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EXPONENTIATED GOMPERTZ EXPONENTIAL (EGoE) DISTRIBUTION: DERIVATION, PROPERTIES AND APPLICATIONS

Johnson Ademola Adewara *
Distance Learning Institute
University of Lagos
Akoka, Lagos, Nigeria.

John Sunday Adeyeye
Department of Mathematics
University of Lagos
Akoka, Lagos, Nigeria.

Mundher Abdullah Khaleel
Department of Mathematics
Faculty of Computer Science and Mathematics
University of Tikrit, Iraq.

Olubisi Lawrence Aako
Department of Mathematics
University of Lagos
Akoka, Lagos, Nigeria.

Abstract: In this paper, a new probability distribution called Exponentiated Gompertz Exponential distribution was introduced which can help researchers to model different types of data sets. In proposed distribution we introduce a new shape parameter to Gompertz Exponential distribution, varied its tail weight such that it enhances its flexibility and performance. Furthermore, the maximum likelihood method was used in estimating the model's parameters. Simulation method was used to investigate the behaviours of the parameters of the proposed distribution; the results showed that the mean square error and standard error for the chosen parameter values decrease as the sample size increases. The proposed distribution was tested on real life data, the results showed that EGoE performed better than the existing distribution in the literature and a strong competitor to other distributions of the same class. The results also showed that the distribution can be used as an alternative model in modelling lifetime processes.

Key words: Exponentiated gompertz exponential distribution, maximum Likelihood, means square error, quantile function, parameters

1. Introduction

The Gompertz distribution is a continuous probability distribution, named after Benjamen Gompertz. It is often used by demographers and actuaries to describe the distribution of adult life spans. It is a two parameter distribution that lies on support $[0, \infty]$. In the fields of Science and Biology, Gompertz distribution was used for survival analysis. This paper proposed a new continuous distribution called Exponentiated Gompertz Exponential distribution with increasing hazard rate. The proposed distribution added a shape parameter to the existing Gompertz Exponential distribution using the gompertz generalized family of distribution to enhance flexibility and better performance. There are so many generalized forms of Gompertz distribution in the literature. For instance, [1] extended Lomax distribution obtained by using Gompertz generalized family of distribution proposed by [3]. Also, [2] studied Gompertz Exponential distribution

* Corresponding author. E-mail address:jadewara@unilag.edu.ng

by extending exponential distribution using Gompertz family. [4] studied generalized Gompertz distribution by generalizing exponential and Gompertz distribution. The main advantage of [4] is that the shape of the hazard function could be increasing, decreasing, constant or bathtub depending on the value of the shape parameters thereby making it a suitable tool for reliability analysis. [5] introduced Beta Gompertz distribution which is quite flexible and can be used effectively in modelling survival data and reliability problems. Beta Gompertz distribution can have a decreasing, increasing, and bathtub-shaped failure rate function depending on its parameters.

In recent time, [8] proposed Gompertz flexible weibull distribution by extending the flexible weibull distribution using the gompertz generalized family of distributions proposed by [3] and used by [1]. The superiority of Gompertz flexible weibull distribution over Gompertz Weibull, Gompertz Burr type XII, Gompertz Lomax, exponentiated flexible weibull, exponentiated flexible weibull extension and Kumaraswamy flexible weibull distributions was demonstrated through its application to real data sets.

The remaining part of this paper is organized as follows: In section 2, the densities of Exponentiated Gompertz Exponential distribution which will henceforth be referred to as EGoE distribution are derived, its statistical properties like reliability function, distribution of order statistics, quantile function, mode, mean and variance (in integral form) are obtained ; including the estimation of the unknown parameters. In section 3, a simulation study was carried out to assess the performance of the unknown parameters of EGoE distribution. Applications to real data sets are provided in section 4 while concluding remark is provided in section 5.

2. The Exponentiated Gompertz Exponential (EGoE) distribution

The cdf of a random variable X from Exponentiated Gompertz Exponential distribution is derived by raising the cdf of Gompertz Exponential distribution to a shape parameter α. The associated expression is given as

$$F(x) = \left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x \gamma}\right]}\right\}^{\alpha} x > 0, \;\; \theta > 0\,, \; \gamma > 0\,, \; \lambda > 0, \;\; \alpha > 0. \tag{2.1}$$

where θ, λ and γ are the shape parameters. The resulting plot is shown in figure 1.

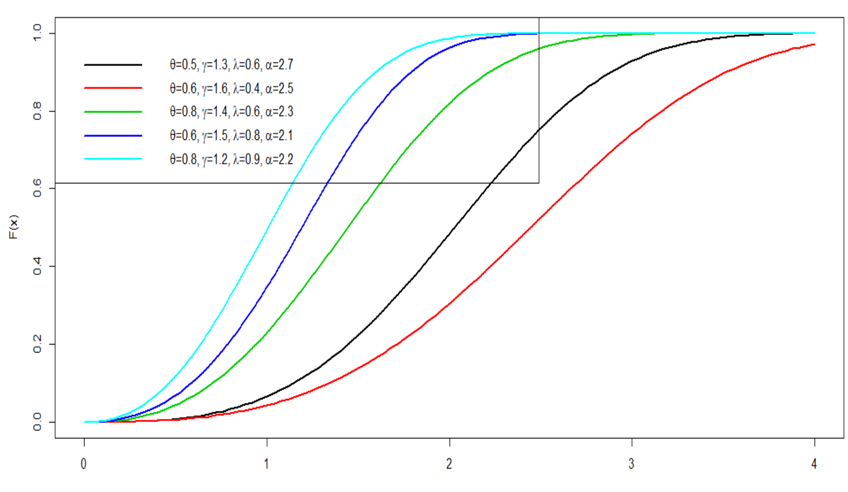

FIGURE 1. The plot of cumulative distribution function of EGoE distribution

Its corresponding pdf is obtained by differentiating equation (2.1) with respect to x

$$f\left(x\right)=\alpha\theta\lambda e^{\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}x>0,\ \ \theta>0\ ,\ \gamma>0\ ,\ \lambda>0,\ \ \alpha>0 \tag{2.2}$$

The expansion of the pdf will be of the form

$$f\left(x\right)=\alpha\theta\lambda\sum_{i,j,s=0}^{\infty}\frac{\left(-1\right)^{i+s}}{j!}\binom{\alpha-1}{i}\binom{j}{s}\left(\frac{i\theta}{\gamma}+\frac{\theta}{\gamma}\right)^{j}e^{(s+1)\lambda x\gamma}\ . \tag{2.3}$$

and its probability density function shown in figure 2.

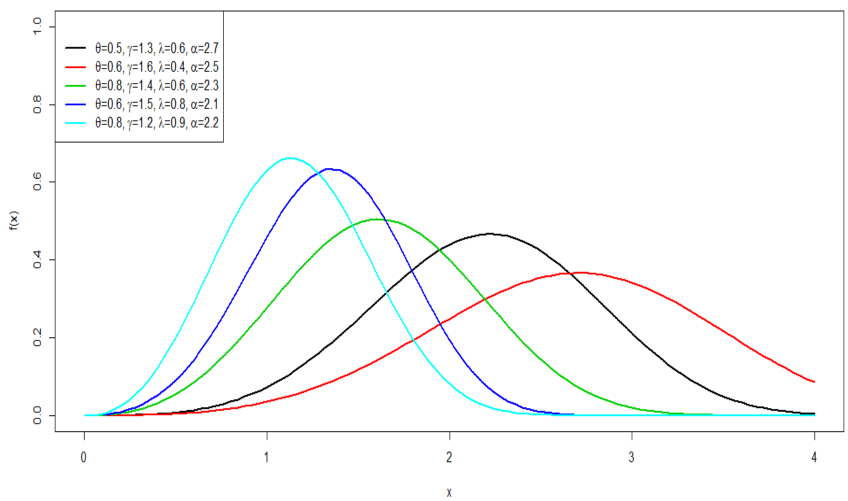

FIGURE 2. Plot of the probability density function of EGoE distribution.

Reliability analysis

The reliability analysis of EGoE distribution discussed in this sub – section are survival function and hazard function. Survival function is the probability that a system will survive beyond a specified time while hazard function also known as failure rate can be interpreted as the conditional probability of failure, given it has survived to time t. Survival and hazard functions are very important in Biological sciences for survival analysis and engineering for reliability analysis.

Reliability or survival function can be obtained mathematically as the complement of the cumulative density function (cdf) as follows:

$$S\left(x\right)=1-F\left(x\right) \tag{2.4}$$

Therefore, the reliability function of EGoE distribution is given by

$$S\left(x\right)=1-\left\{1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha};x>0,\ \ \theta>0\ ,\ \gamma>0\ ,\ \lambda>0,\ \alpha>0 \tag{2.5}$$

Hazard function can be obtained from

$$h\left(x\right)=\frac{f\left(x\right)}{S\left(x\right)}. \tag{2.6}$$

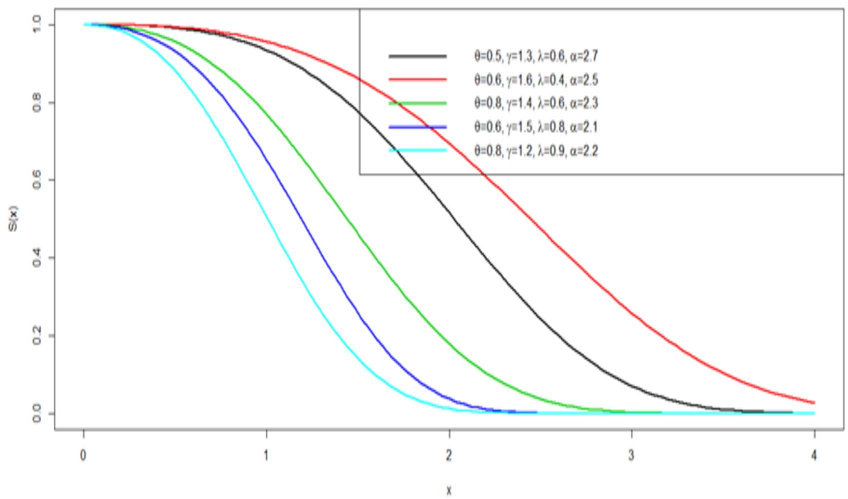

FIGURE 3. Plot of the survival function of EGoE distribution.

Therefore the hazard function of EGoE distribution is given by

$$h(x) = \frac{\alpha\theta\lambda e^{\lambda x\gamma} e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}}{1-\left\{1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}}, x>0, \ \theta>0, \ \gamma>0, \ \lambda>0, \ \alpha>0. \tag{2.7}$$

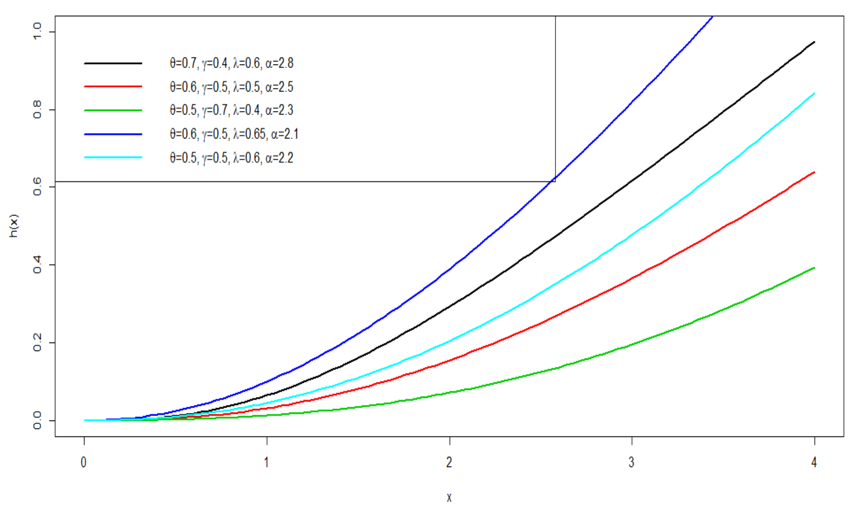

FIGURE 4. Plot of the hazard function of EGoE distribution.

Quantile function and median of EGoE distribution

In this sub-section, quantile and median of EGoE distribution are derived as follows

The quantile X_q of a random variable from EGoE distribution (θ , γ , λ, α) random variable X is given by

$$(X_q)_{EGoE} = \frac{1}{\lambda\gamma}\ln\left[1 - \frac{\gamma}{\theta}ln\left(1 - q^{\frac{1}{\alpha}}\right)\right]$$

The final form of the quantile function of Exponentiated Gompertz Exponential distribution can be written as

$$(X_q)_{EGoE} = \frac{1}{\lambda\gamma}\ln\left[1 - \frac{\gamma}{\theta}ln\left(1 - q^{\frac{1}{\alpha}}\right)\right] \tag{2.8}$$

The median of EGoE can be derived from equation (2.8) by setting $q = 0.5$
Therefore,

$$median = \frac{1}{\lambda\gamma}\ln\left[1 - \frac{\gamma}{\theta}ln\left(1 - (0.5)^{\frac{1}{\alpha}}\right)\right] \tag{2.9}$$

The mode

The mode of EGoE distribution can be derived by first differentiating its probability density function with respect to x and then equating the resulting derivative to zero.

$$\left[\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}\left[\lambda\left(\gamma - \theta e^{2\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right)\right] + \left[\theta\lambda e^{2\lambda x\gamma}e^{\frac{2\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]\right] = 0. \tag{2.10}$$

gives the mode of EGoE distribution.
It should be noted that the non -linear equation (2.10) does not have analytical solution in x but can be solved numerically when data sets are available with the use of statistical packages.

Order statistics

The pdf of the j_{th} order statistics of the EGoE distribution is

$$\begin{gathered} f_{j:n}(x) = \tfrac{n!}{(j-1)!(n-j)!}\alpha\theta\lambda e^{\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1} \\ \left[\left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{j-1}\left[1 - \left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{n-j} \end{gathered} \tag{2.11}$$

The distributions of minimum and maximum order statistics for the Exponentiated Gompertz Exponential distribution are given below.

$$f_{1:n}(x) = n\alpha\theta\lambda e^{\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}\left[1 - \left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{n-1}. \tag{2.12}$$

$$f_{n:n}(x) = n\alpha\theta\lambda e^{\lambda x\gamma}e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}\left[\left\{1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right\}^{\alpha}\right]^{n-1}. \tag{2.13}$$

Moments of EGoE distribution

Let X be a random variable that has the EGoE distribution, then, the r^{th} non-central moments is given by

$$E(X^r) = \int_0^\infty x^r f(x)\,dx \tag{2.14}$$

$$E(X^r) = \int_0^\infty x^r\alpha\theta\lambda\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\binom{\alpha-1}{i}\binom{j}{s}\left(\frac{i\theta}{\gamma}+\frac{\theta}{\gamma}\right)^j e^{(s+1)\lambda x\gamma}\,dx$$

$$= \alpha\theta\lambda\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\binom{\alpha-1}{i}\binom{j}{s}\left(\frac{i\theta}{\gamma}+\frac{\theta}{\gamma}\right)^j\int_0^\infty x^r e^{(s+1)\lambda x\gamma}\,dx.$$

The first moment ($r=1$) which is the mean of the distribution is given by

$$E\left(X\right)=\alpha\theta\lambda\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}\begin{pmatrix}j\\ s\end{pmatrix}\left(\frac{i\theta}{\gamma}+\frac{\theta}{\gamma}\right)^{j}\int_{0}^{\infty}xe^{(s+1)\lambda x\gamma}\,dx. \tag{2.15}$$

The second moment ($r=2$)is given by

$$E\left(X^{2}\right)=\alpha\theta\lambda\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}\begin{pmatrix}j\\ s\end{pmatrix}\left(\frac{i\theta}{\gamma}+\frac{\theta}{\gamma}\right)^{j}\int_{0}^{\infty}x^{2}e^{(s+1)\lambda x\gamma}\,dx. \tag{2.16}$$

Thus, the variance of EGoE distribution using Equations (2.15) and (2.16) is given by

$$Var(X)=E(X^{2})-[E(X)]^{2}$$

$$\begin{aligned}Var\left(X\right)=\alpha\theta\lambda\textstyle\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}\begin{pmatrix}j\\ s\end{pmatrix}\left(\frac{i\theta}{\gamma}+\frac{\theta}{\gamma}\right)^{j}\int_{0}^{\infty}x^{2}e^{(s+1)\lambda x\gamma}\,dx\\ -\left[\alpha\theta\lambda\textstyle\sum_{i,j,s=0}^{\infty}\frac{(-1)^{i+s}}{j!}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}\begin{pmatrix}j\\ s\end{pmatrix}\left(\frac{i\theta}{\gamma}+\frac{\theta}{\gamma}\right)^{j}\int_{0}^{\infty}xe^{(s+1)\lambda x\gamma}\,dx\right]^{2}.\end{aligned} \tag{2.17}$$

Maximum likelihood estimators

The parameters of the EGoE distribution can be estimated using the method of Maximum Likelihood Estimation (MLE) as follows:
Let $x_1, x_2, \ldots, x_n$ be a random sample from the Exponentiated Gompertz Exponential (EGoE) distribution. Then, the likelihood function is given by

Thus,

$$f\left(x_{1},x_{2},\ldots,x_{n};\alpha,\theta,\gamma,\lambda\right)=\alpha^{n}\theta^{n}\lambda^{n}e^{\lambda\gamma\sum_{i=1}^{n}x}e^{\frac{\theta}{\gamma}\sum_{i=1}^{n}\left[\,1-e^{\lambda x\gamma}\right]}\sum_{i=1}^{n}\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]^{\alpha-1}$$

Let $l=logf\left(x_{1},x_{2},\ldots,x_{n};\alpha,\theta,\gamma,\lambda\right)$,

$$l=nlog\alpha+nlog\theta+nlog\lambda+\lambda\gamma\sum_{i=1}^{n}x\;+\frac{\theta}{\gamma}\sum_{i=1}^{n}\left[\,1-e^{\lambda x\gamma}\right]\;+(\alpha-1)\sum_{i=1}^{n}log\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right].$$

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}log\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right].$$

Therefore,

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}+\;\frac{1}{\gamma}\sum_{i=1}^{n}\left[\,1-e^{\lambda x\gamma}\right]\;+\frac{(1-\alpha)(1-e^{\lambda x\gamma})}{\gamma}\sum_{i=1}^{n}\frac{e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}{1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}. \tag{2.18}$$

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+\gamma\sum_{i=1}^{n}x-\theta\sum_{i=1}^{n}xe^{\lambda x\gamma}+\frac{\theta\left(1-\alpha\right)\sum_{i=1}^{n}xe^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}{\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}. \tag{2.19}$$

$$\begin{aligned}\tfrac{\partial l}{\partial\gamma}=\lambda\textstyle\sum_{i=1}^{n}x-\frac{\theta\lambda}{\gamma}\sum_{i=1}^{n}xe^{\lambda x\gamma}-\frac{\theta}{\gamma^{2}}\sum_{i=1}^{n}\left[\,1-e^{\lambda x\gamma}\right]\\ +\frac{\theta(\alpha-1)\sum_{i=1}^{n}\left[\left(\lambda xe^{\lambda x\gamma}+\frac{1}{\gamma}\left(\,1-e^{\lambda x\gamma}\right)\right)e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}{\gamma\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}.\end{aligned} \tag{2.20}$$

Equating $\frac{\partial l}{\partial \alpha} = 0$, $\frac{\partial l}{\partial \theta} = 0$, $\frac{\partial l}{\partial \lambda} = 0$ and $\frac{\partial l}{\partial \gamma} = 0$, which become

$$\frac{n}{\alpha} + \sum_{i=1}^{n} log\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right] = 0. \tag{2.21}$$

$$\frac{n}{\theta} + \frac{1}{\gamma}\sum_{i=1}^{n}\left[\,1 - e^{\lambda x\gamma}\right] + \frac{(1-\alpha)(1-e^{\lambda x\gamma})}{\gamma}\sum_{i=1}^{n}\frac{e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}{1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}} = 0. \tag{2.22}$$

$$\frac{n}{\lambda} + \gamma\sum_{i=1}^{n} x - \theta\sum_{i=1}^{n} xe^{\lambda x\gamma} + \frac{\theta\,(1-\alpha)\sum_{i=1}^{n} xe^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}}{\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]} = 0. \tag{2.23}$$

and

$$\begin{aligned}&\lambda\sum_{i=1}^{n} x - \frac{\theta\lambda}{\gamma}\sum_{i=1}^{n} xe^{\lambda x\gamma} - \frac{\theta}{\gamma^2}\sum_{i=1}^{n}\left[\,1 - e^{\lambda x\gamma}\right]\\ &+\frac{\theta(\alpha-1)\sum_{i=1}^{n}\left[\left(\lambda xe^{\lambda x\gamma} + \frac{1}{\gamma}\left(\,1-e^{\lambda x\gamma}\right)\right)e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}{\gamma\left[1-e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]} = 0.\end{aligned} \tag{2.24}$$

The MLE of α can then be obtained from equation (2.21) for a given θ, λ and γ in the form below.

$$\alpha = \frac{-n}{\sum_{i=1}^{n} log\left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{\lambda x\gamma}\right]}\right]}. \tag{2.25}$$

Substituting equation (2.25) into equations (2.22), (2.23) and (2.24) and by solving the resulting systems of three non-linear equations numerically, we get the MLE of θ, λ and γ.

3. Simulation study

Simulation study was conducted using R statistical software. Data sets were generated from EGoE distribution. Random samples of sizes n=10, 100 and 1000 were used. The simulation was conducted for three different cases using varying true parameters in each case. The selected true parameter values are $\theta = 0.5, \quad \lambda = 0.5, \quad \gamma = 0.5, and\ \alpha = 0.5$; $\theta = 1.0, \quad \lambda = 1.0,\ \gamma = 1.0, and\ \alpha = 1.0$; $\theta = 0.5, \lambda = 0.5, \gamma = 0.5, and\ \alpha = 0.5$ for the first, second and third cases respectively. The maximum likelihood estimates of the true parameters including the Bias, Standard Error and Root Mean Square Error (RMSE) were obtained with the result of the simulation studies shown in Tables 3.1, 3.2, and 3.3.

Table 3.1: Simulation study at $\theta\ = 0.5$, $\lambda = 0.5$, $\gamma = 0.5$ and $\alpha = 0.5$

n	Parameters	Means	Bias	Std. Error	RMSE
10	θ =0.5	2.3028	-1.8028	2.3183	0.4815
	$\lambda = 0.5$	0.7712	-0.2712	7.5652	0.8698
	$\gamma = 0.5$	1.3686	-0.8686	13.2869	1.1527
	$\alpha = 0.5$	0.2217	0.2783	2.2168	0.4708
100	θ=0.5	0.4251	0.0749	0.0681	0.0261
	$\lambda = 0.5$	0.2363	0.2637	0.2333	0.0483
	$\gamma = 0.5$	0.5676	- 0.0676	0.3534	0.0594
	$\alpha = 0.5$	0.6930	- 0.1930	0.3114	0.0558
1000	θ=0.5	0.4972	0.0028	0.0230	0.0048
	$\lambda = 0.5$	0.3693	0.1307	0.0432	0.0066
	$\gamma = 0.5$	0.6537	- 0.1537	0.0357	0.0060
	$\alpha = 0.5$	0.3632	0.1368	0.0483	0.0069

Table 3.2: Simulation study at$\theta = 1.0$,$\lambda = 1.0$, $\gamma = 1.0$ and $\alpha = 1.0$

n	Parameters	Means	Bias	Std. Error	RMSE
10	$\theta = 1.0$	0.9102	0.0898	0.6518	0.2553
	$\lambda = 1.0$	0.6371	0.3629	15.4529	1.2431
	$\gamma = 1.0$	1.1214	- 0.1214	27.0708	1.6453
	$\alpha = 1.0$	1.2441	0.1368	29.9894	1.7317
100	$\theta = 1.0$	0.7499	0.2501	0.1390	0.0373
	$\lambda = 1.0$	0.3780	0.6220	1.9413	0.1393
	$\gamma = 1.0$	1.4201	- 0.4201	7.2655	0.2695
	$\alpha = 1.0$	1.0126	- 0.0126	5.1842	0.2277
1000	$\theta = 1.0$	1.0628	- 0.0628	0.0648	0.0080
	$\lambda = 1.0$	1.1703	-0.1703	4.0535	0.0637
	$\gamma = 1.0$	0.9296	- 0.0704	3.2057	0.0566
	$\alpha = 1.0$	0.9498	0.0502	3.2610	0.0571

Table 3.3: Simulation study at $\theta = 1.5$, $\lambda = 1.5$, $\gamma = 1.5$ and $\alpha = 1.5$

n	Parameters	Means	Bias	Std. Error	RMSE
10	$\theta = 1.5$	1.0776	0.4224	0.9760	0.3124
	$\lambda = 1.5$	0.4382	1.0618	4.7415	0.6886
	$\gamma = 1.5$	1.9782	- 0.4782	21.7080	1.4734
	$\alpha = 1.5$	2.3560	- 0.8560	26.3076	1.6220
100	$\theta = 1.5$	2.0174	- 0.5174	0.4849	0.0696
	$\lambda = 1.5$	1.5719	-0.0719	4.6175	0.2149
	$\gamma = 1.5$	1.5852	- 0.0852	4.5838	0.2141
	$\alpha = 1.5$	1.6677	- 0.1677	4.8031	0.2192
1000	$\theta = 1.5$	1.5388	- 0.0388	0.1102	0.0105
	$\lambda = 1.5$	1.7839	-0.2839	1.3853	0.0372
	$\gamma = 1.5$	1.2857	0.2143	0.9888	0.0314
	$\alpha = 1.5$	1.6850	- 0.1850	1.3079	0.0361

Application to real-life data sets

In this section, Exponentiated Gompertz Exponential distribution was compared with four other four – parameter compound distributions – Gompertz Weibull distribution (GOWE), Gompertz Burrxii distribution (GOBXII), Gompertz Lomax (GOLOM) distribution and Gompertz Flexible Weibull distribution (GOFLWE). The distributions were fitted to three real data sets presented below:

Dataset I: The first data set represents the reproducibility of median – time – to – failure (t 50) measurements. It has been previously used by [1].
6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.12, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.7, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.64, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

Dataset II: This data represents the waiting times (in minutes) before service of 100 Bank customers. It has been used previously by [2], [3] and [4].
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5 .

Dataset III: This data set represents the strength of carbon fibers tested under tension at gauge lengths of 10mm. It has been used previously by [5] and [6]. The observations are as follows:
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Table 3.4: The descriptive statistics for the three data sets above are provided in the table below.

Parameters	N	Min.	Q_1	Median	Q_3	Mean	Max.	Skewness	Kurtosis	Variance
Dataset 1	59	2.997	6.052	6.923	7.941	6.980	11.040	0.1932	3.0874	2.6051
Dataset II	100	0.800	4.675	8.100	13.02	9.877	38.500	1.4728	5.5403	52.3741
Dataset III	63	1.901	2.554	2.996	3.422	3.059	5.020	0.6328	3.2863	0.3855

The goodness-of-fit statistics including Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC),Bayesian Information Criterion (BIC), Log-likelihood (LL), Hannan-Quinn Information Criterion (HQIC), Shapiro – Wilk test (W), Anderson – Darling test (A) and Kolmogorov-Smirnov (K-S) statistics are computed to compare the fitted models.

Table 3.5: Performance rating of EGoE distribution for data set 1

Distribution	Parameter Estimates	- LL	AIC	CAIC	BIC	HQIC
EGoE	$\hat{\alpha}$=1.7868 $\hat{\theta}$=2.4673 $\hat{\lambda}$=7.7670 $\hat{\gamma}$=0.0866	111.2993	230.5985	231.3395	238.9087	233.8425
GOWE	$\hat{\alpha}$=0.3838 $\hat{\theta}$=-0.0589 $\hat{\lambda}$=0.1617 $\hat{\gamma}$=5.3304	111.7834	231.5668	232.3075	239.877	234.8107
GOFLWE	$\hat{\alpha}$ =0.0507 $\hat{\theta}$=15.4192 $\hat{\lambda}$=-0.0115 $\hat{\gamma}$=6.8100	111.8013	231.6027	232.3434	239.9128	234.8466
GOLOM	$\hat{\alpha}$=0.0046 $\hat{\theta}$=3.3722 $\hat{\lambda}$=0.1747 $\hat{\gamma}$=2.3061	114.5715	237.0230	237.7638	245.3332	240.2670
GOBUXII	$\hat{\alpha}$ =0.0027 $\hat{\theta}$=7.5043 $\hat{\lambda}$=0.2720 $\hat{\gamma}$=1.9276	114.5667	237.1335	237.8742	245.4436	240.3774

Table 3.6: Test statistic of EGOE and the competing distributions using data set 1

Distribution	W	A	KS	p - value
EGoE	0.0380	0.2132	0.0664	0.9419
GOWE	0.0456	0.2567	0.0735	0.8842
GOFLWE	0.0511	0.2860	0.0760	0.8589
GOLOM	0.1232	0.7036	0.0979	0.5888
GOBUXII	0.0644	0.3602	0.1339	0.2198

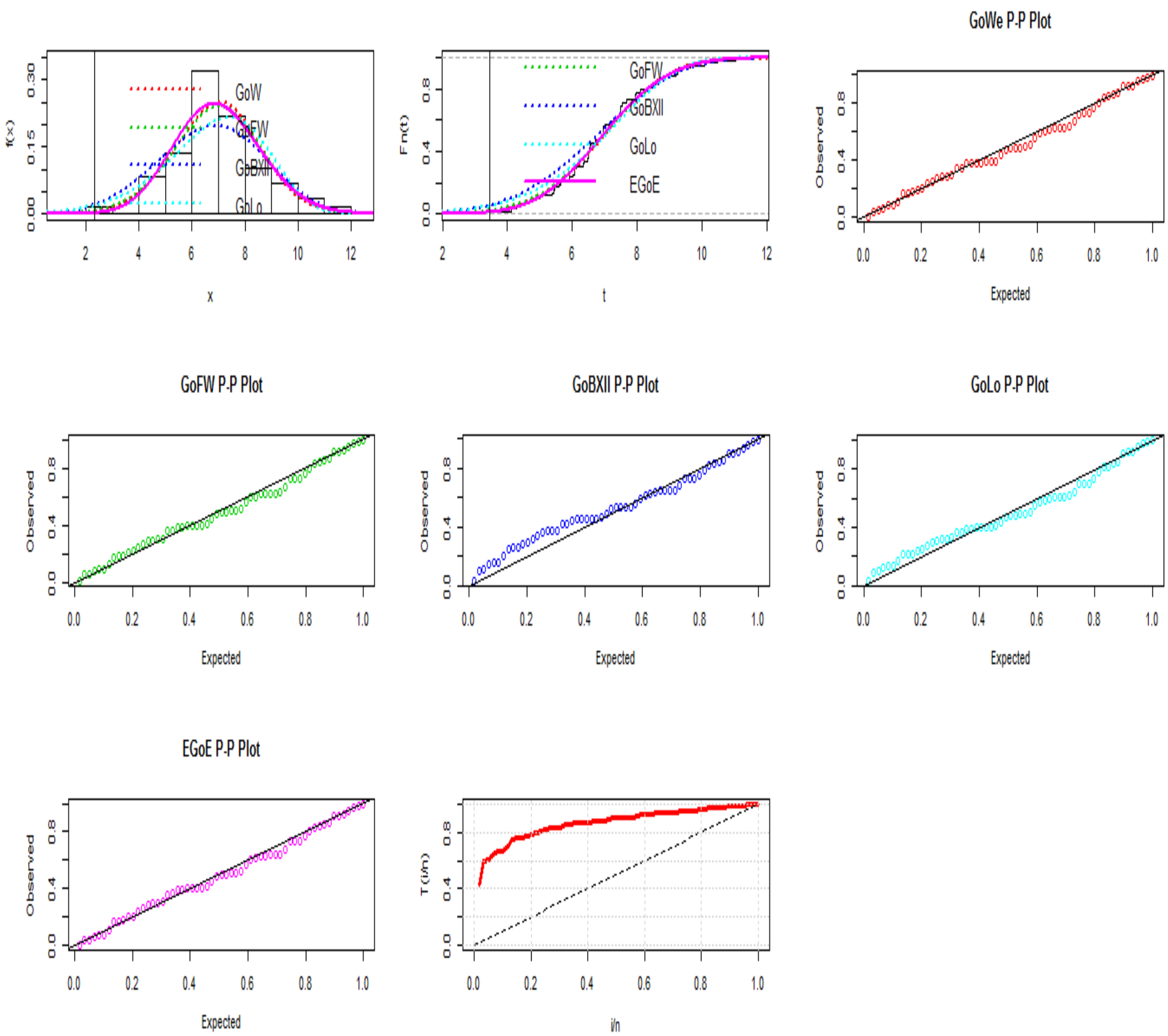

FIGURE 5. Graphical displays of EGoE and the competing distributions with respect to data set 1

Table 3.7: Performance rating of EGoE distribution for data set II

Distribution	Parameter Estimates	- LL	AIC	CAIC	BIC	HQIC
EGoE	$\hat{\alpha}$=0.6558	317.0708	642.1415	642.5626	652.5622	646.3590
	$\hat{\theta}$=-0.0146					
	$\hat{\lambda}$ =2.2605					
	$\hat{\gamma}$=0.2538					
GOWE	$\hat{\alpha}$=1.1742	317.9028	643.8055	644.2266	654.2262	648.0230
	$\hat{\theta}$ =-0.1589					
	$\hat{\lambda}$ =0.0893					
	$\hat{\gamma}$=1.6221					
GOFLWE	$\hat{\alpha}$=0.1224	317.1068	642.2136	642.6347	652.6343	646.4311
	$\hat{\theta}$=4.5968					
	$\hat{\lambda}$=0.0061					
	$\hat{\gamma}$=2.9838					
GOLOM	$\hat{\alpha}$ =0.0159	319.3325	646.6649	647.0860	657.0856	650.8823
	$\hat{\theta}$ =3.0670					
	$\hat{\lambda}$=3.0723					
	$\hat{\gamma}$=0.4841					
GOBUXII	$\hat{\alpha}$ =0.1057	317.3427	642.6854	643.1065	653.1061	646.9028
	$\hat{\theta}$=2.4126					
	$\hat{\lambda}$ =0.1618					
	$\hat{\gamma}$=3.4293					

Considering the values of the AIC, CAIC, BIC and HQIC; Exponentiated Gompertz Exponential distribution seems to perform better than the competing distributions since it has the lowest value. The graphical representations of comparative analysis in table 3.7 are shown in figure 5.

Table 3.8: Test statistic of EGoE and the competing distributions using data set II

Distribution	W	A	KS	p - value
EGoE	0.0187	0.1320	0.0387	0.9983
GOWE	0.0373	0.2346	0.0447	0.9882
GOFLWE	0.0264	0.2074	0.0389	0.9981
GOLOM	0.0728	0.4578	0.0577	0.8926
GOBUXII	0.0321	0.2207	0.0573	0.9551

With EGoE having the lowest value of W, A and KS, it shows that it is the best among the competing distributions.

Table 3.9: Performance rating of EGoE distribution for data set III

Distribution	Parameter Estimates	- LL	AIC	CAIC	BIC	HQIC
EGoE	$\hat{\alpha}$ =1.2366	56.2729	120.5458	121.2354	129.1183	123.9174
	$\hat{\theta}$ =0.2069					
	$\hat{\lambda}$=58.5181					
	$\hat{\gamma}$=0.8998					
GOWE	$\hat{\alpha}$ =0.0081	67.1454	142.2908	142.9805	150.8634	145.6624
	$\hat{\theta}$ =5.6238					
	$\hat{\lambda}$=0.3697					
	$\hat{\gamma}$=0.7116					
GOFLWE	$\hat{\alpha}$ =0.0076	58.3618	124.7236	125.4132	133.2961	128.0952
	$\hat{\theta}$ =27.5730					
	$\hat{\lambda}$=-0.0845					
	$\hat{\gamma}$=3.0426					
GOLOM	$\hat{\alpha}$ =0.0044	64.9569	137.9139	138.6035	146.4864	141.2855
	$\hat{\theta}$ =5.3157					
	$\hat{\lambda}$=0.4533					
	$\hat{\gamma}$=1.4547					
GOBUXII	$\hat{\alpha}$ =0.0074	62.4951	132.9903	133.6799	141.5628	136.3619
	$\hat{\theta}$ =3.6613					
	$\hat{\lambda}$=0.4632					
	$\hat{\gamma}$=3.0294					

Considering the values of the AIC, CAIC, BIC and HQIC; Exponentiated Gompertz Exponential distribution is having the lowest value, therefore, performs better than the competing distributions.

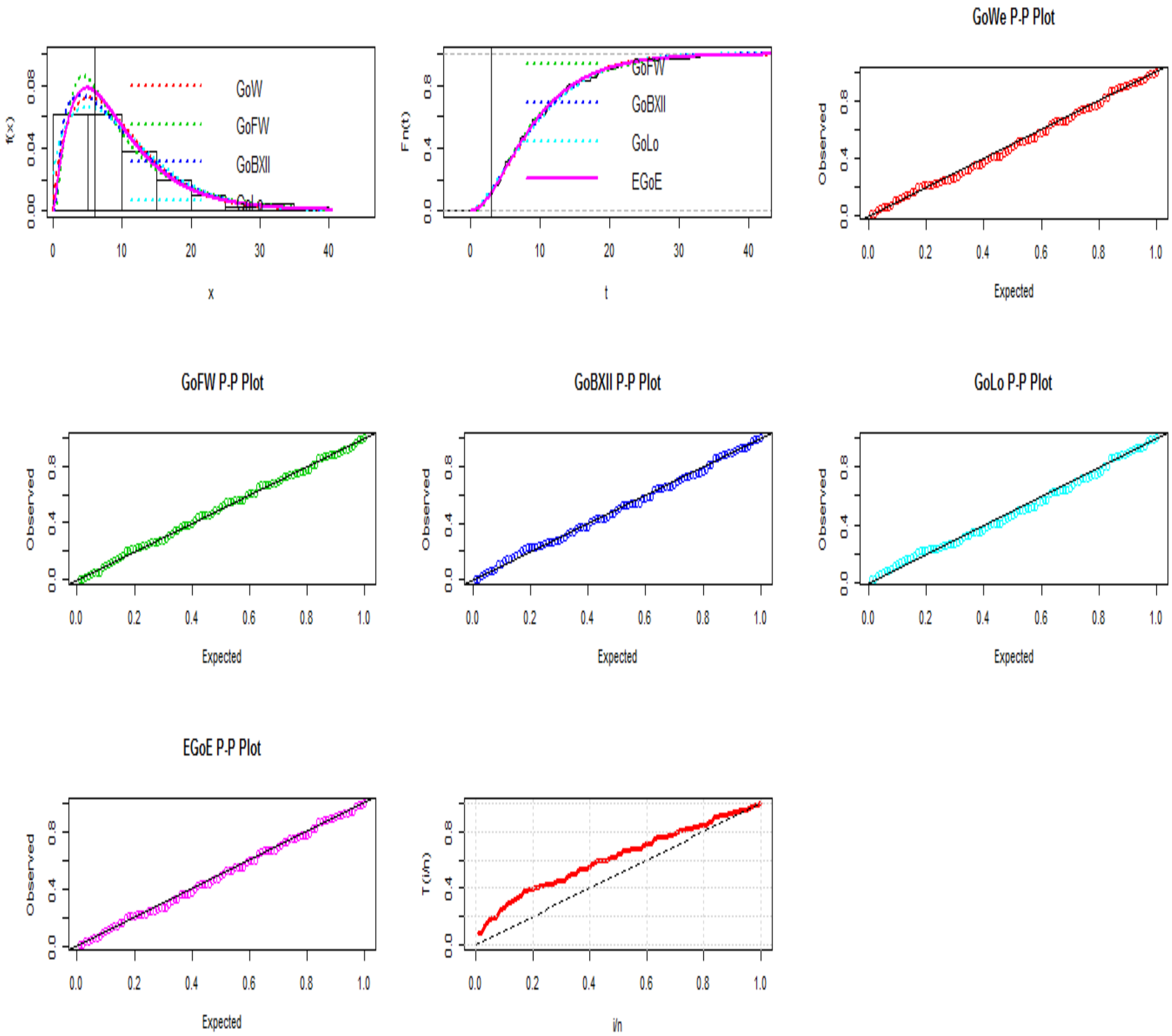

FIGURE 6. Graphical displays of EGoE and the competing distributions with respect to data set II

Table 3.10: Test statistic of EGoE and the competing distributions using data set III

Distribution	W	A	KS	p - value
EGoE	0.0583	0.3166	0.0798	0.8178
GOWE	0.2082	1.3967	0.1370	0.1877
GOFLWE	0.0801	0.5138	0.0914	0.6688
GOLOM	0.1624	1.1128	0.1282	0.2515
GOBUXII	0.1361	0.9429	0.0934	0.6411

With EGoE having the lowest value of W, A and KS, it shows that it is the best among the competing distributions.

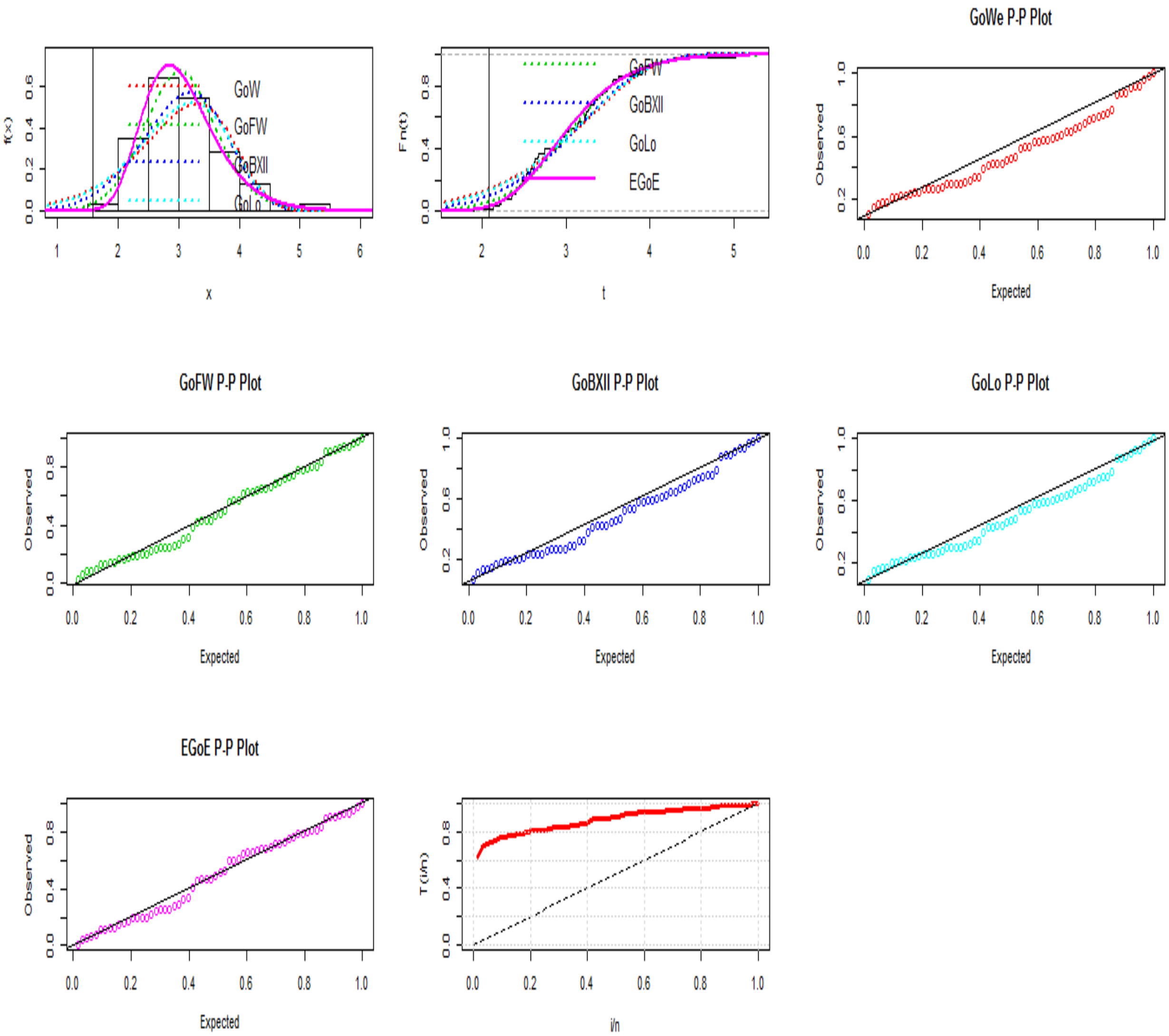

FIGURE 7. Graphical displays of EGoE and the competing distributions with respect to data set III.

4. Discussion of results

Tables 3.1, 3.2 and 3.3 present simulation results performed using R statistical software. The results showed that as parameter $\theta, \lambda, \gamma, and \alpha$; values increases and sample size n also increase then the root mean square error (RMSE) and the standard error of the results decreases. Three different real data sets were used for this study. The results are as shown in the Tables 3.5, 3.7, and 3.9 and Tables 3.6, 3.8 and 3.10.
Tables 3.5, 3.7 and 3.9 the descriptive statistics result showed that life data is positively skewed distribution. Furthermore the test of performance rating was done as shown in the above table the result showed that the proposed distribution Exponentiated Gompertz Exponential distribution has the lowest value of AIC,

CAIC, BIC and HQIC. This implies that the distribution is better fit and performed better than the compared distribution when compared with Gompertz Weibull distribution, Gompertz Flexible Weibull distribution, Gompertz Lomax distribution and Gompertz Burr XII distribution. The goodness of fit for Tables 3.6, 3.8 and 3.10 showed that the performance of EGoE distribution over other distributions is with the lowest values of W, A and KS. This implies that the proposed distribution is a strong competitor to other distribution of the same class and can also be used as alternative model in modeling lifetime processes.

5. Conclusion

In this paper, a new continuous probability distribution with increasing hazard rate is introduced and discussed. Its statistical properties were investigated. The mean and variance of the proposed distribution were obtained in integral form. The maximum likelihood method was used in estimating the parameters of the distribution. Simulation study was carried out to assess the performance of the distribution and its stability. Real life data was carried out on the proposed distribution. The result revealed that EGoE distribution performed better with lower AIC and BIC than the existing distribution when compared with Gompertz Weibull, Gompertz flexible Weibull, Gompertz Lomax and Gompertz Burr XII distributions.

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