



A fixed point theorem for (ϕ, ψ) -convex contraction in metric spaces

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Abstract

In the present paper, we introduce the notion of (ϕ, ψ) -convex contraction mapping of order m and establish a fixed point theorem for such mappings in complete metric spaces. The present result extends and generalizes the well known result of Dutta and Choudhary (Fixed Point Theory Appl. 2008 (2008), Art. ID 406368), Rhoades (Nonlinear Anal., 47(2001), 2683-2693), Istrăţescu (Ann. Mat. Pura Appl., 130(1982), 89-104) and besides many others in the existing literature. An illustrative example is also provided to exhibit the utility of our main results.

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1. Introduction

A mapping $f : X \rightarrow X$, where (X, d) is a metric space, is said to be a contraction mapping if for all $x, y \in X$,

$$d(fx, fy) \leq kd(x, y), \text{ where } 0 \leq k < 1. \quad (1)$$

the Banach contraction principle, which states that every contraction mapping on a complete metric space (X, d) has a unique fixed point, is one of the pivotal result in fixed point theory. This result has been used and generalized by several authors in different directions (see [1], [6, 7, 8, 9, 10], [12], [15], [19], [24] etc.).

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In 1997 Alber and Guerre-Delabariere [1] generalized the Banach contraction principle in setting of Hilbert spaces. Later Rhoades [22] shown that the results which Alber and Guerre-Delabariere have proved in [1] is also valid for arbitrary complete metric space.

Definition 1.1. [22]. A self mapping f on a metric space (X, d) is said to be weakly contractive mapping if for all $x, y \in X$

$$d(fx, fy) \leq d(x, y) - \phi(d(x, y)) \quad (2)$$

where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If one takes $\phi(t) = (1 - k)t$, where $0 \leq k < 1$, then weak contraction (2) reduces to a Banach contraction (1).

The following is the result of Rhoades proved in [22].

Theorem 1.1. If f is a weakly contractive mapping on a complete metric space (X, d) then f has a unique fixed point.

Inspired by the work of Alber and Guerre-Delabariere [1], and Rhoades [22], Dutta and Chaudhary [6] introduced a new generalization of Banach contraction principle.

Theorem 1.2. [6]. Let (X, d) be a complete metric space and let $f : X \rightarrow X$ be a self-mapping satisfying the inequality

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad (3)$$

where $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ are both continuous and monotonic non decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then f has a unique fixed point.

The condition (3) is known as (ϕ, ψ) -weak contraction condition and has been extended by several authors in various papers (see for instance [4, 5] [10, 11], [17], [19, 20, 21], [23, 25]).

On the other hand the study of convex contraction, which does not imply the contraction condition (1) but ensure the existence and uniqueness of the fixed point, was initiated by Istrăţescu (see [12, 13, 14]). In [13], Istrăţescu's introduced a convex contraction of order $m \in \mathbb{N}$ (set of natural numbers) and proved a fixed point theorem which states that every convex contraction of order m has a unique fixed point.

Definition 1.2. [13]. A continuous mapping self mapping f on a complete metric space (X, d) is said to be a convex contraction of order $m \in \mathbb{N}$ (set of natural numbers), if there exist positive numbers a_0, a_1, \dots, a_{m-1} in $(0, 1)$ such that $a_0 + a_1 + \dots + a_{m-1} < 1$ and for all $x, y \in X$,

$$d(f^{[m]}x, f^{[m]}y) \leq a_0d(x, y) + a_1d(fx, fy) + \dots + a_{m-1}d(f^{[m-1]}x, f^{[m-1]}y). \quad (4)$$

For $m = 1$, one can easily see that the convex contraction (4) reduces to Banach contraction (1) therefore Istrăţescu's fixed point theorem is an effective generalization of Banach contraction principle and has been extended by several authors (see for example [2, 3], [18]). Recently Misculescu and Mihali [16] generalized Istrăţescu fixed point theorem for concerning convex contraction in setting of b-metric spaces.

In this paper, motivated by the work of Misculescu and Mihali [16], Istrăţescu [13] and Dutta and Choudhary [6], we introduce a (ϕ, ψ) -convex contraction mapping of order $m \in \mathbb{N}$ and prove a fixed point theorem for such mappings in complete metric spaces. Our result extends and unifies the result of Dutta and Choudhary [6], Rhoades [22] and Istrăţescu [13]. We prove that the Istrăţescu's fixed point theorems (see Theorem 1.2 and Theorem 1.5 in [13]) concerning convex contraction is special case of our main result. Moreover, we also have an illustrative example which shows the validity and utility of our main result.

2. Main Result

In this section firstly, we introduce a (ϕ, ψ) -convex contraction mapping of order $m \in \mathbb{N}$.

Definition 2.1. Let (X, d) be a metric space and a mapping $f : X \rightarrow X$ is said to be a generalize (ϕ, ψ) -convex contraction mapping of order $m \in \mathbb{N}$, if there exist two continuous and monotonic non decreasing functions $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ such that $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$ and satisfies the following inequality

$$\psi(d(f^{[m]}x, f^{[m]}y)) \leq \psi(M_m(x, y)) - \phi(N_m(x, y)), \quad (5)$$

for all $x, y \in X$, where

$$M_m(x, y) = \max\{d(x, y), d(fx, fy), \dots, d(f^{[m-1]}x, f^{[m-1]}y)\}$$

and

$$N_m(x, y) = \min\{d(x, y), d(fx, fy), \dots, d(f^{[m-1]}x, f^{[m-1]}y)\}.$$

Now we state our main result.

Theorem 2.1. Every (ϕ, ψ) -weak convex contraction mapping of order $m \in \mathbb{N}$ in a complete metric space has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X . We construct a sequence $\{x_n\}$ by $x_n = f^{[n]}x_0$, $n = 1, 2, \dots$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of f . Thus we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ and $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. Substituting $x = f^{[n]}x_0$ and $y = f^{[n+1]}x_0$ in (5), we get

$$\psi(d(x_{n+m}, x_{n+m+1})) \leq \psi(M_m(x_n, x_{n+1})) - \phi(N_m(x_n, x_{n+1})) \quad (6)$$

where

$$M_m(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \dots, d(x_{n+m-1}, x_{n+m})\}$$

and

$$N_m(x_n, x_{n+1}) = \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \dots, d(x_{n+m-1}, x_{n+m})\}.$$

From (6), we get

$$\psi(d(x_{n+m}, x_{n+m+1})) \leq \psi(M_m(x_n, x_{n+1}))$$

which implies

$$d(x_{n+m}, x_{n+m+1}) \leq M_m(x_n, x_{n+1}) \quad (\text{by monotone property of } \psi \text{ function}).$$

It follows that the both sequences $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}}$ and $\{M_m(x_n, x_{n+1})\}_{n \in \mathbb{N}}$ are monotonic decreasing and therefore there exists $r \geq 0$ such that

$$d(x_n, x_{n+1}) \rightarrow r \text{ as } n \rightarrow \infty.$$

Making $n \rightarrow \infty$ in (6), we get

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (7)$$

Now we will prove that $\{x_n\}$ is a Cauchy sequence. If possible let $\{x_n\}$ is not a Cauchy sequence, then for each positive integer k , there exists $\epsilon > 0$ and subsequences $\{x_{p(k)}\}$ and $\{x_{q(k)}\}$ of $\{x_{n(k)}\}$ with $k < p(k) < q(k)$ such that

$$d(x_{p(k)}, x_{q(k)}) \geq \epsilon. \quad (8)$$

Let $q(k)$ be the least positive integer exceeds $p(k)$ and satisfies (8), for each positive integer k , then it clear that

$$\begin{aligned} \epsilon \leq d(x_{p(k)}, x_{q(k)}) &\leq d(x_{p(k)}, x_{q(k)-1}) + d(x_{q(k)-1}, x_{q(k)}) \\ &< \epsilon + d(x_{q(k)-1}, x_{q(k)}). \end{aligned} \quad (9)$$

Making $k \rightarrow \infty$ and using (9)

$$\lim_{k \rightarrow \infty} d(x_{p(k)}, x_{q(k)}) = \epsilon. \quad (10)$$

Again

$$d(x_{q(k)-1}, x_{p(k)-1}) \leq d(x_{q(k)-1}, x_{q(k)}) + d(x_{q(k)}, x_{p(k)}) + d(x_{p(k)}, x_{p(k)-1}).$$

Making $k \rightarrow \infty$ in the above inequality and using (7), (10), we get

$$\lim_{k \rightarrow \infty} d(x_{p(k)-1}, x_{q(k)-1}) = \epsilon.$$

Similarly we can prove that

$$\lim_{k \rightarrow \infty} d(x_{p(k)-m}, x_{q(k)-m}) = \epsilon, \quad m \in \mathbb{N}. \quad (11)$$

Now putting $x = x_{p(k)-m}$ and $y = x_{q(k)-m}$ in (5) and using (8) we obtain

$$\psi(\epsilon) \leq \psi(d(x_{p(k)}, x_{q(k)})) \leq \psi(M_m(x_{p(k)-m}, x_{q(k)-m}) - \phi(N_m(x_{p(k)-m}, x_{q(k)-m})))$$

Making $k \rightarrow \infty$ and utilizing (10) and (11), we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) \quad (12)$$

which is a contradiction if $\epsilon > 0$. This shows that $\{x_n\}$ is a Cauchy sequence and hence it is convergent in the complete metric space X . Let

$$\{x_n\} \rightarrow z \quad (\text{say}) \text{ as } n \rightarrow \infty.$$

Substituting $x = x_n$ and $y = z$ in (5), we obtain

$$\psi(d(x_{n+m}, fz)) \leq \psi(M_m(x_n, z) - \phi(N_m(x_n, z))).$$

Making $n \rightarrow \infty$ and using continuity of ϕ and ψ , we have

$$\psi(d(z, fz)) \leq \psi(0) - \phi(0) = 0, \quad (13)$$

which implies $\psi(d(z, fz)) = 0$, that is,

$$d(z, fz) = 0 \quad \text{or} \quad z = fz.$$

To prove the uniqueness of the fixed point, let us suppose that z_1 and z_2 are two fixed points of f . Putting $x = z_1$ and $y = z_2$ in (5), we get

$$\psi(d(f^{[m]}z_1, f^{[m]}z_2)) \leq \psi(M_m(z_1, z_2)) - \phi(N_m(z_1, z_2)) \quad (14)$$

where

$$M_m(z_1, z_2) = \max\{d(z_1, z_2), d(fz_1, fz_2), \dots, d(f^{[m-1]}z_1, f^{[m-1]}z_2)\}$$

and

$$N_m(z_1, z_2) = \min\{d(z_1, z_2), d(fz_1, fz_2), \dots, d(f^{[m-1]}z_1, f^{[m-1]}z_2)\}.$$

implies

$$\phi(d(z_1, z_2)) \leq 0,$$

From (14)

$$\psi(d(z_1, z_2)) \leq \psi(d(z_1, z_2)) - \phi(d(z_1, z_2))$$

or equivalently $d(z_1, z_2) = 0$, that is $z_1 = z_2$. This proves the uniqueness of the fixed point. \square

We observe that several corollaries may be derived from the above theorems by choosing value of m and functions ψ, ϕ suitably. For example, taking $m = 1$ in Theorem 2.1 we obtain the result of Dutta and Choudhary [6]. Similarly taking $m = 1$ and $\psi(t) = t, t \in [0, \infty)$, in Theorem 2.1 we get the result of Rhoades [22].

Example 2.1. Let $X = \{1, 2, 3, 4, 5\}$ be a non-empty set and d be a usual metric on X . Then (X, d) forms a complete metric space. We define a mapping

$$f(x) = \begin{cases} 2, & \text{for } x = 1, \\ 4, & \text{for } x = 2, \\ 5, & \text{for } x = \{3, 4, 5\}. \end{cases}$$

Then

$$f^2(x) = \begin{cases} 4, & \text{for } x = 1, \\ 5, & \text{for } x = \{2, 3, 4, 5\}. \end{cases}$$

Clearly f satisfies all the conditions of Theorem 2.1 for $\phi(t) = (1 - \psi)(t), \psi(t) = t, t \in [0, \infty)$ and $m = 2$. But f does not satisfy the condition (3) for $(x, y) = (1, 2)$ as $d(f(1), f(2)) = 2 > 1 = d(1, 2)$. Hence Theorem 2.1 is an effective generalization of Theorem 1.2.

Corollary 2.1 (Istrăţescu's fixed point theorem concerning convex contraction). Let $f : X \rightarrow X$ is a convex contraction of order $m \in \mathbb{N}$ in a complete metric space (X, d) then there exists a fixed point of f and this is unique.

Proof. Firstly we accept the following notations:

$$M_m(x, y) := \max\{d(x, y), d(fx, fy), \dots, d(f^{[m-1]}x, f^{[m-1]}y)\}$$

and

$$N_m(x, y) := \min\{d(x, y), d(fx, fy), \dots, d(f^{[m-1]}x, f^{[m-1]}y)\}$$

Now we have given that f is convex contraction of order m on complete metric space (X, d) , so there exist positive numbers a_0, a_1, \dots, a_{m-1} in $(0, 1)$ such that $a_0 + a_1 + \dots + a_{m-1} < 1$ and for all $x, y \in X$ satisfy

$$\begin{aligned} d(f^{[m]}x, f^{[m]}y) &\leq a_0d(x, y) + a_1d(fx, fy) + \dots + a_{m-1}d(f^{[m-1]}x, f^{[m-1]}y) \\ &\leq (a_0 + a_1 + \dots + a_{m-1})M_m(x, y) \\ &= (a_0 + a_1 + \dots + a_{m-1})M_m(x, y) + M_m(x, y) - M_m(x, y) \\ &= M_m(x, y) - (1 - (a_0 + a_1 + \dots + a_{m-1}))M_m(x, y) \end{aligned}$$

Considering $\phi(t) = (1 - (a_0 + a_1 + \dots + a_{m-1}))\psi(t)$ and $\psi(t) = t$ for all $t \in [0, \infty)$, we have

$$\psi(d(f^{[m]}x, f^{[m]}y)) \leq \psi(M_m(x, y)) - \phi(M_m(x, y)) \quad (15)$$

As we know that $N_m(x, y) \leq M_m(x, y)$ for $m \in \mathbb{N}$ so

$$\phi(N_m(x, y)) \leq \phi(M_m(x, y)) \quad (\text{using monotonicity of } \phi\text{-function})$$

In view of above inequality and from (15), we have

$$\psi(d(f^{[m]}x, f^{[m]}y)) \leq \psi(M_m(x, y)) - \phi(N_m(x, y))$$

Now applying Theorem (2.1), we get f has a unique fixed point. □

Conflict of Interest The authors declare that they have no conflict of interest.

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