \Gamma(\mathcal{D}_1^R) \tag{3.21}$$

Given $X \in \Gamma(\mu^R)$, by using (2.8) and (3.3), we get

$$\begin{aligned}
& g_M(\nabla_U V, X) \\
&= g_M(\nabla_U \phi_R V, RX) + g_M(\phi_R U, \nabla_{\omega_R V} X) + g_M(\omega_R U, \nabla_{\omega_R V} X) \\
&= g_M(\mathcal{T}_U \phi_R V, RX) + g_M(\phi_R U, \mathcal{A}_{\omega_R V} X) \\
&\quad - \frac{1}{\lambda^2} g_M(\nabla(\ln\lambda), X) g_N(F_* \omega_R V, F_* \omega_R U) + \frac{1}{\lambda^2} g_N(\nabla^F_{\omega_R V} F_* X, F_* \omega_R U) \\
&= g_M(-C_R \mathcal{T}_U \phi_R V - \mathcal{A}_{\omega_R V} \phi_R U - g_M(\omega_R V, \omega_R U)\nabla(\ln\lambda), X) \\
&\quad + \frac{1}{\lambda^2} g_N(\nabla^F_{\omega_R V} F_* X, F_* \omega_R U)
\end{aligned}$$

so that

$$\begin{aligned}
& g_M(\nabla_U V, X) = 0 \\
&\quad \Leftrightarrow g_N(\nabla^F_{\omega_R V} F_* X, F_* \omega_R U) \\
&\qquad = \lambda^2 g_M(C_R \mathcal{T}_U \phi_R V + \mathcal{A}_{\omega_R V} \phi_R U + g_M(\omega_R V, \omega_R U)\nabla(\ln\lambda), X).
\end{aligned} \tag{3.22}$$

Using (3.21) and (3.22), we obtain $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

Therefore, the result follows. □

Definition 3.17. Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then given $R \in \{I, J, K\}$, we call the distribution μ^R *parallel along* $\ker F_*$ if $\nabla_U X \in \Gamma(\mu^R)$ for $X \in \Gamma(\mu^R)$ and $U \in \Gamma(\ker F_*)$.

Lemma 3.18. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Assume that the distribution μ^R is parallel along* $\ker F_*$ *for any $R \in \{I, J, K\}$.*

Then given $R \in \{I, J, K\}$, the following conditions are equivalent:

(a) *Dilation λ is constant on μ^R.*

(b)

$$g_N(\nabla^F_{\omega_R V} F_* X, F_* \omega_R U) = \lambda^2 g_M(C_R \mathcal{T}_U \phi_R V + \mathcal{A}_{\omega_R V} \phi_R U, X)$$

for $X \in \Gamma(\mu^R)$ *and* $U, V \in \Gamma(\ker F_*)$.

Proof. Given $X \in \Gamma(\mu^R)$ and $U, V \in \Gamma(\ker F_*)$, by using the proof of Theorem 3.16 and (3.4), we have

$$g_M(\nabla_U V, X) = g_M(-C_R \mathcal{T}_U \phi_R V - \mathcal{A}_{\omega_R V} \phi_R U - g_M(\omega_R V, \omega_R U)\nabla(\ln\lambda), X) + \frac{1}{\lambda^2} g_N(\nabla^F_{\omega_R V} F_* X, F_* \omega_R U)$$

so that since $g_M(\nabla_U V, X) = -g_M(V, \nabla_U X) = 0$, it is easy to get $(a) \Leftrightarrow (b)$. □

Denote by $M_{\ker F_*}$ and $M_{(\ker F_*)^\perp}$ the integral manifolds of the distributions $\ker F_*$ and $(\ker F_*)^\perp$, respectively.

Using Theorem 3.12 and Theorem 3.16, we have

Theorem 3.19. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *M is locally a product Riemannian manifold $M_{\ker F_*} \times M_{(\ker F_*)^\perp}$.*

(b) $\mathcal{A}_X C_I Y + \mathcal{V}\nabla_X B_I Y \in \Gamma(\mathcal{D}_2^I)$,
$g_N(\nabla^F_X F_* IV, F_* C_I V) = \lambda^2 g_M(\mathcal{A}_X B_I Y - C_I Y(\ln\lambda)X + g_M(X, C_I Y)\nabla(\ln\lambda), IV)$
for $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(\mathcal{D}_2^I)$.
$\mathcal{T}_V \omega_I U + \widehat{\nabla}_V \phi_I U \in \Gamma(\mathcal{D}_1^I)$,
$g_N(\nabla^F_{\omega_I V} F_* X, F_* \omega_I U) = \lambda^2 g_M(C_I \mathcal{T}_U \phi_I V + \mathcal{A}_{\omega_I V} \phi_I U + g_M(\omega_I V, \omega_I U)\nabla(\ln\lambda), X)$
for $U, V \in \Gamma(\ker F_*)$, $X \in \Gamma(\mu^I)$.

(c) $\mathcal{A}_X C_J Y + \mathcal{V}\nabla_X B_J Y \in \Gamma(\mathcal{D}_2^J)$,
$g_N(\nabla^F_X F_* JV, F_* C_J V) = \lambda^2 g_M(\mathcal{A}_X B_J Y - C_J Y(\ln\lambda)X + g_M(X, C_J Y)\nabla(\ln\lambda), JV)$
for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^J)$.
$\mathcal{T}_V \omega_J U + \widehat{\nabla}_V \phi_J U \in \Gamma(\mathcal{D}_1^J)$,
$g_N(\nabla^F_{\omega_J V} F_* X, F_* \omega_J U) = \lambda^2 g_M(C_J \mathcal{T}_U \phi_J V + \mathcal{A}_{\omega_J V} \phi_J U + g_M(\omega_J V, \omega_J U)\nabla(\ln\lambda), X)$
for $U, V \in \Gamma(\ker F_*)$ *and* $X \in \Gamma(\mu^J)$.

(d) $\mathcal{A}_X C_K Y + \mathcal{V}\nabla_X B_K Y \in \Gamma(\mathcal{D}_2^K)$,
$g_N(\nabla^F_X F_* KV, F_* C_K V) = \lambda^2 g_M(\mathcal{A}_X B_K Y - C_K Y(\ln\lambda)X + g_M(X, C_K Y)\nabla(\ln\lambda), KV)$
for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\mathcal{D}_2^K)$.
$\mathcal{T}_V \omega_K U + \widehat{\nabla}_V \phi_K U \in \Gamma(\mathcal{D}_1^K)$,
$g_N(\nabla^F_{\omega_K V} F_* X, F_* \omega_K U) = \lambda^2 g_M(C_K \mathcal{T}_U \phi_K V + \mathcal{A}_{\omega_K V} \phi_K U + g_M(\omega_K V, \omega_K U)\nabla(\ln\lambda), X)$
for $U, V \in \Gamma(\ker F_*)$ *and* $X \in \Gamma(\mu^K)$.

Theorem 3.20. *Let F be an h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $\mathcal{D}_1$ defines a totally geodesic foliation on M.*

(b)
$$(\nabla F_*)(V, IW) \in \Gamma(F_* \mu^I),$$
$$g_N((\nabla F_*)(V, IW), F_* C_I X) = \lambda^2 g_M(W, \mathcal{T}_V \omega_I B_I X)$$
for $V, W \in \Gamma(\mathcal{D}_1)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

(c)
$$(\nabla F_*)(V, JW) \in \Gamma(F_* \mu^J),$$
$$g_N((\nabla F_*)(V, JW), F_* C_J X) = \lambda^2 g_M(W, \mathcal{T}_V \omega_J B_J X)$$
for $V, W \in \Gamma(\mathcal{D}_1)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

(d)

$$(\nabla F_*)(V, KW) \in \Gamma(F_*\mu^K),$$
$$g_N((\nabla F_*)(V, KW), F_*C_KX) = \lambda^2 g_M(W, \mathcal{T}_V\omega_K B_K X)$$

for $V, W \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $U, V \in \Gamma(\mathcal{D}_1)$, $W \in \Gamma(\mathcal{D}_2)$, and $R \in \{I, J, K\}$, we get

$$\begin{aligned} g_M(\nabla_V U, W) &= g_M(\mathcal{H}\nabla_V RU, RW) \\ &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(V, RU), F_*RW) \end{aligned}$$

so that

$$g_M(\nabla_V U, W) = 0 \Leftrightarrow (\nabla F_*)(V, RU) \in \Gamma(F_*\mu^R). \tag{3.23}$$

Given $X \in \Gamma((\ker F_*)^\perp)$, we obtain

$$\begin{aligned} g_M(\nabla_V U, X) &= g_M(U, \nabla_V RB_R X) + g_M(\mathcal{H}\nabla_V RU, C_R X) \\ &= g_M(U, \mathcal{T}_V\omega_R B_R X) - \frac{1}{\lambda^2} g_N((\nabla F_*)(V, RU), F_*C_R X) \end{aligned}$$

so that

$$g_M(\nabla_V U, X) = 0 \Leftrightarrow g_N((\nabla F_*)(V, RU), F_*C_R X) = \lambda^2 g_M(U, \mathcal{T}_V\omega_R B_R X). \tag{3.24}$$

Using (3.23) and (3.24), we have $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

Therefore, we obtain the result. □

Theorem 3.21. *Let F be an h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $\mathcal{D}_2$ defines a totally geodesic foliation on M.*

(b) $(\nabla F_*)(V, IW) \in \Gamma(F_*\mu^I)$,

$$\begin{aligned} -\frac{1}{\lambda^2} g_N(\nabla^F_{IV} F_*IU, F_*IC_I X) &= g_M(V, B_I\mathcal{T}_U B_I X) \\ &\quad + g_M(U, V) g_M(\mathcal{H}\nabla(\ln\lambda), IC_I X) \end{aligned}$$

for $U, V \in \Gamma(\mathcal{D}_2)$, $W \in \Gamma(\mathcal{D}_1)$, and $X \in \Gamma((\ker F_*)^\perp)$.

(c) $(\nabla F_*)(V, JW) \in \Gamma(F_*\mu^J)$,

$$\begin{aligned} -\frac{1}{\lambda^2} g_N(\nabla^F_{JV} F_*JU, F_*JC_J X) &= g_M(V, B_J\mathcal{T}_U B_J X) \\ &\quad + g_M(U, V) g_M(\mathcal{H}\nabla(\ln\lambda), JC_J X) \end{aligned}$$

for $U, V \in \Gamma(\mathcal{D}_2)$, $W \in \Gamma(\mathcal{D}_1)$, and $X \in \Gamma((\ker F_*)^\perp)$.

(d) $(\nabla F_*)(V, KW) \in \Gamma(F_*\mu^K)$,

$$\begin{aligned} -\frac{1}{\lambda^2} g_N(\nabla^F_{KV} F_*KU, F_*KC_K X) &= g_M(V, B_K\mathcal{T}_U B_K X) \\ &\quad + g_M(U, V) g_M(\mathcal{H}\nabla(\ln\lambda), KC_K X) \end{aligned}$$

for $U, V \in \Gamma(\mathcal{D}_2)$, $W \in \Gamma(\mathcal{D}_1)$, and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $U, V \in \Gamma(\mathcal{D}_2)$, $W \in \Gamma(\mathcal{D}_1)$, $R \in \{I, J, K\}$, we get

$$g_M(\nabla_U V, W) = \frac{1}{\lambda^2} g_N((\nabla F_*)(U, RW), F_*RV)$$

so that

$$g_M(\nabla_U V, W) = 0 \Leftrightarrow (\nabla F_*)(U, RW) \in \Gamma(F_*\mu^R). \tag{3.25}$$

Given $X \in \Gamma((\ker F_*)^{\perp})$, by using (2.8), (2.11), (3.4), we obtain

$$\begin{aligned} g_M(\nabla_U V, X) &= -g_M(RV, \mathcal{T}_U B_R X) + g_M(\nabla_{RV} U, C_R X) \\ &= -g_M(RV, \mathcal{T}_U B_R X) + g_M(\nabla_{RV} RU, RC_R X) \\ &= g_M(V, B_R \mathcal{T}_U B_R X) + g_M(U, V) g_M(\mathcal{H}\nabla(\ln \lambda), RC_R X) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^F_{RV} F_* RU, F_* RC_R X) \end{aligned}$$

so that

$$\begin{aligned} & g_M(\nabla_U V, X) = 0 \\ & \Leftrightarrow -\frac{1}{\lambda^2} g_N(\nabla^F_{RV} F_* RU, F_* RC_R X) \\ & = g_M(V, B_R \mathcal{T}_U B_R X) + g_M(U, V) g_M(\mathcal{H}\nabla(\ln \lambda), RC_R X). \end{aligned} \tag{3.26}$$

Using (3.25) and (3.26), we have $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.
Therefore, the result follows. □

Using Theorem 3.20 and Theorem 3.21, we obtain

Theorem 3.22. *Let F be an h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The fibers of F are locally product Riemannian manifolds $M_{\mathcal{D}_1} \times M_{\mathcal{D}_2}$.*

(b)

$$\begin{aligned} & (\nabla F_*)(V, IW) \in \Gamma(F_* \mu^I), \\ & g_N((\nabla F_*)(V, IW), F_* C_I X) = \lambda^2 g_M(W, \mathcal{T}_V \omega_I B_I X) \end{aligned}$$

for $V, W \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma((\ker F_*)^{\perp})$.
$(\nabla F_*)(V, IW) \in \Gamma(F_* \mu^I)$,

$$\begin{aligned} -\frac{1}{\lambda^2} g_N(\nabla^F_{IV} F_* IU, F_* IC_I X) &= g_M(V, B_I \mathcal{T}_U B_I X) \\ &\quad + g_M(U, V) g_M(\mathcal{H}\nabla(\ln \lambda), IC_I X) \end{aligned}$$

for $U, V \in \Gamma(\mathcal{D}_2)$, $W \in \Gamma(\mathcal{D}_1)$, and $X \in \Gamma((\ker F_*)^{\perp})$.

(c)

$$\begin{aligned} & (\nabla F_*)(V, JW) \in \Gamma(F_* \mu^J), \\ & g_N((\nabla F_*)(V, JW), F_* C_J X) = \lambda^2 g_M(W, \mathcal{T}_V \omega_J B_J X) \end{aligned}$$

for $V, W \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma((\ker F_*)^{\perp})$.
$(\nabla F_*)(V, JW) \in \Gamma(F_* \mu^J)$,

$$\begin{aligned} -\frac{1}{\lambda^2} g_N(\nabla^F_{JV} F_* JU, F_* JC_J X) &= g_M(V, B_J \mathcal{T}_U B_J X) \\ &\quad + g_M(U, V) g_M(\mathcal{H}\nabla(\ln \lambda), JC_J X) \end{aligned}$$

for $U, V \in \Gamma(\mathcal{D}_2)$, $W \in \Gamma(\mathcal{D}_1)$, and $X \in \Gamma((\ker F_*)^{\perp})$.

(d)

$$\begin{aligned} & (\nabla F_*)(V, KW) \in \Gamma(F_* \mu^K), \\ & g_N((\nabla F_*)(V, KW), F_* C_K X) = \lambda^2 g_M(W, \mathcal{T}_V \omega_K B_K X) \end{aligned}$$

for $V, W \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma((\ker F_*)^{\perp})$.

$$(\nabla F_*)(V, KW) \in \Gamma(F_*\mu^K),$$

$$-\frac{1}{\lambda^2} g_N(\nabla^F_{KV} F_*KU, F_*KC_KX) = g_M(V, B_K\mathcal{T}_U B_K X) + g_M(U,V)g_M(\mathcal{H}\nabla(\ln\lambda), KC_KX)$$

for $U, V \in \Gamma(\mathcal{D}_2)$, $W \in \Gamma(\mathcal{D}_1)$, *and* $X \in \Gamma((\ker F_*)^\perp)$.

We know

Lemma 3.23 ([4]). *Let F be a horizontally conformal submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) with dilation λ.*

Then the tension field $\tau(F)$ of F is given by

$$\tau(F) = -mF_*H + (2-n)F_*(\nabla(\ln\lambda)), \tag{3.27}$$

where H is the mean curvature vector field of the distribution $\ker F_*$, $m = \dim\ker F_*$, $n = \dim N$.

Using Lemma 3.23, we easily get

Corollary 3.24. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Assume that F is harmonic with* $\dim\ker F_* > 0$ *and* $\dim N > 2$. *Then the following conditions are equivalent:*

(a) *All the fibers of F are minimal.*
(b) *The map F is horizontally homothetic.*

Corollary 3.25. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Assume that* $\dim\ker F_* > 0$ *and* $\dim N = 2$. *Then the following conditions are equivalent:*

(a) *All the fibers of F are minimal.*
(b) *The map F is harmonic.*

We introduce another notion and investigate the condition for such a map to be totally geodesic.

Definition 3.26. Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then given $R \in \{I, J, K\}$, we call the map F a *$(R\mathcal{D}_2^R, \mu^R)$-totally geodesic map* if $(\nabla F_*)(RU, X) = 0$ for $U \in \Gamma(\mathcal{D}_2^R)$ and $X \in \Gamma(\mu^R)$.

Theorem 3.27. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The map F is horizontally homothetic.*
(b) *The map F is a $(I\mathcal{D}_2^I, \mu^I)$-totally geodesic map.*
(c) *The map F is a $(J\mathcal{D}_2^J, \mu^J)$-totally geodesic map.*
(d) *The map F is a $(K\mathcal{D}_2^K, \mu^K)$-totally geodesic map.*

Proof. Given $U \in \Gamma(\mathcal{D}_2^R)$, $X \in \Gamma(\mu^R)$, and $R \in \{I, J, K\}$, we have

$$\begin{aligned}&(\nabla F_*)(RU, X) \\ &= RU(\ln\lambda)F_*X + X(\ln\lambda)F_*RU - g_M(RU, X)F_*\nabla(\ln\lambda) \\ &= RU(\ln\lambda)F_*X + X(\ln\lambda)F_*RU\end{aligned} \tag{3.28}$$

so that we easily get $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$, $(a) \Rightarrow (d)$.

Conversely, from (3.28), we obtain

$$RU(\ln\lambda)F_*X + X(\ln\lambda)F_*RU = 0.$$

Since $\{F_*X, F_*RU\}$ is linearly independent for nonzero X, U, we have $RU(\ln\lambda) = 0$ and $X(\ln\lambda) = 0$, which means $(a) \Leftarrow (b)$, $(a) \Leftarrow (c)$, $(a) \Leftarrow (d)$.

Therefore, the result follows. □

Theorem 3.28. *Let F be an almost h-conformal semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:*

(a) *The map F is totally geodesic.*

(b) (i) $C_I\mathcal{T}_U IV + \omega_I\widehat{\nabla}_U IV = 0$ *for* $U, V \in \Gamma(\mathcal{D}_1^I)$.
(ii) $C_I\mathcal{H}\nabla_U IW + \omega_I\mathcal{T}_U IW = 0$ *for* $U \in \Gamma(\ker F_*)$ *and* $W \in \Gamma(\mathcal{D}_2^I)$.
(iii) *The map F is horizontally homothetic.*
(iv) $\mathcal{T}_U B_I X + \mathcal{H}\nabla_U C_I X \in \Gamma(I\mathcal{D}_2^I)$ *and* $\widehat{\nabla}_U B_I X + \mathcal{T}_U C_I X \in \Gamma(\mathcal{D}_1^I)$ *for* $U \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

(c) (i) $C_J\mathcal{T}_U JV + \omega_J\widehat{\nabla}_U JV = 0$ *for* $U, V \in \Gamma(\mathcal{D}_1^J)$.
(ii) $C_J\mathcal{H}\nabla_U JW + \omega_J\mathcal{T}_U JW = 0$ *for* $U \in \Gamma(\ker F_*)$ *and* $W \in \Gamma(\mathcal{D}_2^J)$.
(iii) *The map F is horizontally homothetic.*
(iv) $\mathcal{T}_U B_J X + \mathcal{H}\nabla_U C_J X \in \Gamma(J\mathcal{D}_2^J)$ *and* $\widehat{\nabla}_U B_J X + \mathcal{T}_U C_J X \in \Gamma(\mathcal{D}_1^J)$ *for* $U \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

(d) (i) $C_K\mathcal{T}_U KV + \omega_K\widehat{\nabla}_U KV = 0$ *for* $U, V \in \Gamma(\mathcal{D}_1^K)$.
(ii) $C_K\mathcal{H}\nabla_U KW + \omega_K\mathcal{T}_U KW = 0$ *for* $U \in \Gamma(\ker F_*)$ *and* $W \in \Gamma(\mathcal{D}_2^K)$.
(iii) *The map F is horizontally homothetic.*
(iv) $\mathcal{T}_U B_K X + \mathcal{H}\nabla_U C_K X \in \Gamma(K\mathcal{D}_2^K)$ *and* $\widehat{\nabla}_U B_K X + \mathcal{T}_U C_K X \in \Gamma(\mathcal{D}_1^K)$ *for* $U \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $U, V \in \Gamma(\mathcal{D}_1^R)$ and $R \in \{I, J, K\}$, we have

$$\begin{aligned}(\nabla F_*)(U, V) &= F_*(R(\mathcal{T}_U RV + \widehat{\nabla}_U RV)) \\ &= F_*(B_R\mathcal{T}_U RV + C_R\mathcal{T}_U RV + \phi_R\widehat{\nabla}_U RV + \omega_R\widehat{\nabla}_U RV) \\ &= F_*(C_R\mathcal{T}_U RV + \omega_R\widehat{\nabla}_U RV)\end{aligned}$$

so that

$$(\nabla F_*)(U, V) = 0 \Leftrightarrow C_R\mathcal{T}_U RV + \omega_R\widehat{\nabla}_U RV = 0. \tag{3.29}$$

Given $U \in \Gamma(\ker F_*)$ and $W \in \Gamma(\mathcal{D}_2^R)$, we get

$$\begin{aligned}(\nabla F_*)(U, W) &= F_*(R(\nabla_U RW)) \\ &= F_*(R(\mathcal{T}_U RW + \mathcal{H}\nabla_U RW)) \\ &= F_*(C_R\mathcal{H}\nabla_U RW + \omega_R\mathcal{T}_U RW)\end{aligned}$$

so that

$$(\nabla F_*)(U, W) = 0 \Leftrightarrow C_R\mathcal{H}\nabla_U RW + \omega_R\mathcal{T}_U RW = 0. \tag{3.30}$$

We claim that

$$\begin{aligned}&(\nabla F_*)(X, Y) = 0 \quad \text{for } X, Y \in \Gamma((\ker F_*)^\perp) \\ &\Leftrightarrow F \text{ is horizontally homothetic.}\end{aligned} \tag{3.31}$$

Given $X, Y \in \Gamma((\ker F_*)^\perp)$, by (2.11), we obtain

$$(\nabla F_*)(X, Y) = X(\ln\lambda)F_*Y + Y(\ln\lambda)F_*X - g_M(X, Y)F_*\nabla(\ln\lambda) \tag{3.32}$$

so that the part from right to left immediately follows.

Conversely, we have

$$0 = X(\ln\lambda)F_*Y + Y(\ln\lambda)F_*X - g_M(X, Y)F_*\nabla(\ln\lambda). \tag{3.33}$$

Applying $Y = RX$, $X \in \Gamma(\mu^R)$, at (3.33), we get

$$\begin{aligned} 0 &= X(\ln\lambda)F_*RX + RX(\ln\lambda)F_*X - g_M(X, RX)F_*\nabla(\ln\lambda) \\ &= X(\ln\lambda)F_*RX + RX(\ln\lambda)F_*X \end{aligned}$$

so that since $\{F_*RX, F_*X\}$ is linearly independent for nonzero X, we obtain $X(\ln\lambda) = 0$ and $RX(\ln\lambda) = 0$, which implies

$$X(\lambda) = 0 \quad \text{for } X \in \Gamma(\mu^R). \tag{3.34}$$

Applying $X = Y = RU$, $U \in \Gamma(\mathcal{D}_2^R)$, at (3.33), we obtain

$$0 = 2RU(\ln\lambda)F_*RU - g_M(RU, RU)F_*\nabla(\ln\lambda). \tag{3.35}$$

Taking inner product with F_*RU at (3.35), we have

$$\begin{aligned} 0 &= 2g_M(RU, \nabla(\ln\lambda))g_N(F_*RU, F_*RU) - g_M(RU, RU)g_N(F_*\nabla(\ln\lambda), F_*RU) \\ &= \lambda g_M(RU, RU)g_M(RU, \nabla(\ln\lambda)), \end{aligned}$$

which implies

$$RU(\lambda) = 0 \quad \text{for } U \in \Gamma(\mathcal{D}_2^R). \tag{3.36}$$

By (3.34) and (3.36), we get the part from left to right.

Given $U \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we obtain

$$\begin{aligned} (\nabla F_*)(U, X) &= F_*(R(\nabla_U RX)) \\ &= F_*(R(\mathcal{T}_U B_R X + \widehat{\nabla}_U B_R X) + R(\mathcal{T}_U C_R X + \mathcal{H}\nabla_U C_R X)) \\ &= F_*(C_R(\mathcal{T}_U B_R X + \mathcal{H}\nabla_U C_R X) + \omega_R(\widehat{\nabla}_U B_R X + \mathcal{T}_U C_R X)) \end{aligned}$$

so that

$$(\nabla F_*)(U, X) = 0 \Leftrightarrow \quad \begin{aligned} &\mathcal{T}_U B_R X + \mathcal{H}\nabla_U C_R X \in \Gamma(R\mathcal{D}_2^R), \\ &\widehat{\nabla}_U B_R X + \mathcal{T}_U C_R X \in \Gamma(R\mathcal{D}_1^R) \end{aligned} \tag{3.37}$$

By (3.29), (3.30), (3.31), (3.37), we have $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

Therefore, we get the result. □

Let $F : (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion. The map F is called a horizontally conformal submersion *with totally umbilical fibers* if

$$\mathcal{T}_X Y = g_M(X, Y)H \quad \text{for } X, Y \in \Gamma(\ker F_*), \tag{3.38}$$

where H is the mean curvature vector field of the distribution $\ker F_*$.

Lemma 3.29. *Let F be an almost h-conformal semi-invariant submersion with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-conformal semi-invariant basis. Then*

$$H \in \Gamma(R\mathcal{D}_2^R) \quad \textit{for } R \in \{I, J, K\}. \tag{3.39}$$

Proof. Given $X, Y \in \Gamma(\mathcal{D}_1^R)$, $W \in \Gamma(\mu^R)$, and $R \in \{I, J, K\}$, we have

$$\begin{aligned} \mathcal{T}_X RY + \widehat{\nabla}_X RY &= \nabla_X RY = R\nabla_X Y \\ &= B_R\mathcal{T}_X Y + C_R\mathcal{T}_X Y + \phi_R\widehat{\nabla}_X Y + \omega_R\widehat{\nabla}_X Y \end{aligned}$$

so that

$$g_M(\mathcal{T}_X RY, W) = g_M(C_R\mathcal{T}_X Y, W) = -g_M(\mathcal{T}_X Y, RW).$$

Using (3.38), we obtain

$$g_M(X, RY)g_M(H, W) = -g_M(X, Y)g_M(H, RW).$$

Interchanging the role of X and Y, we get

$$g_M(Y,RX)g_M(H,W) = -g_M(Y,X)g_M(H,RW).$$

Combining the above two equations, we have

$$g_M(X,Y)g_M(H,RW) = 0,$$

which implies $H \in \Gamma(R\mathcal{D}_2^R)$ (since $R\mu^R = \mu^R$). □

Theorem 3.30. *Let F be an h-conformal semi-invariant submersion with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal semi-invariant basis. Then all the fibers of F are totally geodesic.*

Proof. By Lemma 3.29, we have

$$H \in \Gamma(R\mathcal{D}_2) \quad \text{for } R \in \{I, J, K\}$$

so that

$$\{IH, JH, KH\} \subset \Gamma(\mathcal{D}_2).$$

But

$$KH = IJH = I(JH) \in \Gamma(\mathcal{D}_2) \quad \text{with } JH \in \Gamma(\mathcal{D}_2).$$

Since $I\mathcal{D}_2 \subset (\ker F_*)^\perp$, we must have $H = 0$. By (3.38), we obtain the result. □

4. Examples

Note that given a Euclidean space $\mathbb{R}^{4m}$ with coordinates $(x_1, x_2, \cdots, x_{4m})$, we can canonically choose complex structures I, J, K on $\mathbb{R}^{4m}$ as follows:

$$I(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+2}}, I(\frac{\partial}{\partial x_{4k+2}}) = -\frac{\partial}{\partial x_{4k+1}}, I(\frac{\partial}{\partial x_{4k+3}}) = \frac{\partial}{\partial x_{4k+4}}, I(\frac{\partial}{\partial x_{4k+4}}) = -\frac{\partial}{\partial x_{4k+3}},$$
$$J(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+3}}, J(\frac{\partial}{\partial x_{4k+2}}) = -\frac{\partial}{\partial x_{4k+4}}, J(\frac{\partial}{\partial x_{4k+3}}) = -\frac{\partial}{\partial x_{4k+1}}, J(\frac{\partial}{\partial x_{4k+4}}) = \frac{\partial}{\partial x_{4k+2}},$$
$$K(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+4}}, K(\frac{\partial}{\partial x_{4k+2}}) = \frac{\partial}{\partial x_{4k+3}}, K(\frac{\partial}{\partial x_{4k+3}}) = -\frac{\partial}{\partial x_{4k+2}}, K(\frac{\partial}{\partial x_{4k+4}}) = -\frac{\partial}{\partial x_{4k+1}}$$

for $k \in \{0, 1, \cdots, m-1\}$.

Then we easily check that $(I, J, K, \langle\ ,\ \rangle)$ is a hyperkähler structure on $\mathbb{R}^{4m}$, where $\langle\ ,\ \rangle$ denotes the Euclidean metric on $\mathbb{R}^{4m}$. Throughout this section, we will use these notations.

Example 4.1. Let (M, E, g) be an almost quaternionic Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection [15]. Then the map π is an h-conformal semi-invariant submersion such that $\mathcal{D}_1 = \ker \pi_*$ and dilation $\lambda = 1$.

Example 4.2. Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. Let $F : M \mapsto N$ be a quaternionic submersion [15]. Then the map F is an h-conformal semi-invariant submersion such that $\mathcal{D}_1 = \ker F_*$ and dilation $\lambda = 1$.

Example 4.3. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an h-semi-invariant submersion [21]. Then the map F is an h-conformal semi-invariant submersion with dilation $\lambda = 1$.

Example 4.4. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-semi-invariant submersion [21]. Then the map F is an almost h-conformal semi-invariant submersion with dilation $\lambda = 1$.

Example 4.5. Let (M, E, g_M) be a $4n$-dimensional almost quaternionic Hermitian manifold and (N, g_N) a $(4n-1)$-dimensional Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation λ. Then the map F is an h-conformal semi-invariant submersion such that $\mathcal{D}_2 = \ker F_*$ and dilation λ.

Example 4.6. Let $F : \mathbb{R}^4 \mapsto \mathbb{R}^3$ be a horizontally conformal submersion with dilation λ. Then the map F is an h-conformal semi-invariant submersion such that $\mathcal{D}_2 = \ker F_*$ and dilation λ.

Example 4.7. Define a map $F : \mathbb{R}^4 \mapsto \mathbb{R}^2$ by

$$F(x_1, \cdots, x_4) = e^{1934}(x_1, x_2).$$

Then the map F is an almost h-conformal anti-holomorphic semi-invariant submersion such that $I(\ker F_*) = \ker F_*$, $J(\ker F_*) = (\ker F_*)^\perp$, $K(\ker F_*) = (\ker F_*)^\perp$, and dilation $\lambda = e^{1934}$.

Here, (K, I, J) is an almost h-conformal anti-holomorphic semi-invariant basis.

Example 4.8. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^6$ by

$$F(x_1, \cdots, x_8) = \pi^{1934}(x_3, \cdots, x_8).$$

Then the map F is an almost h-conformal semi-invariant submersion such that $I(\ker F_*) = \ker F_*$, $J(\ker F_*) \subset (\ker F_*)^\perp$, $K(\ker F_*) \subset (\ker F_*)^\perp$, and dilation $\lambda = \pi^{1934}$.

Example 4.9. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, \cdots, x_8) = e^{1968}(x_1, x_2, x_5, x_7).$$

Then the map F is an almost h-conformal semi-invariant submersion such that $\mathcal{D}_1^I = \mathcal{D}_2^J =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$, $\mathcal{D}_2^I = \mathcal{D}_1^J =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} >$, $K(\ker F_*) = (\ker F_*)^\perp$, and dilation $\lambda = e^{1968}$.

Example 4.10. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^3$ by

$$F(x_1, \cdots, x_8) = \pi^{1978}(x_6, x_7, x_8).$$

Then the map F is a h-conformal semi-invariant submersion such that $\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_4} >$, $\mathcal{D}_2 =< \frac{\partial}{\partial x_5} >$, and dilation $\lambda = \pi^{1978}$.

References

[1] M.A. Akyol and B. Sahin, *Conformal anti-invariant submersions from almost Hermitian manifolds*, Turkish J. Math. **40** (1), 43–70, 2016.

[2] M.A. Akyol and B. Sahin, *Conformal semi-invariant submersions*, Commun. Contemp. Math. **19** (2), 1650011, 2017.

[3] D.V. Alekseevsky and S. Marchiafava, *Almost complex submanifolds of quaternionic manifolds*, In: Proceedings of the colloquium on differential geometry, Debrecen (Hungary), 25-30 July 2000, Inst. Math. Inform. Debrecen, 23-38, 2001.

[4] P. Baird and J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, Oxford Science Publications, 2003.

[5] M. Barros, B.Y. Chen and F. Urbano, *Quaternion CR-submanifolds of quaternion manifolds*, Kodai Math. J. **4**, 399–417, 1980.

[6] A.L. Besse, *Einstein manifolds*, Springer Verlag, Berlin, 1987.

[7] J.P. Bourguignon and H.B. Lawson, *Stability and isolation phenomena for Yang-mills fields*, Commum. Math. Phys. **79**, 189–230, 1981.

[8] J.P. Bourguignon and H.B. Lawson, *A mathematician's visit to Kaluza-Klein theory*, Rend. Semin. Mat. Torino Fasc. Spec. **1989**, 143–163, 1989.

[9] B.Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Leuven, 1990.

[10] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry 1. Vector multiplets*, J. High Energy Phys. **3**, 028, 2004.

[11] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16**, 715–737, 1967.

[12] S. Gudmundsson, *The geometry of harmonic morphisms*, Ph.D. thesis, University of Leeds, 1992.

[13] M. Falcitelli, S. Ianus and A.M. Pastore, *Riemannian submersions and related topics*, World Scientific Publishing Co., 2004.

[14] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble), **28** (2), 107–144, 1978.

[15] S. Ianus, R. Mazzocco and G.E. Vilcu, *Riemannian submersions from quaternionic manifolds*, Acta. Appl. Math. **104**, 83–89, 2008.

[16] S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalized Hopf manifolds*, Class. Quantum Gravity, **4**, 1317–1325, 1987.

[17] S. Ianus and M. Visinescu, *Space-time compactification and Riemannian submersions*, In: Rassias, G.(ed.) The Mathematical Heritage of C. F. Gauss, 358-371, World Scientific, River Edge, 1991.

[18] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto University, **19** (2), 215–229, 1979.

[19] M.T. Mustafa, *Applications of harmonic morphisms to gravity*, J. Math. Phys. **41** (10), 6918–6929, 2000.

[20] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13**, 458–469, 1966.

[21] K.S. Park, *H-semi-invariant submersions*, Taiwanese J. Math. **16** (5), 1865–1878, 2012.

[22] K.S. Park, *H-anti-invariant submersions from almost quaternionic Hermitian manifolds*, Czechoslovak Math. J. **67** (142), 557–578, 2017.

[23] K.S. Park and R. Prasad, *Semi-slant submersions*, Bull. Korean Math. Soc. **50** (3), 951–962, 2013.

[24] B. Sahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds*, Cent. Eur. J. Math. **8** (3), 437–447, 2010.

[25] B. Sahin, *Slant submersions from almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie Tome, **54(102)** (1), 93–105, 2011.

[26] B. Sahin, *Semi-invariant submersions from almost Hermitian manifolds*, Canad. Math. Bull. **56** (1), 173–183, 2013.

[27] B. Sahin, *Riemannian submersions from almost Hermitian manifolds*, Taiwanese J. Math. **17** (2), 629–659, 2013.

[28] H. Urakawa, *Calculus of variations and harmonic maps*, Translations of Mathematical Monographs, Amer. Math. Soc. 2013.

[29] B. Watson, *Almost Hermitian submersions*, J. Differential Geometry, **11** (1), 147–165, 1976.

[30] B. Watson, G, G'*-Riemannian submersions and nonlinear gauge field equations of general relativity*, in: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., **57**, 324–349, Teubner, Leipzig, 1983.