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Research Article

On abstract generalized topological spaces generated by the density type operators

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Abstract

In the paper we concentrate on a generalized topological space generated by a density type operator on a measurable space. The properties of such generalized topological space are investigated. Moreover, the properties of nowhere dense sets, meager sets and compact sets in this generalized topological space are studied.

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1. Introduction

Let X be a non-empty set, $\mathcal{S}$ be an algebra of subsets of X (i.e. the empty set belongs to $\mathcal{S}$ and $\mathcal{S}$ is closed under the finite unions of sets and the complements of sets) and $\mathcal{J} \subset \mathcal{S}$ be a ideal of subsets of X (i.e. if $A \in \mathcal{J}$ and $B \subset A$ then $B \in \mathcal{J}$ and $\mathcal{J}$ is closed under the finite unions of sets). We will focus on a measurable space i.e. a triple $\langle X, \mathcal{S}, \mathcal{J} \rangle$, where $\mathcal{J} \subset \mathcal{S}$ is a proper ideal of sets such that all singletons belong to $\mathcal{J}$. Moreover, if it is necessary, we will assume that $\mathcal{J}$ is the σ-ideal of sets it means $\mathcal{J}$ is additionally closed under the countable unions of sets. The density type operators defined on some families of subsets of this space will also play a special role in our considerations.

The family of all subsets of a non-empty set X will be denoted by 2^X. For any $A, B \in 2^X$ the symbol $A \triangle B$ will stand for the set $(A \setminus B) \cup (B \setminus A)$. Moreover, for any measurable space $\langle X, \mathcal{S}, \mathcal{J} \rangle$ and $A, B \subset X$ we will write $A \sim B$ iff $A \triangle B \in \mathcal{J}$.

Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space. A measurable hull of a set $A \subset X$ is any set $B \in \mathcal{S}$ such that $A \subset B$ and for any $C \subset B \setminus A$ if $C \in \mathcal{S}$ then $C \in \mathcal{J}$. The set B described above is called an $\mathcal{S}$-measurable hull of a set A. We will write it simply "a measurable hull of A" when no confusion can arise. The family of all measurable hulls of a set $A \subset X$ will be denoted by $\mathcal{H}(A)$. We shall say that $\langle X, \mathcal{S}, \mathcal{J} \rangle$ has the hull property if $\mathcal{H}(A) \neq \emptyset$ for any set $A \subset X$.

In the next part of the paper a notion of a generalized topological space, introduced in [1] by Á. Császár, will be used. We shall say that a family $\gamma \subset 2^X$ is a generalized topology on X if $\emptyset \in \gamma$ and $\bigcup_{t \in T} G_t \in \gamma$ whenever $\{G_t : t \in T\} \subset \gamma$. The pair (X, γ) is

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called a generalized topological space. If $X \in \gamma$ then we shall say that (X, γ) is a strong generalized topological space.

In the theory of generalized topological spaces almost all notions (e.g. an interior of a set, a closure of a set, a boundary of a set, a compact set) are defined as in standard topological spaces (see [1,2]). The interior, the closure and the boundary of $A \subset X$ will be denoted by $\mathrm{int}_\gamma(A)$, $\mathrm{cl}_\gamma(A)$ and $\mathrm{Fr}_\gamma(A)$, respectively. Moreover, we will write γ-open, γ-closed, etc. if we want to emphasize that the considerations concern the space (X, γ). Similarly to the classical topological space we define a base of a generalized topological space or a connected set in such space (see [3,8]). Separation axioms for a generalized topological space are defined as in the case of the classical topological space [4]. Moreover, the definitions of a separable space, Lindelöf space, first countable and second countable space can be adopt from the classical topological space.

In the case of a topological space, the notion of a nowhere dense set may be introduced by different equivalent definitions. One can say that A is a nowhere dense set if the interior of the closure of A is an empty set. On the other hand one can say that A is a nowhere dense set if any nonempty open set contains a nonempty open subset which is disjoint with A. In the case of a generalized topological space, these two conditions can lead to different notions. In [9] one can find two notions connected with nowhere density in generalized topological space. We say that a set $A \subset X$ is γ-nowhere dense if $\mathrm{int}_\gamma(\mathrm{cl}_\gamma(A)) = \emptyset$. A set $A \subset X$ is γ-strongly nowhere dense if for $V \in \gamma \setminus \{\emptyset\}$ there exists $U \in \gamma \setminus \{\emptyset\}$ such that $U \subset V$ and $A \cap U = \emptyset$. It is easy to see that if A is γ-strongly nowhere dense then it is γ-nowhere dense. The converse theorem is not true in general (see [9]).

At the beginning of this section we mentioned that the particular operators will play a special role in our consideration, so we start with their definitions. First we shortly recall the definition of the lower density operator which is investigated by many mathematicians (e.g. [5,12]).

Definition 1.1. We shall say that an operator $\Phi : \mathcal{S} \to \mathcal{S}$ is the lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ if

1° $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$;
2° $\underset{A\in\mathcal{S}}{\forall}\ \underset{B\in\mathcal{S}}{\forall}\ \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
3° $\underset{A\in\mathcal{S}}{\forall}\ \underset{B\in\mathcal{S}}{\forall}\ A \triangle B \in \mathcal{J} \Rightarrow \Phi(A) = \Phi(B)$;
4° $\underset{A\in\mathcal{S}}{\forall}\ \Phi(A) \triangle A \in \mathcal{J}$.

Obviously, the classical density operator defined in [12] is an example of the lower density operator. If $\langle X, \mathcal{S}, \mathcal{J} \rangle$ is a measurable space with the hull property and Φ is the lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$, then the following theorem is true (see [11], p. 213).

Theorem 1.2. *The family $\mathcal{T}_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\}$ is a topology on X, which is called a topology generated by Φ.*

Proof. One can find the proof of this theorem in [11], but for the convenience of the reader we will present that any union of elements of $\mathcal{T}_\Phi$ belongs to $\mathcal{T}_\Phi$. Let $\{A_w\}_{w\in W} \subset \mathcal{T}_\Phi$. Since $\langle X, \mathcal{S}, \mathcal{J} \rangle$ is a measurable space with the hull property, we obtain that there exists a set B being a measurable kernel of the set $\bigcup_{w\in W} A_w$ (i.e. $B \in \mathcal{S}$, $B \subset \bigcup_{w\in W} A_w$ and for any measurable set $C \subset \bigcup_{w\in W} A_w \setminus B$ we have that $C \in \mathcal{J}$). Obviously $(A_w \cap B) \triangle A_w \in \mathcal{J}$ for any $w \in W$ and

$$B \subset \bigcup_{w\in W} A_w \subset \bigcup_{w\in W} \Phi(A_w) = \bigcup_{w\in W} \Phi(A_w \cap B) \subset \Phi(B).$$

Condition 4° from Definition 1.1 implies that $\Phi(B)\setminus B \in \mathcal{J}$ and, in consequence, $\bigcup_{w\in W} A_w \in \mathcal{S}$. Now, it is easy to see that

$$\bigcup_{w\in W} \Phi(A_w) \subset \Phi(\bigcup_{w\in W} \Phi(A_w)),$$

so $\bigcup_{w\in W} A_w \in \mathcal{T}_\Phi$. □

Obviously topology described in the above theorem is an example of an abstract density topology. The papers [6,7,10,13] contain many results and properties relevant to abstract density topologies and lower density operators. Now, we are following the lower density operator Φ on $\langle X, \mathcal{S}, \mathcal{J}\rangle$. From now on, we will assume that $\langle X, \mathcal{S}, \mathcal{J}\rangle$ has the hull property.

Let Φ be the lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$. Let us consider an operator $\Phi^* : 2^X \to \mathcal{S}$ defined in the following way:

$$\underset{A\subset X}{\forall} \Phi^*(A) = \Phi(B), \tag{1.1}$$

where B is an $\mathcal{S}$-measurable hull of a set A. By condition 3° of Definition 1.1 we have that Φ^* is defined correctly. Clearly, if $A \in \mathcal{S}$ then $\Phi^*(A) = \Phi(A)$. Moreover, we have the following propositions.

Proposition 1.3. *1° $\Phi^*(\emptyset) = \emptyset$ and $\Phi^*(X) = X$;*

2° $\underset{A\subset X}{\forall}\ \underset{B\in\mathcal{S}}{\forall} \Phi^*(A\cap B) = \Phi^*(A) \cap \Phi(B)$;

3° $\underset{A\subset X}{\forall}\ \underset{B\subset X}{\forall} A\triangle B \in \mathcal{J} \Rightarrow \Phi^*(A) = \Phi^*(B)$;

4° $\underset{A\subset X}{\forall} A \setminus \Phi^*(A) \in \mathcal{J}$.

Proof. Condition 1° is obvious. Let $A \subset X$ and $B \in \mathcal{S}$. If $C \in \mathcal{S}$ is a measurable hull of A then $C\cap B$ is a measurable hull of $A\cap B$. Hence $\Phi^*(A\cap B) = \Phi(C\cap B) = \Phi(C)\cap\Phi(B) = \Phi^*(A)\cap\Phi(B)$, so condition 2° is satisfied. To prove 3° let us observe that if $A\triangle B \in \mathcal{J}$ and C_1, C_2 are measurable hulls of A and B, respectively, then $C_1\triangle C_2 \in \mathcal{J}$. It implies that $\Phi(C_1) = \Phi(C_2)$ and finally, $\Phi^*(A) = \Phi^*(B)$. In the case of 4° if C is an $\mathcal{S}$-measurable hull of A then $A \setminus \Phi^*(A) \subset C \setminus \Phi(C) \in \mathcal{J}$. □

Proposition 1.4. *For every $A \subset X$ the following properties hold:*

i) *$\Phi(\Phi^*(A)) = \Phi^*(A)$;*
ii) *$\Phi^*(A \cap \Phi^*(A)) = \Phi^*(A)$.*

Proof. Let B be a measurable hull of A. Then $\Phi(\Phi^*(A)) = \Phi(\Phi(B)) = \Phi(B) = \Phi^*(A)$. It means that i) is satisfied. In the case of ii) we have $\Phi^*(A\cap\Phi^*(A)) = \Phi^*(A\cap\Phi(B)) = \Phi^*(A)\cap\Phi(B) = \Phi^*(A)\cap\Phi^*(A) = \Phi^*(A)$. □

Proposition 1.5. *For every $A \subset X$ we have*

(i) *$A \cap \Phi^*(A) = \emptyset$ iff $A \in \mathcal{J}$;*
(ii) *$A \cap \Phi^*(A) \in \mathcal{S}$ iff $A \in \mathcal{S}$.*

Proof. If $A \in \mathcal{J}$ then by Definition 1.1 we have $\Phi(A) = \emptyset$, so that $A \cap \Phi(A) = \emptyset$. Let $A \notin \mathcal{J}$. Then $A = (A\cap\Phi^*(A)) \cup (A\setminus\Phi^*(A))$. Since, by Proposition 1.3, $A\setminus\Phi^*(A) \in \mathcal{J}$ we get that $A\cap\Phi^*(A) \notin \mathcal{J}$. It implies that $A\cap\Phi^*(A) \neq \emptyset$ and condition (i) is satisfied.

Now, we prove condition (ii). If $A \in \mathcal{S}$ then $\Phi^*(A) = \Phi(A)$ and, by Definition 1.1, we get that $\Phi(A) \in \mathcal{S}$. It implies that $A\cap\Phi^*(A) \in \mathcal{S}$. If $A\cap\Phi^*(A) \in \mathcal{S}$ then, by Proposition 1.3, $A\setminus\Phi^*(A) \in \mathcal{J}$ and we get that $A \in \mathcal{S}$. □

2. A generalized topological space connected with Φ^*

In this section, we will study the family

$$\mathcal{T}_{\Phi^*} = \{A \subset X : A \subset \Phi^*(A)\}$$

generated by the operator Φ^*.

Remark 2.1. The family $\mathcal{T}_{\Phi^*}$ has not to be closed with respect to the finite intersection.

Indeed, let $\mathbb{R}$ be the set of all real numbers, $\mathcal{L}$ be the σ-algebra of Lebesgue measurable sets and $\mathbb{L}$ be the σ-ideal of Lebesgue measure zero sets in $\mathbb{R}$. If Φ is the density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$ and B is a Bernstein set then $\Phi^*(B) = \Phi(\mathbb{R}) = \mathbb{R}$ and $\Phi^*(\mathbb{R} \setminus B) = \Phi(\mathbb{R}) = \mathbb{R}$. Additionally, for every $x \in \mathbb{R}$ we get that $B \cup \{x\} \in \mathcal{T}_{\Phi^*}$ and $(\mathbb{R} \setminus B) \cup \{x\} \in \mathcal{T}_{\Phi^*}$, but $(B \cup \{x\}) \cap ((\mathbb{R} \setminus B) \cup \{x\}) = \{x\} \notin \mathcal{T}_{\Phi^*}$.

However, it is easy to prove the following theorem:

Theorem 2.2. *The family $\mathcal{T}_{\Phi^*}$ is a strong generalized topology on X and $\mathcal{T}_{\Phi} \subset \mathcal{T}_{\Phi^*}$.*

If we consider the classical density operator Φ on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$, then it is easy to see that a Bernstein set belongs to $\mathcal{T}_{\Phi^*} \setminus \mathcal{T}_{\Phi}$. Proposition 1.3 implies that

Remark 2.3. If $W \in \mathcal{T}_{\Phi^*}$ and $A \in \mathcal{J}$ then $W \setminus A \in \mathcal{T}_{\Phi^*}$. Moreover, if $W \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$ then $W \notin \mathcal{J}$.

Proof. Let $W \in \mathcal{T}_{\Phi^*}$, $A \in \mathcal{J}$ and $V = W \setminus A$. Clearly, $V \triangle W \in \mathcal{J}$, so Condition 3° in Proposition 1.3 gives that $\Phi^*(V) = \Phi^*(W)$. Obviously, we have that $W \subset \Phi^*(W)$, because $W \in \mathcal{T}_{\Phi^*}$. Thus $V \subset W \subset \Phi^*(W) = \Phi^*(V)$, which gives that $V \in \mathcal{T}_{\Phi^*}$. Let now $W \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$. Suppose, contrary to our claim that $W \in \mathcal{J}$. Proposition 1.3 Conditions 1° and 3° imply that $\Phi^*(W) = \Phi^*(\emptyset) = \emptyset$. Since $W \subset \Phi^*(W)$ we obtain that $W = \emptyset$, which is impossible. □

By Proposition 1.4 we have that

Remark 2.4. For every $A \subset X$ the sets $\Phi^*(A)$ and $A \cap \Phi^*(A)$ are the members of $\mathcal{T}_{\Phi^*}$.

Moreover, we have the following properties:

Proposition 2.5. *For every $A \subset X$ we have*

$$\text{int}_{\mathcal{T}_{\Phi^*}}(A) = A \cap \Phi^*(A).$$

Proof. By Remark 2.4 we have $A \cap \Phi^*(A) \in \mathcal{T}_{\Phi^*}$, so that $A \cap \Phi^*(A) \subset \text{int}_{\mathcal{T}_{\Phi^*}}(A)$. Let $V \in \mathcal{T}_{\Phi^*}$ and $V \subset A$. Then $\Phi^*(V) \subset \Phi^*(A)$. So that $V \subset A \cap \Phi^*(A)$. Finally, $\text{int}_{\mathcal{T}_{\Phi^*}}(A) \subset A \cap \Phi^*(A)$. □

Proposition 2.6. $\text{Fr}_{\mathcal{T}_{\Phi^*}}(A) \in \mathcal{J}$ *for every set $A \subset X$.*

Proof. Obviously, $\text{Fr}_{\mathcal{T}_{\Phi^*}}(A) = \text{cl}_{\mathcal{T}_{\Phi^*}}(A) \setminus \text{int}_{\mathcal{T}_{\Phi^*}}(A) = [A \cup (X \setminus \Phi^*(X \setminus A)] \setminus (A \cap \Phi^*(A)) = (A \setminus \Phi^*(A)) \cup [(X \setminus \Phi^*(X \setminus A)) \cap ((X \setminus A) \cup (X \setminus \Phi^*(A))] \subset (A \setminus \Phi^*(A)) \cup ((X \setminus A) \setminus \Phi^*(X \setminus A)) \in \mathcal{J}$. □

The next property is the characterization of nowhere dense sets in the generalized topological space $\langle X, \mathcal{T}_{\Phi^*} \rangle$.

Theorem 2.7. *Let $A \subset X$. Then the following conditions are equivalent:*

i) $\underset{W \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}}{\forall} \ \underset{V \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}}{\exists} (V \subset W \wedge V \cap A = \emptyset)$;

ii) $\text{int}_{\mathcal{T}_{\Phi^*}}(\text{cl}_{\mathcal{T}_{\Phi^*}}(A)) = \emptyset$;

iii) $A \in \mathcal{J}$.

Proof. i) $\Rightarrow$ ii). Let us suppose that $\mathrm{int}_{\mathcal{T}_{\Phi^*}}(\mathrm{cl}_{\mathcal{T}_{\Phi^*}}(A)) \neq \emptyset$. Hence there exists $W \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$ such that $W \subset \mathrm{int}_{\mathcal{T}_{\Phi^*}}(\mathrm{cl}_{\mathcal{T}_{\Phi^*}}(A))$. By condition i) there exists $V \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$ such that $V \subset W$ and $V \cap A = \emptyset$. At the same time we get contradiction with the fact that $V \subset \mathrm{cl}_{\mathcal{T}_{\Phi^*}}(A)$.

Now, we shall prove that ii) $\Rightarrow$ iii). Let us suppose that $A \notin \mathcal{J}$. Then, by Proposition 1.5, $A \cap \Phi^*(A) \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$ and we have the contradiction with the fact that $\mathrm{int}_{\mathcal{T}_{\Phi^*}}(\mathrm{cl}_{\mathcal{T}_{\Phi^*}}(A)) = \emptyset$.

The implication iii) $\Rightarrow$ i) left to complete the proof. Let $A \in \mathcal{J}$ and $W \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$. Remark 2.3 gives that $V = W \setminus A \in \mathcal{T}_{\Phi^*}$. Moreover, since $W \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$, we obtain, by Remark 2.3, that $W \notin \mathcal{J}$ and, in consequence, that $V \neq \emptyset$. Clearly, $V \cap A = \emptyset$, so the proof is finished. □

The above theorem gives that the notions of a nowhere dense set and a strong nowhere dense set in the space $\langle X, \mathcal{T}_{\Phi^*}\rangle$ are equivalent. What is more, we see at once that

Corollary 2.8. *If $\mathcal{N}(\mathcal{T}_{\Phi^*})$ is the family of all nowhere dense sets in $\langle X, \mathcal{T}_{\Phi^*}\rangle$ then $\mathcal{N}(\mathcal{T}_{\Phi^*}) = \mathcal{J}$.*

Proposition 2.9. *Let $\mathcal{B}a(\mathcal{T}_{\Phi^*})$ be the smallest σ-algebra of sets containing the family $\mathcal{N}(\mathcal{T}_{\Phi^*}) \cup \mathcal{T}_{\Phi^*}$. Then*

$$\mathcal{B}a(\mathcal{T}_{\Phi^*}) = 2^X.$$

Proof. Let $A \subset X$. Then $A = (A \setminus \Phi^*(A)) \cup (A \cap \Phi^*(A))$. By Proposition 1.3 and Corollary 2.8, $A \setminus \Phi^*(A) \in \mathcal{N}(\mathcal{T}_{\Phi^*})$ and by Proposition 2.5, $A \cap \Phi^*(A) \in \mathcal{T}_{\Phi^*}$, so that $\mathcal{B}a(\mathcal{T}_{\Phi^*}) = 2^X$. □

Proposition 2.10. *Let $A \subset X$. Then $A \in \mathcal{J}$ if and only if A is $\mathcal{T}_{\Phi^*}$-closed and $\mathcal{T}_{\Phi^*}$-nowhere dense.*

Proof. Let $A \in \mathcal{J}$. Obviously $X \in \mathcal{T}_{\Phi^*}$, so, by Remark 2.3, we have that $X \setminus A \in \mathcal{T}_{\Phi^*}$. Thus A is $\mathcal{T}_{\Phi^*}$-closed and evidently, by Corollary 2.8, A is $\mathcal{T}_{\Phi^*}$-nowhere dense. Sufficiency is the consequence of Corollary 2.8. □

As the consequence of this property we have

Proposition 2.11. *If $A \in \mathcal{J}$ then A is $\mathcal{T}_{\Phi^*}$-closed and $\mathcal{T}_{\Phi^*}$-discrete.*

Also the following property is obvious.

Property 2.12. *If $\mathcal{J}$ is a σ-ideal and $A \in \mathcal{T}_{\Phi^*} \setminus \{\emptyset\}$ then A is $\mathcal{T}_{\Phi^*}$-second category, i.e. $A \notin \mathbb{K}(\mathcal{T}_{\Phi^*})$.*

Moreover, we have

Proposition 2.13. *If $\mathcal{J}$ is a σ-ideal then a set $A \subset X$ is $\mathcal{T}_{\Phi^*}$-compact if and only if A is finite.*

Proof. Sufficiency is obvious. Let us assume that $A \subset X$ is $\mathcal{T}_{\Phi^*}$-compact and infinite. Let $B \subset A$ be infinite and countable. Then $(X \setminus B) \cup \{x\} \in \mathcal{T}_{\Phi^*}$ for every $x \in B$. Indeed, clearly $B \in \mathcal{J}$ and $\{x\} \in \mathcal{J}$ for any $x \in B$, so $X \triangle ((X \setminus B) \cup \{x\}) \in \mathcal{J}$ and, in consequence, Proposition 1.3 Conditions 1° and 3° give that $(X \setminus B) \cup \{x\} \subset X = \Phi^*((X \setminus B) \cup \{x\})$. Thus $(X \setminus B) \cup \{x\} \in \mathcal{T}_{\Phi^*}$. Obviously the family $\{(X \setminus B) \cup \{x\}\}_{x \in B}$ is an open cover of A which does not contain a finite subcover of A. It contradicts the fact that A is $\mathcal{T}_{\Phi^*}$-compact. □

Proposition 2.14. *If $\mathcal{J}$ is a σ-ideal then the space $\langle X, \mathcal{T}_{\Phi^*}\rangle$ neither fulfills the first nor the second axiom of countability and is not separable.*

Proof. Let us suppose that $\langle X, \mathcal{T}_{\Phi^*}\rangle$ fulfills the first axiom of countability. Let $x \in X$ and $\{V_n\}_{n \in \mathbb{N}}$ be a sequence of all $\mathcal{T}_{\Phi^*}$-open sets from a countable base of $\mathcal{T}_{\Phi^*}$ at x. Let $x_n \in V_n \setminus \{x\}$ for $n \in \mathbb{N}$. Putting $V = V_1 \setminus \{x_n : n \in \mathbb{N}\}$ we get that $V \in \mathcal{T}_{\Phi^*}$, $x \in V$ and V

does not contain any set V_n for $n \in \mathbb{N}$. Hence $\langle X, \mathcal{T}_{\Phi^*}\rangle$ does not fulfill the first countability axiom and therefore does not fulfill the second countability axiom. Since every countable set belongs to $\mathcal{J}$ so thus we infer that $\langle X, \mathcal{T}_{\Phi^*}\rangle$ is not separable. □

Proposition 2.15. *If $\mathcal{J}$ contains an uncountable set then $\langle X, \mathcal{T}_{\Phi^*}\rangle$ is not a Lindelöf space.*

Proof. Let $D \in \mathcal{J}$ be an uncountable set then $(X \setminus D) \cup \{x\} \in \mathcal{T}_{\Phi^*}$ for every $x \in D$ and $\{(X \setminus D) \cup \{x\}\}_{x\in D} \in \mathcal{T}_{\Phi^*}$ is an open cover of X which does not contain a countable subcover. □

Since $V = X \setminus \{x\} \in \mathcal{T}_{\Phi^*}$ for any $x \in X$, we see at once

Proposition 2.16. *The space $\langle X, \mathcal{T}_{\Phi^*}\rangle$ is a T_1-space.*

We end this section with the interesting property of the functions continuous with respect to the generalized topology $\mathcal{T}_{\Phi^*}$.

Theorem 2.17. *If $\mathcal{J}$ is a σ-ideal then for an arbitrary function $f : X \to Y$, where $\langle Y, \mathcal{T}\rangle$ satisfies the second countability axiom, there exists a set $A \in \mathcal{J}$ such that for every $x \in X\setminus A$ the function f is $\mathcal{T}_{\Phi^*}$-continuous at x.*

Proof. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable base of $\langle Y, \mathcal{T}\rangle$. For every $n \in \mathbb{N}$ we have that $f^{-1}(B_n) = C_n \cup D_n$, where $C_n = \Phi^*(f^{-1}(B_n)) \cap f^{-1}(B_n)$ and $D_n = f^{-1}(B_n) \setminus \Phi^*(f^{-1}(B_n))$. By Proposition 2.5, $C_n \in \mathcal{T}_{\Phi^*}$ for any $n \in \mathbb{N}$. Moreover, by Proposition 1.3, $A = \bigcup_{n=1}^{\infty} D_n \in \mathcal{J}$. Let $x_0 \in X \setminus A$ and $W \in \mathcal{T}_{\Phi^*}$ be such that $f(x_0) \in W$. Thus there exists $n_0 \in N$ such that $B_{n_0} \subset W$ and $x_0 \in f^{-1}(B_{n_0})$. Hence $x_0 \in C_{n_0} \in \mathcal{T}_{\Phi^*}$ and $f(C_{n_0}) \subset W$. It means that f is $\mathcal{T}_{\Phi^*}$-continuous for every $x \in X \setminus A$. □

3. The ($\star$) property and the ($\star\star$) property

In this section we concentrate on the family $\mathcal{T}_{\Phi^*}$ in a space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ having two special properties: the ($\star$) property and the ($\star\star$) property. We start with the definition of these properties.

Definition 3.1. We shall say that $\langle X, \mathcal{S}, \mathcal{J}\rangle$ has

- the ($\star$) property if there exist $B \subset X$ such that $X \in \mathcal{H}(B) \cap \mathcal{H}(X \setminus B)$;
- the ($\star\star$) property if for every $A \subset X$ there exist $B \subset A$ and $C \in \mathcal{H}(B)$ such that $C \in \mathcal{H}(A \setminus B) \cap \mathcal{H}(A)$.

It is easy to see that $\langle X, \mathcal{S}, \mathcal{J}\rangle$ has the ($\star\star$) property if for every $A \subset X$ there exists $B \subset A$ such that $\mathcal{H}(B) \cap \mathcal{H}(A \setminus B) \cap \mathcal{H}(A) \neq \emptyset$. Moreover, we see at once that if $\langle X, \mathcal{S}, \mathcal{J}\rangle$ has the ($\star$) property and $B \subset X$ is such that $\mathcal{H}(B) \cap \mathcal{H}(X \setminus B) \neq \emptyset$ then $\mathcal{H}(B) \cap \mathcal{H}(X \setminus B) = \{X\}$.

Let $\mathcal{B}a$, $\mathcal{B}$ and $\mathbb{K}$ be the family of all sets having the Baire property, the family of Borel sets and the family of all meager sets with respect to the natural topology τ_0, respectively. Note that the measurable space $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$ has the ($\star$) property. Indeed, if $\mathfrak{C} \subset \mathbb{R}$ is a Bernstein set then $\mathbb{R} \in \mathcal{H}(\mathfrak{C}) \cap \mathcal{H}(\mathbb{R} \setminus \mathfrak{C})$.

Moreover if additivity of σ-ideal $\mathbb{K}$ is equal to $\mathfrak{c}$ then the measurable space $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$ has the ($\star\star$) property. Indeed, if $A \subset \mathbb{R}$ and $A \in \mathbb{K}$, then for any $B \subset A$ we have that $A \in \mathcal{H}(B) \cap \mathcal{H}(A \setminus B) \cap \mathcal{H}(A)$. If $A \subset \mathbb{R}$ and $A \notin \mathbb{K}$, then the cardinality of the family $\mathfrak{F} = \{F \in \mathcal{B} : A \cap F \notin \mathbb{K}\}$ equals $\mathfrak{c}$. Therefore, one can find sets $P_1 = \{x_\alpha : \alpha < \mathfrak{c}\}$ and $P_2 = \{y_\alpha : \alpha < \mathfrak{c}\}$ such that $P_1 \cup P_2 \subset A$, $P_1 \cap P_2 = \emptyset$, the cardinality of P_1 and P_2 is equal to $\mathfrak{c}$ and $P_1 \cap F \neq \emptyset \neq P_2 \cap F$ for any $F \in \mathfrak{F}$. Putting $B = P_1$ we obtain that $\mathcal{H}(B) \cap \mathcal{H}(A \setminus B) \cap \mathcal{H}(A) \neq \emptyset$. Indeed, let $V \in \mathcal{H}(A)$. Let $W \subset V \setminus B$ have the Baire property. Suppose that $W \cap A \notin \mathbb{K}$. Obviously, one can find a set $Z \in \mathcal{B}$ such that $Z \subset W$ and $Z \cap A \notin \mathbb{K}$. Thus $\emptyset \neq Z \cap P_1 \subset W \cap P_1$, which is impossible. Therefore, we have that $W \cap A \in \mathbb{K}$ and, in consequence, $W \setminus A$ has the Baire property. Since $V \in \mathcal{H}(A)$, we obtain that $W \setminus A \in \mathbb{K}$. Hence $W \in \mathbb{K}$. Finally, we have that $V \in \mathcal{H}(B)$. By a similar argument, $V \in \mathcal{H}(A \setminus B)$.

Theorem 3.2. *If $\langle X, \mathcal{S}, \mathcal{J} \rangle$ has the $(\star)$ property then the smallest topology $\sigma(\mathcal{T}_{\Phi^*})$ containing $\mathcal{T}_{\Phi^*}$ is equal to 2^X.*

Proof. Let $x \in X$ and $B \subset X$ be such that $\mathcal{H}(B) \cap \mathcal{H}(X \setminus B) \neq \emptyset$. Thus $B \cup \{x\} \in \mathcal{T}_{\Phi^*}$ and $(X \setminus B) \cup \{x\} \in \mathcal{T}_{\Phi^*}$. It implies that $(B \cup \{x\}) \cap ((X \setminus B) \cup \{x\}) \in \sigma(\mathcal{T}_{\Phi^*})$, so that $\sigma(\mathcal{T}_{\Phi^*}) = 2^X$. □

Theorem 3.3. *If $\langle X, \mathcal{S}, \mathcal{J} \rangle$ has the $(\star)$ property then $\mathcal{T}_{\Phi^*}$ does not include the supremum of the topologies included in $\mathcal{T}_{\Phi^*}$.*

Proof. Let us suppose that $\mathcal{T}$ is the supremum of the topologies included in $\mathcal{T}_{\Phi^*}$. Let $B \subset X$ be such that $X \in \mathcal{H}(B) \cap \mathcal{H}(X \setminus B)$. Let $x_0 \in X$. Put $\mathcal{T}_1 = \{\emptyset, B \cup \{x_0\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, (X \setminus B) \cup \{x_0\}, X\}$. It is easy to see that $\mathcal{T}_1, \mathcal{T}_2$ are topologies contained in $\mathcal{T}_{\Phi^*}$, so $\mathcal{T}_1 \cup \mathcal{T}_2 \subset \mathcal{T}$. Moreover, $(B \cup \{x_0\}) \cap ((X \setminus B) \cup \{x_0\}) = \{x_0\} \in \mathcal{T} \subset \mathcal{T}_{\Phi^*}$. It is a contradiction with the fact that $\{x_0\} \notin \mathcal{T}_{\Phi^*}$. □

However, we have the following property.

Theorem 3.4. *There exists a maximal topology in the family $\mathcal{A}$ of all topologies contained in $\mathcal{T}_{\Phi^*}$ and ordered by the inclusion.*

Proof. Let $\{\mathcal{T}_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary chain in $\mathcal{A}$. Put $\mathcal{T} = \{ \bigcup_{w \in W} A_w : \{A_w\}_{w \in W} \subset \bigcup_{\lambda \in \Lambda} \mathcal{T}_\lambda \}$. We see at once that $\emptyset, X \in \mathcal{T}$, $\mathcal{T}$ is closed under arbitrary unions and $\mathcal{T} \subset \mathcal{T}_{\Phi^*}$. Since $\{\mathcal{T}_\lambda\}_{\lambda \in \Lambda}$ is a chain we obtain that $\mathcal{T}$ is closed under finite intersections. Therefore $\mathcal{T}$ is a topology contained in $\mathcal{T}_{\Phi^*}$ and simultaneously, it is the upper bound of $\{\mathcal{T}_\lambda\}_{\lambda \in \Lambda}$. By Kuratowski-Zorn Lemma we get the existence of a maximal topology in $\mathcal{A}$. □

Proposition 3.5. *If $\langle X, \mathcal{S}, \mathcal{J} \rangle$ has the $(\star)$ property then $\langle X, \mathcal{T}_{\Phi^*} \rangle$ is a Hausdorff space.*

Proof. Let $x, y \in X$ and $x \neq y$. Let $B \subset X$ be such that $X \in \mathcal{H}(B) \cap \mathcal{H}(X \setminus B)$. If $x \in B$ and $y \in X \setminus B$ then putting $V_1 = B$ and $V_2 = X \setminus B$ we get that $V_1, V_2 \in \mathcal{T}_{\Phi^*}$, $V_1 \cap V_2 = \emptyset$, $x \in V_1$ and $y \in V_2$. If $x, y \in B$ then it is enough to consider the sets $V_1 = B \setminus \{y\} \in \mathcal{T}_{\Phi^*}$ and $V_2 = (X \setminus B) \cup \{y\} \in \mathcal{T}_{\Phi^*}$. If $x, y \in X \setminus B$ the proof runs in the similar way. □

Proposition 3.6. *If $\langle X, \mathcal{S}, \mathcal{J} \rangle$ has the $(\star\star)$ property then $\langle X, \mathcal{T}_{\Phi^*} \rangle$ is a normal space.*

Proof. Let F_1, F_2 be nonempty and disjoint $\mathcal{T}_{\Phi^*}$-closed subsets of X. If $A = X \setminus (F_1 \cup F_2) \in \mathcal{J}$ then putting $V_1 = (X \setminus F_2) \setminus A$ and $V_2 = (X \setminus F_1) \setminus A$ we get that $F_1 \subset V_1$, $F_2 \subset V_2$, $V_1, V_2 \in \mathcal{T}_{\Phi^*}$ and $V_1 \cap V_2 = \emptyset$.

If $A \notin \mathcal{J}$ then by the $(\star\star)$ property there exist $B \subset A$ and $C \in \mathcal{H}(B)$ such that $C \in \mathcal{H}(A \setminus B) \cap \mathcal{H}(A)$. Let $V_1 = F_1 \cup B$, $V_2 = F_2 \cup (A \setminus B)$. Evidently, $F_1 \subset V_1$, $F_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$. We prove first that $V_1 \in \mathcal{T}_{\Phi^*}$. In this purpose we show that $F_1 \cup B \subset \Phi^*(F_1 \cup B)$. Since $\Phi^*(F_1 \cup B) = \Phi^*(F_1 \cup C) = \Phi^*(F_1 \cup A) = \Phi^*(X \setminus F_2)$. Suppose that $x \in (F_1 \cup B) \setminus \Phi^*(F_1 \cup B) \subset X \setminus \Phi^*(X \setminus F_2)$. Because F_2 is $\mathcal{T}_{\Phi^*}$-closed it means that $X \setminus F_2 \subset \Phi^*(X \setminus F_2)$ and finally $x \in (F_1 \cup B) \setminus (X \setminus F_2) = (F_1 \cup B) \setminus (F_1 \cup A) = \emptyset$. This contradiction infer that $V_1 \in \mathcal{T}_{\Phi^*}$. Similarly, we can prove that $V_2 \in \mathcal{T}_{\Phi^*}$. It ends the proof. □

Theorem 3.7. *If $\langle X, \mathcal{S}, \mathcal{J} \rangle$ has the $(\star\star)$ property then every $\mathcal{T}_{\Phi^*}$-closed subset of X is G_δ-set in the space $\langle X, \mathcal{T}_{\Phi^*} \rangle$.*

Proof. Let $F \subset X$ be $\mathcal{T}_{\Phi^*}$-closed subset of X. Let $A = X \setminus F$. If $A \in \mathcal{J}$ then $F = X \setminus A \in \mathcal{T}_{\Phi^*}$.

Let us assume that $A \notin \mathcal{J}$. By the $(\star\star)$ property there exist $B \subset A$ and $C \in \mathcal{H}(B)$ such that $C \in \mathcal{H}(A \setminus B) \cap \mathcal{H}(A)$. Let $V_1 = F \cup B$ and $V_2 = F \cup (A \setminus B)$. Simultaneously as in the proof of the previous theorem we get that $V_1, V_2 \in \mathcal{T}_{\Phi^*}$. Since $F = V_1 \cap V_2$, we get that F is G_δ-set in the space $\langle X, \mathcal{T}_{\Phi^*} \rangle$. □

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