Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (6) (2020), 1926 – 1943
DOI : 10.15672/hujms.573315

Research Article

Radio k-labeling of paths

Laxman Saha[1], Satyabrata Das[1], Kinkar Chandra Das[2*], Kalishankar Tiwary[3]

[1] *Department of Mathematics, Balurghat College, Balurghat 733101, India*
[2] *Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea*
[3] *Department of Mathematics, Raiganj University, Raiganj 733134, India*

Abstract

The Channel Assignment Problem (CAP) is the problem of assigning channels (non-negative integers) to the transmitters in an optimal way such that interference is avoided. The problem, often modeled as a labeling problem on the graph where vertices represent transmitters and edges indicate closeness of the transmitters. A radio k-labeling of graphs is a variation of CAP. For a simple connected graph $G = (V(G), E(G))$ and a positive integer k with $1 \leq k \leq \text{diam(G)}$, a radio k-labeling of G is a mapping $f : V(G) \to \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geq k + 1 - d(u, v)$ for each pair of distinct vertices u and v of G, where diam(G) is the diameter of G and $d(u, v)$ is the distance between u and v in G. The *span* of a radio k-labeling f is the largest integer assigned to a vertex of G. The *radio k-chromatic number* of G, denoted by $rc_k(G)$, is the minimum of spans of all possible radio k-labelings of G. This article presents the exact value of $rc_k(P_n)$ for even integer $k \in \left\{\left\lceil \frac{2(n-2)}{5} \right\rceil, \ldots, n-2\right\}$ and odd integer $k \in \left\{\left\lceil \frac{2n+1}{7} \right\rceil, \ldots, n-1\right\}$, i.e., at least 65% cases the radio k-chromatic number of the path P_n are obtain for fixed but arbitrary values of n. Also an improvement of existing lower bound of $rc_k(P_n)$ has been presented for all values of k.

Mathematics Subject Classification (2010). 05C12, 05C15, 05C78

Keywords. channel assignment, radio k-labeling, radio k-chromatic number, span

1. Introduction

The Channel Assignment Problem (CAP) is the problem of assigning channels (non-negative integers) to the stations in an optimal way such that interference is avoided. CAP plays an important role in wireless network and a well-studied interesting problem. Many researchers have modeled CAP as an optimization problem as follows: Given a collection of transmitters to be assigned operating frequencies and a set of interference constraints on transmitter pairs, find an assignment that satisfies all the interference constraints and minimizes the value of a given objective function. In 1980, Hale [11] has modeled FAP as a Graph labeling problem (in particular as a generalized graph labeling problem) and is an active area of research now. Griggs and Yeh [10] concentrated on the fundamental case

*Corresponding Author.
Email addresses: laxman.iitkgp@gmail.com (L. Saha), sdas1012@gmail.com (S. Das), kinkardas2003@gmail.com (K.C. Das), tiwarykalishankar@yahoo.com (K. Tiwary)
Received: 01.06.2019; Accepted: 25.02.2020

of $L(2,1)$-labelings. The $L(p, q)$-labeling problem (p, $q > 0$) and its variants have been studied extensively (see e.g. [1, 2, 7–12, 14, 20, 29–32, 34, 35]).

Motivated by FM channel assignments, a new model, namely the radio k-labeling problem was introduced in [4, 15] and studied further in [22, 25, 33]. For a simple connected graph $G = (V(G), E(G))$ and a positive integer k with $1 \le k \le \text{diam(G)}$, a radio k-labeling of G is a mapping $f \colon V(G) \to \{0, 1, 2, \ldots\}$ such that

$$|f(u) - f(v)| \ge k + 1 - d(u, v) \tag{1.1}$$

for each pair of distinct vertices u and v of G, where diam(G) is the diameter of G and $d(u, v)$ is the shortest distance between u and v in G. The *span* of a radio k-labeling f, denoted by $\text{span}_f(G)$, is the largest integer assigned to a vertex of G. The *radio k-chromatic number of* G, denoted by $rc_k(G)$, is the minimum of spans of all possible radio k-labelings of G. A radio k-labeling f of G is called minimal if $\text{span}_f(G) = rn(G)$. Without loss of generality, for a minimal radio labeling f we assume that $\min_{v \in V(G)} f(v) = 0$, otherwise the span of f can be reduced further by subtracting the positive integer $\min_{v \in V(G)} f(v)$ from all the labels of the vertices of the graph. For some specific values of k there are specific names for radio k-labelings as well as the radio k-chromatic number in the literature, which are given in Table 1:

Table 1. Special names of radio k-labelings and radio k-chromatic number.

k	**Name of labeling**	$rc_k(G)$
1	Vertex coloring	Chromatic number, $\chi(G)$
diam(G)	Radio	Radio number, $rn(G)$
diam(G) − 1	Antipodal	Antipodal number, $ac(G)$

The radio k-labeling problem can be viewed as an instance of the $L(p_1, \ldots, p_m)$-labeling problem (see e.g. [10, 36]), where m, p_1, $p_2, \ldots, p_m \ge 1$ are given integers, which aims at minimizing the span of a labeling $f \colon V(G) \to \{0, 1, 2, \ldots\}$ subject to $|f(u) - f(v)| \ge p_i$ whenever $d(u, v) = i$, $1 \le i \le m$. In the special case where $m = k$ and $p_i = \max\{k + 1 - i, 0\}$ for each i, the minimum span of such a labeling is exactly the radio k-chromatic number of G.

Determining the radio k-chromatic number of a graph is an interesting yet difficult combinatorial problem with potential applications to CAP. So far it has been explored for a few basic families of graphs and values of k near to diameter. The radio number of any hypercube was determined in [16] by using generalized binary Gray codes. Ortiz et al. [27] have studied the radio number of generalized prism graphs and have computed the exact value of radio number for some specific types of generalized prism graphs. For two positive integers $m \ge 3$ and $n \ge 3$, the Toroidal grids $T_{m,n}$ are the cartesian product of cycle C_m with cycle C_n. Morris et al. [26] have determined the radio number of $T_{n,n}$ and Saha et al. [30] have given exact value for radio number of $T_{m,n}$ when $mn \equiv 0 \pmod 2$. The radio numbers of the square of paths and cycles were studied in [23, 24]. For a cycle C_n, the radio number was determined by Liu and Zhu [25], and the antipodal number is known only for $n = 1, 2, 3 \pmod 4$ (see [3, 13]).

Surprisingly, even for paths finding the radio number was a challenging task. It is envisaged that in general determining the radio number would be difficult even for trees, despite a general lower bound for trees given in [22]. Till now, the radio number is known for very limited of families of trees. For path P_n, complete m-ary trees the exact values of radio number were determined in [21, 25]. The results for path were generalized [25] to spiders, leading to the exact value of the radio number in certain special cases. In [28], Reddy et al. gave an upper bound for the radio number of some special type of trees. For an n-vertex path P_n, the exact value of $rc_k(P_n)$ is known only for $k = n-1$ [25], $n-2$ [16], $n-3$ [18], and $n-4$ (n odd) [19].

In literature, the exact value of $rc_k(G)$ are known only when $k \in \{\text{diam(G)}, \text{diam(G)}-1, diam(G)-2\}$ and G belong to some specific class of graphs. For path P_n, the radio k-chromatic numbers $(rc_k(P_n))$ are known for relatively more values of k, namely, $k = n-1, n-2, n-3$ and $k = n-4$ (odd n). This article presents the exact value of $rc_k(P_n)$ for even integer $k \in \left\{\left\lceil\frac{2(n-2)}{5}\right\rceil, \ldots, n-2\right\}$ and odd integer $k \in \left\{\left\lceil\frac{2n+1}{7}\right\rceil, \ldots, n-1\right\}$, i.e., at least 65% cases the radio k-chromatic of the path P_n are obtain for fixed but arbitrary values of n. Also an improvement of existing lower bound of $rc_k(P_n)$ has been presented for all values of k. In Table 2, we summarize the existing results and our results on $rc_k(P_n)$.

Table 2. Existing Results and Our Result on radio k-chromatic number of P_n. Here LB and UB denotes the lower and upper bounds for $rc_k(P_n)$.

Author	**Values of k**	$rc_k(P_n)$
Liu and Zhu [25]	$n-1$	Exact value
Khennoufa and Togni [17]	$n-2$	Exact value
Kola and Panigrahi [18]	$n-3$	Exact value
Kola and Panigrahi [19]	$n-4$ (odd)	Exact value
Chartrand et al. [5]	$\leq n-3$	LB and UB
Current article	65% cases	Exact value
Current article	other cases	Improve LB

2. Preliminaries

Let $V(P_n) = \{0, 1, \ldots, n-1\}$ be the vertex set of an n-vertex path P_n. The path P_n has *length* $n-1$. For a fixed vertex $w \in V(P_n)$, *level function* is defined by $L_w(u) = d(w, u)$ for any $u \in V(P_n)$ and the *weight of* P_n *at* w is defined by $W_{P_n}(w) = \sum_{u \in V(P_n)} L_w(u)$. The *weight* $\omega(P_n)$ of P_n is the smallest weight among all vertices of P_n, i.e., $\omega(P_n) = \min\{W_{P_n}(w) : w \in V(P_n)\}$. A vertex C is said to be weight center of P_n if $W_{P_n}(C) = \omega(P_n)$.

Notation 2.1. We shall always fix a weight center C for the path P_n. Then $P_n \setminus C$ consists of two branches (components), called the *left* and *right* branches of $P_n \setminus C$. The left branch and right branch of P_n with respect to C are denoted by $L(P_n)$ and $R(P_n)$, respectively. From here to onwards by $L(u)$ we mean $d(C, u)$ and called it the *level* of the vertex u with respect to the weight center C. We denote the length of the common part of the paths from C to u and C to v by $\phi(u, v)$. Clearly, $\phi(u, v) = 0$ if and only if u and v are in opposite sides of C.

Definition 2.2. For an n-vertex path P_n, by *highest level vertex* of P_n we mean a vertex whose distance is maximum from a specified weight center C.

Observation 2.3. For an n-vertex path P_n the following hold :

1. If C is the weight center of P_n, then $L(P_n)$ and $R(P_n)$ have maximum $\lfloor\frac{n}{2}\rfloor$ number of vertices.
2. If n is odd, then P_n has exactly one weight center.
3. If n is even, then P_n has two weight centers.

Lemma 2.4. *Let P_n be a path of n vertices with weight center at C. Then for distinct u, $v \in V(P_n)$ the following hold :*

(1) $d(u,v) = L(u) + L(v) - 2\phi(u,v)$.

(2) $\phi(u,v) = 0$ *if and only if* $C \in \{u,v\}$ *or* u, v *are in different branch.*

(3) $\omega(P_n) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even,} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$

In this article one target is to determine the radio k-chromatic number of path P_n. For this we need to determine the minimum span of a radio labeling of path P_n in terms of some parameters like number of vertices, distances of minimum and maximum labeled (colored) vertices from the centroid. In Section 3, we discuss about the span of a radio labeling in terms of these parameters.

3. Radio labeling of path

Let f be any radio labeling of P_n. So f is injective and f induces a linear order

$$u_0,\, u_1,\, u_2, \ldots,\, u_{n-1} \tag{3.1}$$

of the vertices of P_n with $f(u_0) < f(u_1) < f(u_2) < \cdots < f(u_{n-1})$. Clearly span of f is $f(u_{n-1})$. Now from the radio conditions we have the following for $0 \le i \le n-2$,

$$f(u_{i+1}) - f(u_i) \ge n - d(u_i, u_{i+1}). \tag{3.2}$$

To make it an equality, we add a positive quantity $J_f(u_i, u_{i+1})$, called *jump* of f from u_i to u_{i+1}, in right side of the inequality (3.2). Therefore,

$$f(u_{i+1}) - f(u_i) = n - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1}).$$

Summing up these $n-1$ equations, we get

$$\begin{aligned} f(u_{n-1}) &= \sum_{i=0}^{n-2}[f(u_{i+1}) - f(u_i)] + f(u_0) \\ &= \sum_{i=0}^{n-2}[n - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1})] + f(u_0) \\ &\ge n(n-1) - 2\sum_{i=0}^{n-1} L(u_i) + L(u_0) + L(u_{n-1}) + \sum_{i=0}^{n-2}[J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})] \\ &\qquad + f(u_0) \qquad (3.3) \\ &= n(n-1) - 2\omega(P_n) + f(u_0) + L(u_0) + L(u_{n-1}) + \sigma(f) \qquad (3.4) \end{aligned}$$

where $\sigma(f) = \sum_{i=0}^{n-2} \sigma_f(u_i, u_{i+1})$ and $\sigma_f(u_i, u_{i+1}) = J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})$. Here *total jump* $J(f) = \sum_{i=0}^{n-2} J_f(u_i, u_{i+1})$. So the relationship between $\sigma(f)$ and $J(f)$ is $\sigma(f) = J(f) + 2\sum_{i=0}^{n-2} \phi(u_i, u_{i+1})$. If u_t, u_{t+1} are in same branch then it is clear that $\sigma_f(u_t, u_{t+1}) \ge 2$.

Now we calculate the jumps from u_i to u_{i+1} and u_{i+1} to u_{i+2} under the following assumption:

Assumption 3.1. Vertices u_i and u_{i+2} are in the same branch of P_n and vertex u_{i+1} is in a different branch.

Lemma 3.2. *Let u_i and u_{i+2} be in the same branch of P_n and let u_{i+1} be in a different branch of P_n. Then*

$$J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq \max\{2L(u_{i+1}) + 2\phi(u_i, u_{i+2}) - n, 0\}.$$

Proof. We have $f(u_{i+1}) - f(u_i) = n - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1}) = n - L(u_i) - L(u_{i+1}) + 2\phi(u_i, u_{i+1}) + J_f(u_i, u_{i+1})$ and $f(u_{i+2}) - f(u_{i+1}) = n - d(u_{i+1}, u_{i+2}) + J_f(u_{i+1}, u_{i+2}) = n - L(u_{i+1}) - L(u_{i+2}) + 2\phi(u_{i+1}, u_{i+2}) + J_f(u_{i+1}, u_{i+2})$. Summing up we get

$$f(u_{i+2}) - f(u_i) = 2n - L(u_i) - L(u_{i+2}) - 2L(u_{i+1}) + J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2})$$

where $J_f(u_t, u_{t+1}) = J_f(u_t, u_{t+1}) + 2\phi(u_t, u_{t+1})$ for $t = i,\ i+1$. On the other hand, since f is a radio labeling, we have

$$f(u_{i+2}) - f(u_i) \geq n - d(u_i, u_{i+2}) = n - L(u_i) - L(u_{i+2}) + 2\phi(u_i, u_{i+2}).$$

Combining the two expressions above, we get

$$J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq 2L(u_{i+1}) + 2\phi(u_i, u_{i+2}) - n.$$

Since the value $J_f(u_t, u_{t+1}) \geq 0$ for $t = i,\ i+1$, the result follows immediately. □

Definition 3.3. For a radio k-labeling f of P_n and a linear ordering $u_0, u_1, \ldots, u_{n-1}$ as in (3.1), two vertices u_i and u_{i+1} are called *consecutive colored vertices under* f and their labels $f(u_i)$, $f(u_{i+1})$ are called *consecutive radio k-coloring numbers.* A radio labeling f is said to be an *alternating radio k-labeling* if two consecutive colored vertices are in different branches.

Observation 3.4. From the above discussion, we may observe the following points under the Assumption 3.1:

(1) For an alternating labeling f, $\sigma(f) = J(f)$. Also if f is not an alternating radio labeling, then $\sigma(f) \geq 2$ because in this case there exist at least one pair u_t, u_{t+1} of vertices which are in same branch, i.e., $\phi(u_t, u_{t+1}) \geq 1$.
(2) If n is even and the vertex u_{i+1} is in highest level, then $J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq 2$ when $C \notin \{u_i, u_{i+2}\}$ and $i \neq n-2$ (by Lemma 3.2 using $L(u_{i+1}) = \frac{n}{2}$).
(3) If n is odd and the vertex u_{i+1} is in highest level, then $J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq 1$ when $C \notin \{u_i, u_{i+2}\}$ (by Lemma 3.2 using $L(u_{i+1}) = \frac{n-1}{2}$).
(4) For odd integer n, if $C \in \{u_0, u_{n-1}\}$ and $\{u_0, u_{n-1}\} \setminus C$ is not in highest level then $\sigma(f) \geq 1$ because in this case there exist at least one highest level vertex u_t in the segment $u_2, u_3, \ldots, u_{n-2}$ such that $J_f(u_{t-1}, u_t) + J_f(u_t, u_{i+2}) \geq 1$.
(5) If $C \notin \{u_0, u_{n-1}\}$, then $L(u_0) + L(u_{n-1}) \geq 2$.

Theorem 3.5. *Let P_n be a path of odd number of vertices n and let f be any radio labeling of P_n with first and last colored vertices are u_0 and u_m, respectively. Then*

$$\text{span}_f(P_n) \geq \frac{(n-1)^2}{2} + f(u_0) + L(u_0) + L(u_m) + \sigma(f).$$

Proof. From Equation (3.3), the result follows immediately. □

Corollary 3.6. *Let P_n be a path of odd number of vertices n and let f be any radio labeling of P_n with first and last colored vertices are u and v, respectively. If u and v are in same branch of P_n and none of them are neither weight center nor highest level vertices, then*

$$\text{span}_f(P_n) \geq \frac{(n-1)^2}{2} + f(u) + L(u) + L(v) + 1.$$

Proof. Let the radio labeling f induces the vertices of P_n as $u = u_0, u_1, \ldots, u_{n-1} = v$. Without loss of generality, we take $u, v \in L(P_n)$. As n is odd, $|L(P_n)| = |R(P_n)|$. Let C be the weight center of P_n. Here $C \notin \{u_0, u_{n-1}\}$. Thus we consider $C = u_r$ for some $r \in \{1, 2, \ldots, n-2\}$. Let $D_1 = \{u_0, u_1, \ldots, u_{r-1}\}$ and $D_2 = \{u_{r+1}, u_{r+2}, \ldots, u_{n-1}\}$. If one of D_1 or D_2 contains two consecutive colored vertices u_i, u_{i+1} from same branch, then $\phi(u_i, u_{i+1}) \geqslant 1$. So $\sigma(f) \geqslant 2$. Now we consider the case when both D_1 and D_2 are alternating sequence of vertices. In this case as $u_0, u_{n-1} \in L(P_n)$ and $|L(P_n)| = |R(P_n)|$, so $u_{r-1}, u_{r+1} \in R(P_n)$. If none of u_0 or u_{n-1} is the highest level $\frac{n-1}{2}$, then the $\frac{n-1}{2}$-level vertex in left branch, say, $u_p \in D_1 \cup D_2 \setminus \{u_1, u_{r-1}, u_{r+1}, u_{n-2}\}$. So applying Lemma 3.2, we have $J_f(u_{p-1}, u_p) + J_f(u_p, u_{p+1}) \geqslant 1$. Thus $\sigma(f) \geqslant 1$. □

Theorem 3.7. *Let P_n be a path of even number of vertices n and let f be any radio labeling of P_n with first and last colored vertices u_0 and u_m, respectively. Then*

$$\text{span}_f(P_n) \geq \frac{(n-1)^2 - 1}{2} + f(u_0) + L(u_0) + L(u_m) + \sigma(f).$$

Proof. From Equation (3.3), the result follows immediately. □

Remark 3.8. Liu and Zhu [25] have determined the exact value of radio number of path P_n $(n \geq 4)$ as $\frac{n^2}{2} - n + 1$ if n is even and $\frac{n^2+1}{2} - n + 2$ if n is odd. Thus the lower bound given in Theorems 3.5 and 3.7 coincide with the radio number of P_n.

Definition 3.9. Let $f : E \to F$ be a mapping from a set E to a set F. For a set $A \subset E$, we call the mapping $f|_A \colon A \to F$ as the *restriction of f on A*.

Lemma 3.10. *Let f be any radio k-labeling of an n-vertex path P_n with $n \geq k+1$. Then for any sub-path P_{k+1} of P_n,* $\text{span}_f(P_n) \geq \text{span}_f(P_{k+1})$.

Proof. Let f be a radio k-labeling of P_n. Since P_{k+1} is a subpath of P_n, $V(P_{k+1}) \subset V(P_n)$. Let $g = f|_{V(P_{k+1})}$ be the restriction of f on $V(P_{k+1})$. Then $\text{span}_f(P_n) \geqslant \text{span}_g(P_{k+1})$ and this is true for any radio k-labeling of P_n and its restriction $g = f|_{V(P_{k+1})}$. □

4. Lower bound of $rc_k(P_n)$

Theorem 4.1. *Let P_n be a path of order n. If $n \leq \lfloor \frac{3k+1}{2} \rfloor$, then $rc_k(P_n) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$.*

Proof. The main key for the proof of this theorem is to search a sub-path P^*_{k+1} of length k whose radio number is at least $\lfloor \frac{k^2}{2} \rfloor + n - k$. Let f be a radio-k-labeling of an n-vertex path P_n : 0, 1, 2, …, $n-1$. As $1 \leq k \leq n-2$, it is always possible to find a sub-path P_{k+1} of length k. Now we consider a sub-path P^0_{k+1} : 0, 1, 2, …, k of length k as in Fig. 1. Rest of path P_n is of length ℓ, say. Then $n - 1 = k + \ell$. We also construct a sub-path P^ℓ_{k+1} : ℓ, $\ell+1$, $\ell+2$, …, $\ell+k\,(= n-1)$ as in Fig. 1. If k is even, then every sub-path of length k has exactly one position of weight center. Otherwise there are two position of weight center. When k is even, the weight centers C_0 and C_ℓ of sub-paths P^0_{k+1} and P^ℓ_{k+1} are at the vertices $\frac{k}{2}$ and $\frac{k}{2} + \ell$ of path P_n, respectively (see Fig. 1 for an illustration). When k is odd, position of C_0 is at the vertex $\frac{k-1}{2}$ or $\frac{k+1}{2}$ and position of C_ℓ is at the vertex $\frac{k-1}{2} + \ell$ or $\frac{k+1}{2} + \ell$ of path P_n (see Fig. 3 for an illustration).

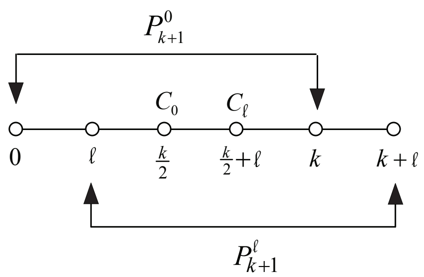

Figure 1. Sub-path construction of P_n when k is even.

Since $n-1=k+\ell$ and $n \leq \lfloor\frac{3k+1}{2}\rfloor$, thus $\ell \leq \lfloor\frac{k-1}{2}\rfloor$ and hence the weight center C_0 of sub-path P_{k+1}^0 belongs to the sub-path P_{k+1}^ℓ. For radio-k-labeling f of the path P_n, let u be the initial colored vertex and v be the maximum colored vertex. Then $f(u)=0$ and $\text{span}_f(P_n)=f(v)$. Now we consider the positions of u and v on the path $P_n: 0, 1, 2, \ldots, n-1$.

Case I: u **or** $v \in \{0, 1, 2, \ldots, \ell\}$. If $u \in \{0, 1, 2, \ldots, \ell\}$, then we can construct a sub-path $P_{k+1}^u: u, u+1, u+2, \ldots, u+k$ of length k as in Fig. 2. As $u \leq \ell$, so $u+k \leq k+\ell = n-1$. Hence sub-path P_{k+1}^u always exist in this case.

Subcase (a): k **is even.** Here weight center, say, C_u of P_{k+1}^u is $u+\frac{k}{2}$. So $d(C_u,u)=\frac{k}{2}=L(u)$ and $L(v)=d(C_u,v)\geq 0$. Applying Theorem 3.5 to path P_{k+1}^u, we have $\text{span}_f(P_{k+1}^u)\geq \frac{k^2}{2}+\frac{k}{2}\geq \frac{k^2}{2}+n-k$ as $n\leq \frac{3k}{2}=\lfloor\frac{3k+1}{2}\rfloor$. If $v\in\{0, 1, 2, \ldots, \ell\}$, then for

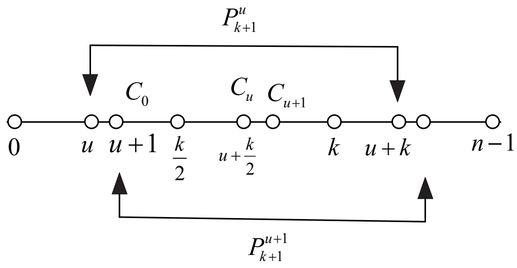

Figure 2. The sub-paths P_{k+1}^u and P_{k+1}^{u+1} of P_n.

the sub-path $P_{k+1}^v: v, v+1, v+2, \ldots, v+k$ of length k, by the same argument, one can easily prove that $\text{span}_f(P_{k+1}^v)\geq \frac{k^2}{2}+\frac{k}{2}\geq \frac{k^2}{2}+n-k$.

Subcase (b): k **is odd.** Here weight centers of P_{k+1}^u are $u+\frac{k-1}{2}$ and $u+\frac{k+1}{2}$. Let us denote the weight center of P_{k+1}^u by C_u. First we assume that $C_u=u+\frac{k-1}{2}$. If $C_u=v$ (the maximum colored vertex by f), then $d(C_u,u)+d(C_u,v)=\frac{k-1}{2}$. As the left branch $L(P_{k+1}^u)$ of P_{k+1}^u has less number of vertices than the right branch $R(P_{k+1}^u)$ and first color vertex is in left branch, so f can not be alternating radio labeling for the path P_{k+1}^u. Thus from Observation 3.4 (1), $\sigma(f)\geq 2$ and hence $\text{span}_f(P_{k+1}^u)\geq \frac{k^2-1}{2}+\frac{k-1}{2}+2>\lfloor\frac{k^2}{2}\rfloor+n-k$. Otherwise, $C_u\neq v$. Then $d(C_u,v)\geq 1$ and from Theorem 3.7 we obtain, $\text{span}_f(P_{k+1}^u)\geq \frac{k^2-1}{2}+\frac{k+1}{2}\geq \lfloor\frac{k^2}{2}\rfloor+n-k$.

Next we assume that $C_u=u+\frac{k+1}{2}$. Then $d(C_u,u)+d(C_u,v)\geq \frac{k+1}{2}$. By applying Theorem 3.7 to the path P_{k+1}^u, we have $\text{span}_f(P_{k+1}^u)\geq \frac{k^2-1}{2}+\frac{k+1}{2}\geq \lfloor\frac{k^2}{2}\rfloor+n-k$.

If $v\in\{0,1,2,\ldots,\ell\}$, then by the argument above for the sub-path $P_{k+1}^v: v, v+1, v+2, \ldots, v+k$; we can easily prove that $\text{span}_f(P_{k+1}^v)\geq \frac{k^2-1}{2}+\frac{k+1}{2}\geq \lfloor\frac{k^2}{2}\rfloor+n-k$.

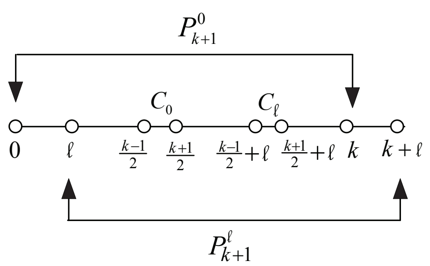

Figure 3. Sub-paths P_{k+1}^0 and P_{k+1}^ℓ of P_n.

Case II: u **or** $v \in \{\ell+1, \ell+2, \dots, \lfloor \frac{k}{2} \rfloor\}$. If $u \in \{\ell+1, \ell+2, \dots, \lfloor \frac{k}{2} \rfloor\}$, then we can construct a sub-path $P^{\ell}_{k+1} : \ell, \ell+1, \ell+2, \dots, \ell+k$ of length k.

Subcase (a): k **is even.** Here weight center C_ℓ of path P^{ℓ}_{k+1} is $\ell + \frac{k}{2}$. So $d(C_\ell, u) = \frac{k}{2} + \ell - u = L(u)$. Thus applying Theorem 3.5 to path P^{ℓ}_{k+1}, we have $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2}{2} + \frac{k}{2} + \ell - u + L(v) + \sigma(f)$. Since $\ell = n-k-1$ and $u \leq \frac{k}{2}$, $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2}{2} + n - k - 1 + L(v) + \sigma(f)$. If $L(v) = d(C_\ell, v) \geq 1$, then $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2}{2} + n - k + \sigma(f) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$. Otherwise, $d(C_\ell, v) = 0$. Thus we have $v = \frac{k}{2} + \ell$. Therefore the maximum colored vertex v is the weight center of path P^{ℓ}_{k+1} and the minimum colored vertex u is not the highest level. Thus from Observation 3.4 (4), we have $\sigma(f) \geq 1$ for sub-path P^{ℓ}_{k+1}. Therefore for the sub-path P^{ℓ}_{k+1}, we have $\text{span}_f(P^{\ell}_{k+1}) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$.

If $v \in \{\ell+1, \ell+2, \dots, \frac{k}{2}\}$, then for the sub-path P^{ℓ}_{k+1}, by the same argument, one can easily prove that $span_f(P^{\ell}_{k+1}) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$.

Subcase (b): k **is odd.** In this subcase weight center C_ℓ of P^{ℓ}_{k+1} is $\ell + \frac{k-1}{2}$ or $\ell + \frac{k+1}{2}$. First we assume that weight center $C_\ell = \ell + \frac{k-1}{2}$. Then the left branch $L(P^{\ell}_{k+1})$ has less number of vertices than the right branch $R(P^{\ell}_{k+1})$ of P^{ℓ}_{k+1}. Now, $d(C_\ell, u) + d(C_\ell, v) = \frac{k-1}{2} + \ell - u + d(C_\ell, v)$. By Theorem 3.7, we have $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2-1}{2} + \frac{k-1}{2} + \ell - u + d(C_\ell, v)$. If $d(C_\ell, v) \geq 1$, then we get $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2-1}{2} + \ell + 1 = \lfloor \frac{k^2}{2} \rfloor + n - k$ as $u \leq \frac{k-1}{2}$. Otherwise, $d(C_\ell, v) = 0$, i.e., $C_\ell = v$. Then $d(C_\ell, u) + d(C_\ell, v) = \frac{k-1}{2} + \ell - u$. Since the first colored vertex u is in the left branch $L(P^{\ell}_{k+1})$ of P^{ℓ}_{k+1} and the maximum colored vertex v is the centroid, so f can not be an alternating radio labeling of P^{ℓ}_{k+1} due to the same fact as described in *Case I*. By applying Theorem 3.7 to P^{ℓ}_{k+1} with $\sigma(f) \geq 2$, we have $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2-1}{2} + \frac{k-1}{2} + \ell - u + 2$. As $u \leq \frac{k-1}{2}$, thus $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2-1}{2} + \ell + 2 > \lfloor \frac{k^2}{2} \rfloor + n - k$.

Next we assume that weight center C_ℓ of P^{ℓ}_{k+1} is $\ell + \frac{k+1}{2}$. Then $d(C_\ell, u) + d(C_\ell, v) \geq \frac{k+1}{2} + \ell - u$. As $u \leq \frac{k-1}{2}$, hence $\text{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2-1}{2} + \ell + 1 = \lfloor \frac{k^2}{2} \rfloor + n - k$.

If $v \in \{\ell+1, \ell+2, \dots, \frac{k-1}{2}\}$, then for the sub-path P^{ℓ}_{k+1}, by the same argument we can easily prove that $\text{span}_f(P^{\ell}_{k+1}) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$.

Case III: Both u **and** v **lie in** $\{\lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor + 2, \dots, \lfloor \frac{k}{2} \rfloor + \ell\}$. In this case both u and v are in P^{0}_{k+1} as well as P^{ℓ}_{k+1}.

Subcase (a): k **is even.** We have $L(u) = d(C_0, u) = u - \frac{k}{2}$ and $L(v) = d(C_0, v) = v - \frac{k}{2}$. As both first and last colored vertices are in same side (right side) of P^{0}_{k+1} and none of them are neither the weight center nor the highest level vertices, so by Corollary 3.6, we have

$$\text{span}_f(P^{0}_{k+1}) \geq \frac{k^2}{2} + u + v - k + 1.$$

Again for the sub-path P^{ℓ}_{k+1}, the weight center C_ℓ is the vertex $\frac{k}{2} + \ell$. So $L(u) = d(C_\ell, u) = \frac{k}{2} + \ell - u$ and $L(v) = d(C_\ell, v) = \frac{k}{2} + \ell - v$. First we assume that $v \neq \frac{k}{2} + \ell$. Since both first and last colored vertices are in the same side (left side) of path P^{ℓ}_{k+1} and none of them are neither the weight center nor the highest level vertices, so by Corollary

3.6, we have

$$\mathrm{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2}{2} + k + 2\ell - (u+v) + 1.$$

Next we assume that $v = \frac{k}{2} + \ell$ of path P^{ℓ}_{k+1}. Then by the similar argument as in *Case II*, we can show that $\sigma(f) \geq 1$. By applying Theorem 3.5 to path P^{ℓ}_{k+1}, we have

$$\mathrm{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2}{2} + 1 + k + 2\ell - (u+v).$$

By simple calculations one can easily prove that $\max\Big\{u+v-k, k+2\ell-(u+v)\Big\} \geq \ell = n-k-1$. Thus we have

$$\begin{aligned} \mathrm{span}_f(P_n) &\geq \max\Big\{\mathrm{span}_f(P^{0}_{k+1}),\, \mathrm{span}_f(P^{\ell}_{k+1})\Big\} \\ &\geq \frac{k^2}{2} + 1 + \max\Big\{u+v-k,\, k+2\ell-(u+v)\Big\} \\ &\geq \left\lfloor \frac{k^2}{2} \right\rfloor + n - k. \end{aligned}$$

Subcase (b): k is odd. If the weight center C_0 is $\frac{k-1}{2}$ of the path P^{0}_{k+1}, then $d(C_0,u) + d(C_0,v) = u+v-k+1$. Then by Theorem 3.7, we have

$$\mathrm{span}_f(P^{0}_{k+1}) \geq \frac{k^2-1}{2} + u + v - k + 1.$$

Otherwise, the weight center C_0 is $\frac{k+1}{2}$. Then the right branch $R(P^{0}_{k+1})$ has less number of vertices than the left branch $L(P^{0}_{k+1})$ of P^{0}_{k+1} and $d(C_0,u) + d(C_0,v) = u+v-k-1$. As both u and v are in $R(P^{0}_{k+1})$, so $\sigma_f(P^{0}_{k+1}) \geq 2$ (since f can not be alternating radio labeling for the path P^{0}_{k+1} as described in *Case II* of this theorem). By Theorem 3.7, we have

$$\mathrm{span}_f(P^{0}_{k+1}) \geq \frac{k^2-1}{2} + u + v - k + 1.$$

By the similar argument to the path P^{ℓ}_{k+1} with the weight centers $\frac{k-1}{2} + \ell$ and $\frac{k+1}{2} + \ell$, we obtain

$$\mathrm{span}_f(P^{\ell}_{k+1}) \geq \frac{k^2-1}{2} + k + 1 + 2\ell - u - v.$$

It is easy to prove that $\max\{u+v-k+1, k-1+2\ell-(u+v)\} \geq \ell+1 = n-k$. Thus we have

$$\begin{aligned} \mathrm{span}_f(P_n) &\geq \max\Big\{\mathrm{span}_f(P^{0}_{k+1}),\, \mathrm{span}_f(P^{\ell}_{k+1})\Big\} \\ &\geq \frac{k^2-1}{2} + 1 + \max\Big\{u+v-k,\, k+2\ell-(u+v)\Big\} \\ &\geq \left\lfloor \frac{k^2}{2} \right\rfloor + n - k. \end{aligned}$$

Case IV: u **or** $v \in \{\lfloor \frac{k}{2} \rfloor + \ell + 1,\ \lfloor \frac{k}{2} \rfloor + \ell + 2, \ldots,\ k\}$. This case is similar to **Case II**. For the sub-path P^0_{k+1}, by the same argument as used in **Case II**, one can easily prove that $\mathrm{span}_f(P^0_{k+1}) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$.

Case V: u **or** $v \in \{k+1,\ k+2, \ldots,\ n-1\}$. For the sub-path $P^{u-k}_{k+1} :\ u-k,\ u-k+1,\ u-k+2, \ldots,\ u$; by the same argument as used in **Case I**, we have $\mathrm{span}_f(P^{u-k}_{k+1}) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$.

Finally we conclude that for any radio-k-labeling f of path P_n, $\mathrm{span}_f(P_n) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$ and hence $rc_k(P_n) \geq \lfloor \frac{k^2}{2} \rfloor + n - k$. □

Corollary 4.2. *For an n-vertex path P_n with even integer k,*

$$rc_k(P_n) \geq \frac{k^2}{2} + \min\left\{n - k, \frac{k}{2}\right\}.$$

Proof. The value of $\min\left\{n - k, \frac{k}{2}\right\}$ is $n - k$ or $\frac{k}{2}$ according as $n \leq \frac{3k}{2}$ and $n \geq \frac{3k}{2}$. Since $rc_k(P_n) \geq rc_k(P_m)$ for $n \geq m$, the result is follows from Theorem 4.1. □

Corollary 4.3. *For an n-vertex path P_n and odd integer k, $rc_k(P_n) \geq \frac{k^2-1}{2} + \min\left\{n - k, \frac{k+1}{2}\right\}$.*

Proof. The value of $\min\left\{n - k, \frac{k+1}{2}\right\}$ is $n-k$ or $\frac{k+1}{2}$ according as $n \leq \frac{3k+1}{2}$ and $n \geq \frac{3k+1}{2}$. Since $rc_k(P_n) \geq rc_k(P_m)$ for $n \geq m$, the result follows from Theorem 4.1. □

Theorem 4.4. *Let P_n be a path of order n with even integer k. If $n \geq \frac{3k}{2} + 2$, then $rc_k(P_n) \geq \frac{k^2}{2} + \frac{k}{2} + 1$.*

Proof. We have $rc_k(P_n) \geq rc_k(P_m)$ for $n \geq m$. Thus we prove this theorem for $n = \frac{3k}{2}+2$. Let f be an optimal radio k-labeling of path P_n, where $n = \frac{3k}{2} + 2$. Also let the minimum color and the maximum color (say, m and M, respectively) assigned by f for the path $P_n :\ 0,\ 1,\ 2, \ldots,\ n-1$ are attained at vertices u and v of the path P_n, respectively. Now we consider the following cases depending on the positions of u and v in the path P_n.

Case I: u or $v \in \{0,\ 1,\ 2, \ldots,\ \frac{k}{2}\}$. If $u \in \{0,\ 1, \ldots,\ \frac{k}{2}\}$, then we can construct a sub-path $P^u_{k+1} :\ u,\ u+1,\ u+2, \ldots,\ u+k$ of length k as in Fig. 2. Since $u \leq \frac{k}{2}$, we have $u+k \leq n-2$. Hence the sub-path P^u_{k+1} always exist in this case. Let C_u $(= u+\frac{k}{2})$ be the weight center of path P^u_{k+1}. So $d(C_u, u) = \frac{k}{2}$ and $d(C_u, v) \geq 0$. If $d(C_u, v) \geq 1$, then applying Theorem 3.5 to path P^u_{k+1}, we have $\mathrm{span}_f(P^u_{k+1}) \geq \frac{k^2}{2} + \frac{k}{2} + 1$ and hence the result follows. Otherwise, $d(C_u, v) = 0$. Therefore the maximum color M attains at C_u, weight center of path P^u_{k+1}. Now we construct an another sub-path $P^{u+1}_{k+1} :\ u+1,\ u+2,\ u+3, \ldots,\ u+k+1$ of length k with starting vertex $u+1$ of path P_n (see Fig. 2). For this sub-path the weight center, denoted by C_{u+1}, is the vertex $u + \frac{k}{2} + 1$ and the maximum color attains at the vertex $u + \frac{k}{2} = v$. Since the minimum colored vertex u of path P_n is not in the sub-path P^{u+1}_{k+1}, so let m' be the minimum color assigned by f for the sub-path P^{u+1}_{k+1} and m' attains at the vertex $u' \in V(P^{u+1}_{k+1})$, say. Here obviously $m' \geq m$. We now consider the following two subcases:

Subcase (a): u' is in right half of C_{u+1}. Let $u' = u + \frac{k}{2} + 1 + t$ with $0 \leq t \leq \frac{k}{2}$. Then $d(u, u') = t + \frac{k}{2} + 1$ and $d(C_{u+1}, u') = t$. The color difference of the vertices u and u' is

$m' - m$. From the radio k-labeling condition $m' - m \geq \frac{k}{2} - t$. Thus applying Theorem 3.5 to path P_{k+1}^{u+1}, we have $\text{span}_f(P_{k+1}^{u+1}) \geq \frac{k^2}{2} + d(C_{u+1}, u') + d(C_{u+1}, v) + m' \geq \frac{k^2}{2} + \frac{k}{2} + 1$.

Subcase (b): u' is in left half of C_{u+1}. Let $u' = u + \frac{k}{2} + 1 - t$ with $2 \leq t \leq \frac{k}{2}$. Then $d(u, u') = \frac{k}{2} + 1 - t$ and $d(C_{u+1}, u') = t$. The color difference between u and u' is $m' - m$. From the radio k-labeling condition $m' - m \geq \frac{k}{2} + t$. Thus applying Theorem 3.5 to path P_{k+1}^{u+1}, we have $\text{span}_f(P_{k+1}^{u+1}) \geq \frac{k^2}{2} + d(C_{u+1}, u') + d(C_{u+1}, v) + m' \geq \frac{k^2}{2} + \frac{k}{2} + 2t + 1$.

Thus if $u \in \{0, 1, 2, \ldots, \frac{k}{2}\}$, there always exist a sub-path P of length k with $\text{span}_f(P) \geq \frac{k^2}{2} + \frac{k}{2} + 1$.

If $v \in \{0, 1, 2, \ldots, \frac{k}{2}\}$, then for the sub-path $P_{k+1}^{v} : v, v+1, v+2, \ldots, v+k$, by the same argument, one can easily prove the required result.

Case II: Both $u, v \in \{\frac{k}{2} + 1, \frac{k}{2} + 2, \ldots, k\}$. Construct two sub-paths $P_{k+1}^{0} : 0, 1, \ldots, k$ and $P_{k+1}^{\frac{k}{2}+1} : \frac{k}{2} + 1, \frac{k}{2} + 2, \ldots, \frac{3k}{2} + 1$. Here the weight centers of P_{k+1}^{0} and $P_{k+1}^{\frac{k}{2}+1}$ are at

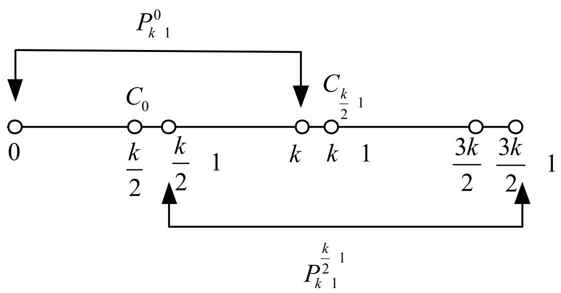

Figure 4. The sub-paths P_{k+1}^{0} and $P_{k+1}^{\frac{k}{2}+1}$.

the vertices $\frac{k}{2}$ and $k+1$ of P_n, respectively. Let us denote these weight centers by C_0 and $C_{\frac{k}{2}+1}$, respectively. In this case both u and v are in P_{k+1}^{0} as well as $P_{k+1}^{\frac{k}{2}+1}$. Here $d(C_0, u) = d(\frac{k}{2}, u) = u - \frac{k}{2}$; $d(C_0, v) = d(\frac{k}{2}, v) = v - \frac{k}{2}$; $d(C_{\frac{k}{2}+1}, u) = d(k+1, u) = k + 1 - u$ and $d(C_{\frac{k}{2}+1}, v) = d(k+1, v) = k + 1 - v$. Then applying Theorem 3.5 to the paths P_{k+1}^{0} and $P_{k+1}^{\frac{k}{2}+1}$ each of length k, we have

$$\text{span}_f(P_{k+1}^{0}) \geq \frac{k^2}{2} + u + v - k \quad \text{and} \quad \text{span}_f\left(P_{k+1}^{\frac{k}{2}+1}\right) \geq \frac{k^2}{2} + 2(k+1) - u - v.$$

By simple calculations, one can easily prove that $\max\{u + v - k,\, 2(k+1) - u - v\} \geq \frac{k}{2} + 1$. Since $\text{span}_f(P_n) \geq \max\left\{\text{span}_f(P_{k+1}^{0}),\, \text{span}_f(P_{k+1}^{\frac{k}{2}+1})\right\}$, therefore

$$\text{span}_f(P_n) \geq \frac{k^2}{2} + \max\{u + v - k, 2(k+1) - u - v\} \geq \frac{k^2}{2} + \frac{k}{2} + 1.$$

Case III: u or $v \in \{k+1,\, k+2, \ldots,\, \frac{3k}{2} + 1\}$. This case is similar to **Case I** if we reverse the vertex labeling of path P_n by the operation $j = \frac{3k}{2} + 1 - i$, $0 \leq i \leq \frac{3k}{2} + 1$. This completes the proof of the theorem. □

Theorem 4.5. *Let P_n be a path of order n with odd integer k. If $n \geq \frac{5k+1}{2}$, then $rc_k(P_n) \geq \frac{k^2+k}{2} + 1$.*

Proof. This theorem can be prove by similar argument as given in Theorem 4.4. □

Remark 4.6. The existing lower bound of $rc_k(P_n)$ is $\frac{k^2+4}{2}$ for even integer k and $\frac{k^2+1}{2}$ for odd integer k (see, [14]). But the results presented in Section 4 gives an improved lower bound of $rc_k(P_n)$ for even integer k as $\frac{k^2}{2} + \min\{n-k, \frac{k}{2}\}$ or $\frac{k^2}{2} + \frac{k}{2} + 1$ according as $n \leq \frac{3k}{2}$ and $n \geq \frac{3k}{2} + 2$. For odd integer k, improved lower bound presented here as $\frac{k^2-1}{2} + \min\{n-k, \frac{k+1}{2}\}$ or $\frac{k^2+k}{2} + 1$ according as $n \leq \frac{5k-1}{2}$ and $n \geq \frac{5k+1}{2}$.

5. Radio k-chromatic number of P_n when k is even and $n \leq \frac{5k}{2} + 2$

In this section, we give the exact value of $rc_k(P_n)$ when k is even and $n \leq \frac{5k}{2} + 2$.

Theorem 5.1. *For an n-vertex path P_n and an even integer k,*

$$rc_k(P_n) = \begin{cases} \frac{k^2}{2} + n - k & \text{if } n \leq \frac{3k}{2}, \\ \frac{k^2}{2} + \frac{k}{2} & \text{if } n = \frac{3k}{2} + 1, \\ \frac{k^2}{2} + \frac{k}{2} + 1 & \text{if } \frac{3k}{2} + 2 \leq n \leq \frac{5k}{2} + 2. \end{cases}$$

Proof. Let $V(P_n) = \{0, 1, 2, \ldots, n-1\}$ be the vertex set of an n-vertex path P_n. By Theorems 4.1 and 4.4 with Corollary 4.2, we have

$$rc_k(P_n) \geq \begin{cases} \frac{k^2}{2} + n - k & \text{if } n \leq \frac{3k}{2}, \\ \frac{k^2}{2} + \frac{k}{2} & \text{if } n = \frac{3k}{2} + 1, \\ \frac{k^2}{2} + \frac{k}{2} + 1 & \text{if } \frac{3k}{2} + 2 \leq n \leq \frac{5k}{2} + 2. \end{cases} \tag{5.1}$$

To prove the equality we have to give an optimal k-labeling with this required span. To define optimal k-labelings we consider the following three cases depending on the values of n as stated in this theorem.

Case I: $n \leq \frac{3k}{2}$. Let $k = 2p$ and ℓ be a positive integer such that $k + \ell = n - 1$ with $0 < \ell < p$. Define a mapping $f : V(P_n) \to \{0, 1, 2, \ldots\}$ as follows:

$$\begin{aligned} f(i) &= p + 1 + i(2p+1), \ 0 \leq i \leq \ell - 2; \\ f(\ell - 1) &= 2p^2 - p + \ell + 1; \\ f(\ell + j) &= p + 2 + (\ell + j - 1)(2p + 1), \ 0 \leq j \leq p - \ell - 1; \\ f(p + m) &= m(2p + 1), \ 0 \leq m \leq \ell - 1; \\ f(p + \ell) &= 2p^2 + \ell + 1; \\ f(p + \ell + 1 + t) &= (\ell + t)(2p + 1) + 1, \ 0 \leq t \leq p - \ell - 1; \\ f(2p + 1 + r) &= p + r(2p + 1), \ 0 \leq r \leq \ell - 1. \end{aligned}$$

One can easily show that f satisfies the radio k-labeling condition. Thus we have $\text{span}_f(P_n) = 2p^2 + \ell + 1 = \frac{k^2}{2} + n - k$.

Case II: $n = \frac{3k}{2} + 1$. In this case we define a mapping $f : V(P_n) \to \{0, 1, 2, \ldots\}$ as follows:

$$\begin{aligned} f(i) &= \frac{k}{2}+1+i(k+1),\ 0\le i\le \frac{k}{2}-1;\\ f\left(\frac{k}{2}+j\right) &= j(k+1),\ 0\le j\le \frac{k}{2};\\ f(k+\ell+1) &= \frac{k}{2}+\ell(k+1),\ 0\le \ell\le \frac{k}{2}-1. \end{aligned}$$

It is easy to show that f satisfies the radio-k-labeling condition. Here clearly $\text{span}_f(P_n)=\frac{k^2}{2}+\frac{k}{2}$.

Case III: $\frac{3k}{2}+2\le n\le \frac{5k}{2}+2$. Define a mapping $f: V(P_n)\to\{0,1,2,\ldots\}$ as follows:

$$\begin{aligned} f(i) &= \frac{k}{2}+2+i(k+1),\ 0\le i\le \frac{k}{2}-1;\\ f\left(\frac{k}{2}+j\right) &= j(k+1)+1,\quad 0\le j\le \frac{k}{2};\\ f(k+\ell+1) &= f(\ell)-1,\quad 0\le \ell\le \frac{k}{2}-1;\\ f\left(\frac{3k}{2}+m\right) &= f\left(\frac{k}{2}+m-1\right)-1,\ 1\le m\le n-1-\frac{3k}{2}. \end{aligned}$$

It is easy to show that f satisfies the radio-k-labeling condition. Thus we have $\text{span}_f(P_n)=\frac{k^2}{2}+\frac{k}{2}+1$. This completes the proof of the theorem. □

Example 5.2. An optimal radio 8-labeling of P_{20} and radio 14-labeling of P_{19} have given in Fig. 5 and Fig. 6, respectively.

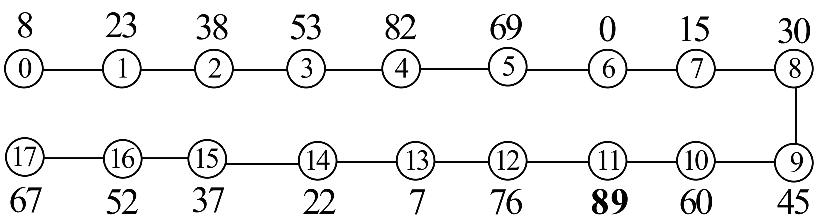

Figure 5. A radio 8-labeling of P_{20} with span 37.

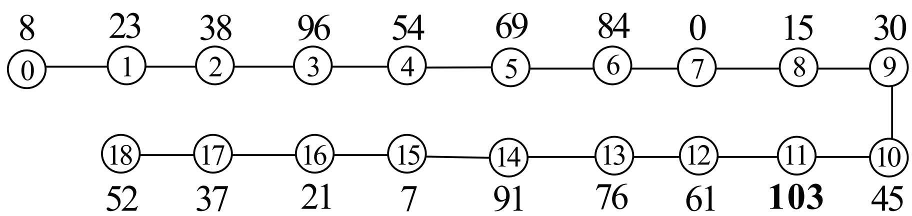

Figure 6. A radio 14-labeling of P_{19} with span 103.

6. Radio k-chromatic number of P_n when k is odd and $k+2 \le n \le \frac{7k-1}{2}$

In this section, we give the exact value of $rc_k(P_n)$ when k is odd and $k+2 \le n \le \frac{7k-1}{2}$.

Theorem 6.1. *For an n-vertex path P_n and an odd integer k,*

$$rc_k(P_n) = \begin{cases} \frac{k^2-1}{2} + n - k & \text{if } k+2 \le n \le \frac{3k-1}{2}, \\ \frac{k^2+k}{2} & \text{if } \frac{3k+1}{2} \le n \le \frac{5k-1}{2}, \\ \frac{k^2+k}{2} + 1 & \text{if } \frac{5k+1}{2} \le n \le \frac{7k-1}{2}. \end{cases}$$

Proof. From Corollary 4.3 and Theorem 4.5, we have the following:

$$rc_k(P_n) \ge \begin{cases} \frac{k^2-1}{2} + n - k & \text{if } k+2 \le n \le \frac{3k-1}{2}, \\ \frac{k^2+k}{2} & \text{if } \frac{3k+1}{2} \le n \le \frac{5k-1}{2}, \\ \frac{k^2+k}{2} + 1 & \text{if } n \ge \frac{5k+1}{2}. \end{cases} \tag{6.1}$$

To prove equality in (6.1), we have to define a radio k-labeling with the span as specified in this theorem. For the cases $\frac{3k+1}{2} \le n \le \frac{5k-1}{2}$ and $\frac{5k+1}{2} \le n \le \frac{7k-1}{2}$, it is sufficient to show that there exist radio k-labelings f and g for the paths $P_{\frac{5k-1}{2}}$ and $P_{\frac{7k-1}{2}}$ with spans $\frac{k^2+k}{2}$ and $\frac{k^2+k}{2}+1$, respectively (because these are the lower bounds and $rc_k(P_n) \ge rc_k(P_m)$ for $n \ge m$).

Let the vertex set of an n-vertex path P_n be $V(P_n) = \{0, 1, \ldots, n-1\}$. Also let $k = 2p+1$ and ℓ be a positive integer such that $k + \ell = n-1$. We consider the following three cases:

Case I: $k+2 \le n \le \frac{3k-1}{2}$. Since $n \le \frac{3k-1}{2}$, we have $\ell < p$. Define a mapping $f : V(P_n) \to \{0, 1, 2, \ldots\}$ as follows:

$$\begin{aligned} f(i) &= p+2+i(2p+3),\ 0 \le i \le \ell-1; \\ f(\ell) &= 2p^2+p+\ell; \\ f(\ell+1+j) &= p+3+(\ell+j)(2p+3),\ 0 \le j \le p-\ell-2; \\ f(p+m) &= m(2p+3),\ 0 \le m \le \ell; \\ f(p+\ell+1) &= 2p^2+2p+\ell+1; \\ f(p+\ell+2+t) &= (\ell+t+1)(2p+3)+1,\ 0 \le t \le p-\ell-2; \\ f(2p+1+r) &= p+1+r(2p+3),\ 0 \le r \le \ell. \end{aligned}$$

One can easily check that f satisfies the radio k-labeling condition. Thus we have $\mathrm{span}_f(P_n) = 2p^2 + 2p + \ell + 1 = \frac{k^2-1}{2} + n - k$ as $k = 2p + 1$.

Case II: $\frac{3k+1}{2} \leq n \leq \frac{5k-1}{2}$. As discuss above it is sufficient to define a radio k-labeling of P_n only for $n = \frac{5k-1}{2}$. We construct a radio k-labeling f of $P_{\frac{5k-1}{2}}$ as follows:

$$
\begin{aligned}
f(i) &= \frac{k+5}{2} + i(k+2),\ 0 \leq i \leq \frac{k-3}{2};\\
f\left(\frac{k-1}{2} + j\right) &= j(k+2) + 1,\ 0 \leq j \leq \frac{k-1}{2};\\
f(k+\ell) &= \frac{k+3}{2} + \ell(k+2),\ 0 \leq \ell \leq \frac{k-3}{2};\\
f\left(\frac{3k-1}{2} + m\right) &= m(k+2),\ 0 \leq m \leq \frac{k-1}{2};\\
f(2k+p) &= \frac{k+1}{2} + p(k+2),\ 0 \leq p \leq \frac{k-3}{2}.
\end{aligned}
$$

It is easy to see that f satisfy the radio-k-labeling condition. Here clearly $\mathrm{span}_f(P_n) = f(k-1) = \frac{k^2+k}{2}$. It is also noted that this radio k-labeling scheme will work for any path P_n with $\frac{3k+1}{2} \leq n \leq \frac{5k-1}{2}$.

Case III: $\frac{5k+1}{2} \leq n \leq \frac{7k-1}{2}$. As discuss above it is sufficient to define a radio k-labeling of P_n only for $n = \frac{7k-1}{2}$. We construct a radio k-labeling f of $P_{\frac{7k-1}{2}}$ as follows:

$$
\begin{aligned}
f(i) &= \frac{k+7}{2} + i(k+2),\ 0 \leq i \leq \frac{k-3}{2};\\
f\left(\frac{k-1}{2} + j\right) &= j(k+2) + 2,\ 0 \leq j \leq \frac{k-1}{2};\\
f(k+\ell) &= \frac{k+5}{2} + \ell(k+2),\ 0 \leq \ell \leq \frac{k-3}{2};\\
f\left(\frac{3k-1}{2} + m\right) &= m(k+2) + 1,\ 0 \leq m \leq \frac{k-1}{2};\\
f(2k+p) &= \frac{k+3}{2} + p(k+2),\ 0 \leq p \leq \frac{k-3}{2};\\
f\left(\frac{5k-1}{2} + q\right) &= q(k+2),\ 0 \leq q \leq \frac{k-1}{2};\\
f(3k+r) &= \frac{k+1}{2} + r(k+2),\ 0 \leq r \leq \frac{k-3}{2}.
\end{aligned}
$$

Here clearly $\mathrm{span}_f(P_n) = f(k-1) = \frac{k^2+k}{2} + 1$. It is easy to show that f satisfy the radio-k-labeling condition. As maximum color attained at $(k-1)th$ vertex, so this radio k-labeling scheme will work for any path P_n with $\frac{5k+1}{2} \leq n \leq \frac{7k-1}{2}$. □

Example 6.2. An optimal radio 13-labeling of P_{18} and radio 9-labeling of P_{20} has been given in Fig. 7 and Fig. 8, respectively.

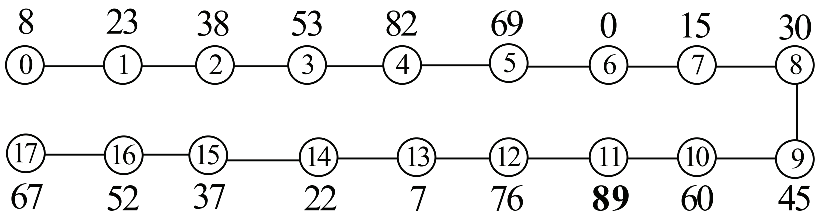

Figure 7. A radio 13-labeling of P_{18} with span 89.

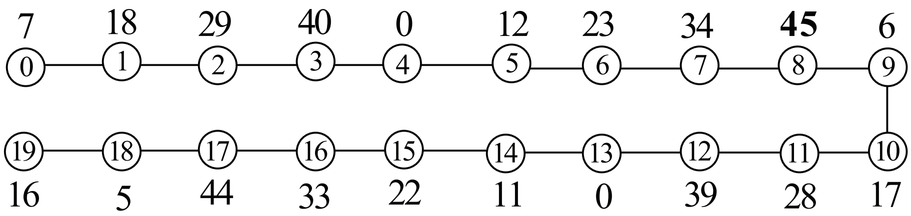

Figure 8. A radio 9-labeling of P_{20} with span 45.

7. Concluding Remark

Consequences of Theorem 5.1 and Theorem 6.1 include the radio k-chromatic number of P_n for $k \in \{n-4, n-3, n-2, n-1\}$ (which were settled in [5, 17–19, 25] by different approaches). Not only that these theorem determines the radio k-chromatic number of P_n for even integer $k \in \left\{\left\lceil \frac{2(n-2)}{5} \right\rceil, \ldots, n-1\right\}$ and odd integer $k \in \left\{\left\lceil \frac{2n+1}{7} \right\rceil, \ldots, n-1\right\}$ that is at least 65% cases the radio k-chromatic of the path P_n are obtained for fixed but arbitrary values of n. For example, if we take $n = 1000$, then this article determines the exact value of $rc_k(P_{1000})$ for even $k \in \{400, 402, \ldots, 998\}$ and odd $k \in \{287, 289, \ldots, 999\}$ where as the existing results are only for $k \in \{997, 998, 999\}$.

Acknowledgment. The first author is thankful to the National Board for Higher Mathematics (NBHM), India for its financial support (Grant No. 2/48(22)/R & D II/4033). The third author is supported by the National Research Foundation of the Korean government with grant No. 2017R1D1A1B03028642.

References

[1] G.J. Chang and D. Kuo, *The $L(2,1)$-labelling problem on graphs*, SIAM J. Discrete Math. **9**, 309–316, 1996.

[2] G.J. Chang, C. Lu, and S. Zhou, *Distance-two labellings of Hamming graphs*, Discrete Appl. Math. **157**, 1896–1904, 2009.

[3] G. Chartrand, D. Erwin, and P. Zhang, *Radio antipodal colorings of cycles*, Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000) **144**, 129–141, 2000.

[4] G. Chartrand, D. Erwin, and P. Zhang, *A graph labeling problem suggested by FM channel restrictions*, Bull. Inst. Combin. Appl. **43**, 43–57, 2005.

[5] G. Chartrand, L. Nebesky, and P. Zhang, *Radio k-colorings of paths*, Discuss. Math. Graph. Theory, **24**, 5–21, 2004.

[6] G. Chartrand, D. Erwin, F. Harary, and P. Zhang, *Radio labelings of graphs*, Bull. Inst. Combin. Appl. **33**, 77–85, 2001.

[7] J.P. Georges, D.W. Mauro, and M.I. Stein, *Labelling products of complete graphs with a condition at distance two*, SIAM J. Discrete Math. **14**, 28–35, 2001.

[8] J.R. Griggs and X.T. Jin, *Real number graph labelling with distance conditions*, SIAM J. Discrete Math. **20**, 302–327, 2006.

[9] J.R. Griggs, and D. Král', *Graph labellings with variable weights*, a survey, Discrete Appl. Math. **157**, 2646–2658, 2009.

[10] J.R. Griggs and R.K. Yeh, *Labelling graphs with a condition at distance 2*, SIAM J. Discrete Math. **5**, 586–595, 1992.

[11] W.K. Hale, *Frequency assignment, theory and application*, Proc. IEEE, **68**, 1497–1514, 1980.

[12] J.v.d. Heuvel, R. Leese, and M. Shepherd, *Graph labelling and radio channel assignment*, J. Graph Theory, **29**, 263–283, 1998.

[13] J.S.-T. Juan and D.D.-F. Liu, *Antipodal labelings for cycles*, Ars Combin. **103**, 81–96, 2012.

[14] M. Kchikech, R. Khennoufa, and O. Togni, *Linear and cyclic radio k-labelings of trees*, Discuss. Math. Graph Theory, **27** (1), 105–123, 2007.

[15] G. Chartrand, D. Erwin, F. Harary, and P. Zhang, *Radio labelings of graphs*, Bull. Inst. Combin. Appl. **33**, 77–85, 2001.

[16] R. Khennoufa and O. Togni, *The radio antipodal and radio numbers of the hypercube*, Ars Combin. **102**, 447–461, 2011.

[17] R. Khennoufa and O. Togni, *A note on radio antipodal colourings of paths*, Math. Bohem. **130** (3), 277–282, 2005.

[18] S.R. Kola and P. Panigrahi, *Nearly antipodal chromatic number* $ac'(P_n)$ *of the path*, Math. Bohem. **134** (1), 77–86, 2009.

[19] S.R. Kola and P. Panigrahi, *On radio* $(n-4)$*-chromatic number of the path* P_n, AKCE Int. J. Graphs Comb. **6** (1), 209–217, 2009.

[20] D. Král', *The channel assignment problem with variable weights*, SIAM J. Discrete Math. **20**, 690–704, 2006.

[21] X. Li, V. Mak, and S. Zhou, *Optimal radio labellings of complete m-ary trees*, Discrete Appl. Math. **158**, 507–515, 2010.

[22] D.D.-F. Liu, *Radio number for trees*, Discrete Math. **308**, 1153–1164, 2008.

[23] D.D.-F. Liu and M. Xie, *Radio number for square of cycles*, Congr. Numer. **169**, 105–125, 2004.

[24] D.D.-F. Liu and M. Xie, *Radio number for square paths*, Ars Combin. **90**, 307–319, 2009.

[25] D.D.-F. Liu and X. Zhu, *Multi-level distance labelings for paths and cycles*, SIAM J. Discrete Math. **19**, 610–621, 2005.

[26] M. Morris-Rivera, M. Tomova, C. Wyels, and Y. Yeager, *The radio number of* $C_n \times C_n$, Ars Combin. **103**, 81–96, 2012.

[27] J.P. Ortiz, P. Martinez, M. Tomova, and C. Wyels, *Radio numbers of some generalized prism graphs*, Discuss. Math. Graph Theory, **31** (1), 45–62, 2011.

[28] V.S. Reddy and V.K. Iyer, *Upper bounds on the radio number of some trees*, Int. J. Pure Applied Math. **71** (2), 207–215, 2011.

[29] L. Saha and P. Panigrahi, *A graph Radio k-coloring algorithm*, Lecture notes in Computer Science **7643**, 125–129, 2012.

[30] L. Saha and P. Panigrahi, *On the Radio number of Toroidal grids*, Australian J. Combin. **55**, 273–288, 2013.

[31] L. Saha and P. Panigrahi, *A lower bound for radio k-chromatic number*, Discrete Appl. Math. **192**, 87–100, 2015.

[32] U. Sarkar and A. Adhikari, *On characterizing radio k-coloring problem by path covering problem*, Discrete Math **338** (4), 615–620, 2015.

[33] P. Zhang, *Radio labellings of cycles*, Ars Combin **65**, 21–32, 2002.

[34] S. Zhou, *A channel assignment problem for optical networks modelled by Cayley graphs*, Theoret. Comput. Sci. **310**, 501–511, 2004.

[35] S. Zhou, *Labelling Cayley graphs on abelian groups*, SIAM J. Discrete Math. **19**, 985–1003, 2006.
[36] S. Zhou, *A distance-labelling problem for hypercubes*, Discrete Appl. Math. **156**, 2846–2854, 2008.