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On b_2-Metric Spaces

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Abstract

The target of this paper is to induce a topology from a given b_2-metric and study the properties of the topology induced by this way. We first define the notion of ε-ball in b_2-metric spaces and consider the topology induced by a given b_2-metric via ε-balls. We study some properties of this topological space such as separation axioms and semi-metrizability. Also, we show with the examples that some known properties for ε-balls in metric spaces have not existed in b_2-metric spaces. Then we introduce the concept of strong b_2-metric spaces in which these known properties are provided. Finally, we show that every strong b_2-metric topological space is normal, metrizable and of second category.

Keywords: *b_2-metric space, strong b_2-metric space, ε-ball, Hausdorff space, metrizable topological space.*
2010 Mathematics Subject Classification: *06D72, 54A20, 54E35, 54E50.*

1. Introduction and Preliminaries

In general topology, various fundamental and considerable results are obtained from the class of metric spaces. Because of this aspect and other important features such to have a key role in mathematics and quantitative sciences, a lot of works related to metric spaces have been in the literature. Due to the importance of this notion, it has been extended and generalized in several distinct directions by many authors. For instance, the notion of b-metric space is one of the generalizations of metric space and defined by Bakhtin [2] and Czerwik [3] as follows:

Definition 1.1. *[2, 3] A mapping $D : U \times U \to \mathbb{R}^+$ is said to be a b-metric on the universe set U if the following conditions are satisfied:*
(BM1) $D(x,y) = 0$ if and only if $x = y$,
(BM2) $D(x,y) = D(y,x)$ for all $x,y \in U$,
(BM3) $D(x,y) \leq K[D(x,z)+D(z,y)]$ for all $x,y,z \in U$ and for some constant $K \geq 1$.
When D is a $b-metric$ on U, the pair (U,D,K) is called a $b-metric\ space$.

One of the main differences between metric and b-metric is that the metric is always continuous in each variables but generally the b-metric is not continuous. Bakhtin [2] and Czerwik [3] also proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in such spaces. Then, Khamsi and Hussein [9] gave the definitions of convergent sequence, Cauchy sequence, ε-balls, obtained some theorems related to KKM mappings, studied compactness and discussed a natural topology induced by b-metric:

Definition 1.2. *[9] Let (U,D,K) be a b-metric space and (x_n) be a sequence in U.*
(i) The sequence (x_n) is said to converge to $x \in U$ if $D(x_n,x) \to 0$ as $n \to \infty$. This is denoted by $x_n \to x$ as $n \to \infty$ or by $\lim_{n\to\infty} x_n = x$.
(ii) The sequence (x_n) is said to be a Cauchy sequence in (U,D,K) if $D(x_n,x_m) \to 0$ as $n,m \to \infty$.
(iii) (U,D,K) is called $complete$ if every Cauchy sequence in U converges to some point of U.
A subset A in a b-metric space (U,D,K) is called open if for all $x \in A$ there exists $\varepsilon > 0$ such that $B_D(x,\varepsilon) \subseteq A$, where $B_D(x,\varepsilon) = \{y | D(x,y) < \varepsilon\}$. The set $B_D(x,\varepsilon)$ is called an ε-ball at centered $x \in U$ with radius $\varepsilon > 0$ in (U,D,K). If τ_D is the family of all open subsets of U, then τ_D is also a topology on U.

In [14], the authors showed that to be an open set which is one of the most important properties of ε-balls is not satisfied in b-metric spaces. Also, they proved the Stone-type theorem on b-metric spaces, showed that each b-metric space is semi-metrizable and obtained a sufficient condition for a b-metric space to be metrizable:

Proposition 1.3. *[14] Let (U,D,K) be a b-metric space.*
(1) (U,τ_D) is semi-metrizable.
(2) If D is continuous in one variable, then D is continuous in each variables. Moreover, for all $x \in X$ and $\varepsilon > 0$, $B_D(x,\varepsilon)$ is an open set in (U,D,K).
(3) If D is continuous in one variable, then (U,τ_D) is metrizable.

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Since b-metric space is always understood to be a topological space and a b-metric need not be continuous, there was needed strengthening of the notion of b- metric spaces which remedies this defect. For this reason, Kirk and Shahzad [10] introduced the notion of a strong b-metric space and studied some properties of this spaces:

Definition 1.4. *[10] A mapping $D_s : U \times U \to \mathbb{R}^+$ is said to be a strong b-metric on the set U if (BM1), (BM2) and the following condition are satisfied:*
(BM3') $D_s(x,y) \leq KD_s(x,z) + D_s(z,y)$ for all $x,y,z \in U$ and for some constant $K \geq 1$.
In this case the ordered pair (U,D_s,K) is called strong $b-metric$ space.

Proposition 1.5. *[10] (1) Every strong b-metric is continuous in each variables.*
(2) Every ε-ball $B_{D_s}(x,\varepsilon)$ is open in strong b-metric spaces (U,D_s,K).
(3) The collection $\mathfrak{B} = \{B_{D_s}(x,\varepsilon) : x \in U, \varepsilon > 0\}$ of ε-balls is a base for a topology on U and we denote this topology by τ_{D_s}.
(4) (U,τ_{D_s}) is metrizable.

In the 1960s, Gähler introduced the notion of 2-metric space which is topologically different from the other generalizations of metric space. Geometrically $d(x,y,z)$ represents the area of a triangle formed by the x, y and z in universal set U as its vertices (see [7]). Theory of 2-metric spaces has been extensively studied and developed by some authors [1, 4, 5, 6, 8, 11, 13].

Definition 1.6. *[5] A mapping $d : U \times U \times U \to \mathbb{R}^+$ is said to be a 2-metric on the set U if the following conditions are satisfied:*
(2M1) For all $x,y \in U$ $(x \neq y)$, there exists a point $z \in U$ such that $d(x,y,z) \neq 0$,
(2M2) $d(x,y,z) = 0$ when at least two of x,y,z are equal,
(2M3) $d(x,y,z) = d(x,z,y) = d(y,z,x)$ for all $x,y,z \in U$,
(2M4) $d(x,y,z) \leq d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x,y,z,w \in U$.
We denote the structure of 2-metric space by the ordered pair (U,d).

Definition 1.7. *[5] Let (U,d) be a 2-metric space and (x_n) be a sequence in U.*
(i) The sequence (x_n) is said to converge to $x \in U$, if $d(x_n,x,z) \to 0$ as $n \to \infty$ for all $z \in U$. This is denoted by $x_n \to x$ as $n \to \infty$ or by $\lim_{n\to\infty} x_n = x$.
(ii) The sequence (x_n) is said to be a Cauchy sequence in (U,d) if $d(x_n,x_m,z) \to 0$ as $n,m \to \infty$ for all $z \in U$.
(iii) (U,d) is called complete if every Cauchy sequence in U converges to some point of U.

Remark 1.8. *[5, 13] (1) It is clear from definition of 2-metric that every 2-metric is non-negative and every 2-metric space contains at least three distinct points.*
(2) A 2-metric d is continuous in only one variable. Moreover, if a 2-metric d is continuous in two variables, then it continuous in each variables.
(3) A convergent sequence in a 2-metric space need not be a Cauchy sequence.
(4) In a 2-metric space (U,d), if d is continuous, then every convergent sequence is a Cauchy sequence.
(5) There exists a 2-metric space (U,d) such that every convergent sequence in this space is a Cauchy sequence but d is not continuous.

In 2014, Mustafa et al. [12] introduced a new metric structure, called b_2-metric, as a generalization of both 2-metric and b-metric. Then they obtained some fixed point theorems under different contractive conditions in ordered b_2-metric spaces.

Definition 1.9. *[12] A mapping $d : U \times U \times U \to \mathbb{R}^+$ is said to be a b_2-metric on the set U if (2M1)-(2M3) and the following condition are satisfied:*
(2M4′) $d(x,y,z) \leq K(d(x,y,w) + d(x,w,z) + d(w,y,z))$ for all $x,y,z,w \in U$ and for some constant $K \geq 1$.
If d is a $b_2-metric$ on U, then the ordered pair (U,d,K) is called a $b_2-metric$ space with parameter K.
It is clear that each 2-metric is a b_2-metric and b_2-metric is coincident with 2-metric when the parameter $K = 1$.

Example 1.10. *[12] Let $U = \{0,1,\frac{1}{2},\frac{1}{3},...,\frac{1}{n},...\}$ and the mappings $d_1,d_2 : U \times U \times U \to \mathbb{R}^+$ defined by*

$$d_1(x,y,z) = \begin{cases} (xy+yz+xz)^2, & x \neq y \neq z \\ 0, & otherwise \end{cases}$$

and $d_2(x,y,z) = (x-y)^2(y-z)^2(x-z)^2$ respectively. Then d_1 and d_2 are b_2-metrics on U with $K = 3$.

In this manuscript, we define the notion of ε-ball in b_2-metric spaces and induce a topology from a given b_2-metric via ε-balls. We study some properties of this topological space such as separation axioms and semi-metrizability. Also, we show with the examples that some known properties for ε-balls in metric spaces have not existed in b_2-metric spaces. Then we introduce the concept of strong b_2-metric spaces in which these known properties are provided. Finally, we show that every strong b_2-metric topological space is a normal space and metrizable.

2. b_2-Metric Topology

In this section, we define the notion of ε-balls in b_2-metric spaces. Then, we show that every ε-ball may not be an open set and every b_2-metric may not be continuous in each variable. Also, we study some properties of the topology induced by a b_2-metric.

Definition 2.1. *Let (U,d,K) be a b_2-metric space. If there exists a non-negative real number M such that $d(x,y,z) \leq M$, for all $x,y,z \in U$, then (U,d,K) is called a bounded b_2-metric space. Otherwise we consider it unbounded.*

Example 2.2. *Let (U,d,K) be a b_2-metric space and $\bar{d} : U \times U \times U \to \mathbb{R}^+$ be a mapping defined by $\bar{d}(x,y,z) = \min\{d(x,y,z),1\}$. Then $(U,\bar{d},K)$ is a bounded b_2-metric space.*

In b_2-metric spaces, the definitions of convergent sequence, Cauchy sequence and completeness are defined similar with 2-metric spaces. Now, we define the notion of ε-ball in b_2-metric spaces and give some examples.

Definition 2.3. *Let (U,d,K) be a b_2-metric space, $x \in U$ and $\varepsilon > 0$. Then the subset $B_d(x,\varepsilon) = \{y \in X : sup_{z\in X} d(x,y,z) < \varepsilon\}$ of U is said to be an ε-ball centered at x with radius ε.*
A subset A in a b_2-metric space (U,d,K) is said to be an open set if for all $x \in A$ there is an ε-ball $B_d(x,\varepsilon)$ such that $B_d(x,\varepsilon) \subseteq A$ and A is called a closed set if $U \setminus A$ is open. The family of all open subsets of U induces a topology on U. This topology is called b_2-metric topology and denoted by τ_d.

Proposition 2.4. *Every b_2-metric topological space (U,τ_d) is a T_1-space.*

Proof. Let $x,y \in U$ and $x \neq y$. Take $G = U \setminus \{y\}$. For all $z \in G$, we have $\varepsilon_z = d(x,y,z) > 0$ and so $B_d(z,\varepsilon_z) \subset G$. Hence G is an open neighborhood of x and $y \notin G$. Therefore, (U,τ_d) is T_1-space. □

The following examples show that there exists an ε-ball $B_d(x,\varepsilon)$ which is not open set and a b_2-metric $d : U \times U \times U \to \mathbb{R}^+$ may not be continuous in each variables.

Example 2.5. *Let us consider the b_2-metric space (U,d_2) given in Example 1.10. Here, we have $B_{d_2}(1,1) = \{0,1\}$. Therefore, $0 \in B_{d_2}(1,1)$. Note that $\frac{1}{2n} \in B_{d_2}(0,\varepsilon)$ for all $\varepsilon > 0$. But $\frac{1}{2n} \notin B_{d_2}(1,1)$ since $d(1,\frac{1}{2n},z) \not< 1$ for all $z \in U$. Hence, $B_{d_2}(0,\varepsilon) \not\subset B_{d_2}(1,1)$ and this proves that $B_{d_2}(1,1)$ is not an open set in this space.*

Example 2.6. *Let $U = \{0,1,\frac{1}{2},\frac{1}{3},...,\frac{1}{n},...\}$. Consider the b_2-metric spaces (U,d_1) and (U,d_2) given in Example 1.8 and define the mapping $d : U \times U \times U \to \mathbb{R}^+$ by*

$$d(x,y,z) = \begin{cases} 0, & \text{when at least two of } x,y,z \text{ are equal} \\ d_1(x,y,z), & x \neq y \neq z \in \{1\} \cup \{\frac{1}{2n} : n \in \mathbb{N}\} \\ d_2(x,y,z), & \text{otherwise} \end{cases}$$

for all $x,y,z \in X$. Then (U,d,K) is a b_2-metric space with $K = 3$. Now, consider the sequence $(\frac{1}{2n})_{n\in\mathbb{N}}$ in U. We have $\lim_{n\to\infty} d(\frac{1}{2n},0,z) = 0$ for all $z \in U$. Hence, $\lim_{n\to\infty} \frac{1}{2n} = 0$ in $(U,d,3)$. However, we also obtain

$$\lim_{n\to\infty} d(\frac{1}{2n},1,\frac{1}{4}) = \frac{1}{16} \neq \frac{9}{16} = d(0,1,\frac{1}{4}).$$

This shows that d is not continuous in its variable.

The following proposition shows that there is a sufficient condition that an ε-ball $B_d(x,\varepsilon)$ is an open set.

Proposition 2.7. *Let (U,d,K) be a b_2-metric space. If d is continuous in one variable, then d is continuous in each variable. Also, $B_d(x,\varepsilon)$ is an open set for all $\varepsilon > 0$ and $x \in U$.*

Proof. Let us suppose that d is continuous with respect to the first variable. If $lim_{n\to\infty} x_n = x$, then we obtain

$$lim_{n\to\infty} d(y,x_n,z) = lim_{n\to\infty} d(x_n,y,z) = d(x,y,z) = d(y,x,z)$$

for all $y,z \in U$. This means that d is continuous with respect to the second variable. Also, it can be seen easily with the similar way that d is continuous with respect to the third variable.
Now, to show that $B_d(x,\varepsilon)$ is an open set where $\varepsilon > 0$ and $x \in U$, assume that $x_n \in X \setminus B_d(x,\varepsilon)$ and $lim_{n\to\infty} x_n = a$. This follows that $lim_{n\to\infty} d(x_n,a,z) = 0$ for all $z \in U$ and $sup_{z\in U} d(x_n,x,z) \geq \varepsilon$. Also, we have

$$\begin{array}{rcl} d(x_n,x,z) & \leq & d(x_n,x,a) + d(x_n,a,z) + d(x,a,z) \\ sup_{z\in U} d(x_n,x,z) & \leq & d(x_n,x,a) + sup_{z\in U} d(x_n,a,z) + sup_{z\in U} d(x,a,z) \\ r & \leq & sup_{z\in U} d(x,a,z) \end{array}$$

which means that $x \in X \setminus B(x,\varepsilon)$. This proves that $B_d(x,\varepsilon)$ is an open set. □

Theorem 2.8. *Let (U,d,K) be a b_2-metric space such that d is continuous in one variable. Then b_2-metric topological space (U,τ_d) is Hausdorff space.*

Proof. Let $x,y \in U$ $(x \neq y)$. Then there is a point $z \in U$ such that $d(x,y,z) \neq 0$. Hence $d(x,y,z) > 0$. Now, consider the ε-balls $B_d(x,\frac{\varepsilon}{4K})$ and $B_d(y,\frac{\varepsilon}{4K})$ centered at x and y, respectively, with $\varepsilon > 0$ and $K \geq 1$. Since d is continuous in one variable, then $B_d(x,\frac{\varepsilon}{4K})$ and $B_d(y,\frac{\varepsilon}{4K})$ are open sets. It is clear that $x \in B_d(x,\frac{\varepsilon}{4K})$ and $y \in B_d(y,\frac{\varepsilon}{4K})$. Now, we show that $B_d(x,\frac{\varepsilon}{4K}) \cap B_d(y,\frac{\varepsilon}{4K}) = \emptyset$. Suppose that $a \in B_d(x,\frac{r}{4K}) \cap B_d(y,\frac{r}{4K})$. Since d is a b_2-metric on U, from the condition (2M4$'$) we have

$$\begin{array}{rcl} d(x,y,z) & \leq & K(d(x,y,a) + d(x,a,z) + d(a,y,z)) \\ & < & K(\frac{\varepsilon}{4K} + \frac{\varepsilon}{4K} + \frac{\varepsilon}{4K}) = \frac{3\varepsilon}{4} \end{array}$$

for $K \geq 1$. Therefore,

$$sup_{z\in U} d(x,y,z) < \varepsilon$$

which implies that $d(x,y,z) < \varepsilon$ for all $z \in U$. Therefore, we have that $d(x,y,z) = 0$ which is a contradiction. Hence, (U,τ_d) is a Hausdorff space when d is continuous in one variable. □

Theorem 2.9. *Let (U,d,K) be a b_2-metric space such that d is continuous in one variable. Then b_2-metric topological space (U,τ_d) is reguler topological space.*

Proof. Let O be an open set such that $x \in O$. Since O is open in U, there is an ε-ball $B_d(x,\varepsilon)$ with radius $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subseteq O$. Since d is continuous in one variable, then $B_d(x,\varepsilon)$ is an open set. Also, it can be easily seen that $\overline{B_d(x,\frac{\varepsilon}{2K})} \subseteq B_d(x,\varepsilon) \subseteq U$, where $\overline{B_d(x,\frac{\varepsilon}{2K})}$ is closure of $B_d(x,\frac{\varepsilon}{2K})$ with respect to τ_d. Since $x \in B_d(x,\frac{\varepsilon}{2})$ and $B_d(x,\frac{\varepsilon}{2})$ is an open set, the proof is completed. □

Theorem 2.10. *Let (U,d,K) be a b_2-metric space such that d is continuous in one variable. Then b_2-metric topological space (U,τ_d) is normal topological space.*

Proof. Let A and B be closed sets with $A \cap B = \emptyset$. Let $G = \bigcup_{a\in A} B_d(a,\frac{\varepsilon_a}{4K})$ and $H = \bigcup_{b\in B} B_d(b,\frac{\varepsilon_b}{4K})$ where $K \geq 1$, $\varepsilon_a, \varepsilon_b > 0$ and $B_d(a,\varepsilon_a) \cap B = B_d(b,\varepsilon_b) \cap A = \emptyset$. It is clear that $A \subseteq G$ and $B \subseteq H$. Now, we show that $G \cap H = \emptyset$. Suppose $x \in G \cap H$. Then there are $a \in A,\ b \in B$ such that $x \in B_d(a,\frac{\varepsilon_a}{4K})$ and $x \in B_d(b,\frac{\varepsilon_b}{4K})$. We can assume that $\varepsilon_a < \varepsilon_b$, then we have

$$\begin{aligned} d(a,b,z) &\leq K(d(a,b,x)+d(a,x,z)+d(x,b,z)) \\ &< K(\tfrac{\varepsilon_b}{4K}+\tfrac{\varepsilon_b}{4K}+\tfrac{\varepsilon_b}{4K}) = \tfrac{3\varepsilon_b}{4} \end{aligned}$$

Hence $sup_{z\in U} d(a,b,z) \leq \frac{3\varepsilon_b}{4}$. It means that $a \in B_d(b,\varepsilon_b)$ which is a contradiction. So, we have $G \cap H = \emptyset$. □

Now, we show that every b-metrizable topological spaces are coarser than b_2-metrizable topological space:

Theorem 2.11. *Let $|U| \geq 3$ and (U,D,K) be a b-metric space. Then d_D is a b_2-metric on U with the same parameter K, where $d_D : U\times U\times U \to \mathbb{R}^+$ is defined by $d_D(x,y,z) = min\{D(x,y),D(x,z),D(y,z)\}$ for all $x,y,z \in U$. Also, the b-metric topology τ_D is coarser than the b_2-metric topology τ_{d_D}.*

Proof. First, we show that d_D is a b_2-metric on U.
(2M1) Let $x,y \in U$ $(x \neq y)$. Since $|U| \geq 3$, there is a point $z \in U$ such that $z \neq x$ and $z \neq y$. Hence, we obtain that $D(x,z) > 0, D(x,y) > 0$ and $D(y,z) > 0$. Therefore, we obtain that $d_D(x,y,z) = min\{D(x,y),D(x,z),D(y,z)\} > 0$.
It is clear that (2M2) and (2M3) are satisfied.
(2M4) Let $x,y,z,w \in U$. Without loss of generality, we can assume that $d_D(x,y,z) = D(x,y)$.
Case I: Let $d_D(x,y,w) = D(x,y)$. Then we obtain

$$d_D(x,y,z) = D(x,y) \leq K(d_D(x,y,w)+d_D(x,w,z)+d_D(w,y,z)).$$

Case II: Let $d_D(x,y,w) = D(x,w)$ and $d_D(w,y,z) = D(w,y)$. Then,

$$\begin{aligned} d_D(x,y,z) = D(x,y) &\leq K(D(x,w)+D(w,y)) = K(d_D(x,y,w)+d_D(w,y,z)) \\ &\leq K(d_D(x,y,w)+d_D(x,w,z)+d_D(w,y,z)). \end{aligned}$$

Case III: Let $d_D(x,y,w) = D(x,w)$ and $d_D(w,y,z) = D(w,z)$. Since $d_D(x,y,z) = D(x,y)$, then

$$\begin{aligned} d_D(x,y,z) = D(x,y) &\leq D(x,z) \leq K(D(x,w)+D(w,z)) = K(d_D(x,y,w)+d_D(w,y,z)) \\ &\leq K(d_D(x,y,w)+d_D(x,w,z)+d_D(w,y,z)). \end{aligned}$$

For the other cases, it can be completed the proof with the same procedure. Also, it is obvios that $B_{d_D}(x,\varepsilon)$ is a subset of $B_D(x,\varepsilon)$. Hence, we obtain that τ_D is coarser than τ_{d_D}. □

Proposition 2.12. *Let (U,d,K) be a bounded b_2-metric space and $D_d : U\times U \to \mathbb{R}^+$ be a mapping defined by $D_d(x,y) = \sup_{z\in X} d(x,y,z)$. Then*
(i) D_d is a b-metric on U with $2K$.
(ii) $x_n \to x$ in $(U,D_d,2K)$ if and only if $x_n \to x$ in (U,d,K).
(iii) The topology τ_d induced by the bounded b_2-metric d is coincident with the topology τ_{D_d} induced by the b-metric D_d.

Corollary 2.13. *Every b_2-metric topological space (U,τ_d) is a semi-metrizable space.*

Proof. If τ_d is a b_2-metric topology on U, then there exists a bounded b_2-metric d^* such that $\tau_d = \tau_{d^*}$. From Proposition 2.12, we obtain that $\tau_{D_{d^*}}$ is a b-metric topology and $\tau_{d^*} = \tau_{D_{d^*}}$. Since $(U,\tau_{D_{d^*}})$ is semi-metrizable, from Proposition 1.3, and $\tau_d = \tau_{D_{d^*}}$, (U,τ_d) is a semi-metrizable space. □

3. Strong b_2-Metric (Topological) Spaces

In this section, we define the notion of strong b_2-metric space and propose fundamental definitions of this new kind of metric space such as open (closed) ball, convergence of a sequence, completeness and etc. We also introduce induced topology from a given strong b_2-metric and study some important properties.

Definition 3.1. *A mapping $d_s : U \times U \times U \to \mathbb{R}^+$ is said to be a strong b_2-metric on the set U if the following conditions are satisfied:*
(S2M1) For all $x,y \in U$ $(x \neq y)$, there exists a point $z \in U$ such that $d_s(x,y,z) \neq 0$,
(S2M2) $d_s(x,y,z) = 0$ when at least two of x,y,z are equal,
(S2M3) $d_s(x,y,z) = d_s(y,x,z) = d_s(z,y,x)$ for all $x,y,z \in U$,
(S2M4) $d_s(x,y,z) \leq d_s(x,y,w) + d_s(x,w,z) + K.d_s(w,y,z)$ for all $x,y,z,w \in U$ and for some constant $K \geq 1$.
If d_s is a strong $b_2-metric$ on U, then the ordered pair (U,d_s,K) is called a strong $b_2-metric$ space with parameter K.

Remark 3.2. *It is obvious that strong b_2-metric is coincident with the 2-metric given by Gähler [5] for $K = 1$.*

Example 3.3. *Let $U = \{1,2,3,4\}$ and $d_s : U \times U \times U \to \mathbb{R}^+$ defined by as follows:*

$$d_s(x,y,z) = \begin{cases} 0, & when\ at\ least\ two\ of\ x,y,z\ are\ equal \\ 1, & x \neq y \neq z \in \{1,2,4\} \\ 2, & x \neq y \neq z \in \{2,3,4\} \\ 3, & x \neq y \neq z \in \{1,3,4\} \\ 8, & x \neq y \neq z \in \{1,2,3\} \end{cases}$$

Then, we have d_s is not a 2-metric on U, but d_s is a strong b_2-metric on U with $K = 3$.

Example 3.4. *Let (U_1,d_{s_1},K_1) and (U_2,d_{s_2},K_2) be two strong b_2-metric spaces. Define a mapping $d_s((x_1,x_2),(y_1,y_2),(z_1,z_2)) = d_{s_1}(x_1,y_1,z_1) + d_{s_2}(x_2,y_2,z_2)$ for all $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in U_1 \times U_2$. Then d_s is a strong b_2-metric on $U_1 \times U_2$ with $K = \max\{K_1,K_2\}$.*

Proposition 3.5. *If (U,d_s,K) is a strong b_2-metric space, then d_s is continuous in each variables.*

Proof. Suppose that $lim_{n\to\infty} x_n = x$. Then we obtain $lim_{n\to\infty} d_s(x_n,x,z) = 0$ for all $z \in U$. Also,

$$d_s(x_n,y,z) \leq d_s(x_n,y,x) + K.d_s(x_n,x,z) + d_s(x,y,z)$$

for all $y,z \in U$ where $K \geq 1$. This means that $lim_{n\to\infty} d_s(x_n,y,z) \leq d_s(x,y,z)$. With similar procedure, we obtain that $d_s(x,y,z) \leq lim_{n\to\infty} d_s(x_n,y,z)$. This follows that $lim_{n\to\infty} d_s(x_n,y,z) = d_s(x,y,z)$ which means that d_s is continuous with respect to the first variable. As a consequence of the similar consideration, it can be easily seen that d_s is continuous with respect to other variables. □

In strong b_2-metric spaces, the notions of ε-ball, convergent sequence, Cauchy sequence, complete and bounded space are defined similar with b_2-metric spaces:

Definition 3.6. *Let (U,d_s,K) be a strong b_2-metric space and (x_n) be a sequence in U.*
(i) The sequence (x_n) is said to converge to $x \in U$ if $d_s(x_n,x,z) \to 0$ as $n \to \infty$ for all $z \in U$.
We denote this by $x_n \to x$ as $n \to \infty$ or by $\lim_{n\to\infty} x_n = x$.
(ii) The sequence (x_n) is said to be a Cauchy sequence in (U,d_s,K) if $d_s(x_n,x_m,z) \to 0$ as $n,m \to \infty$ for all $z \in U$.
(iii) (U,d_s,K) is called complete if every Cauchy sequence in U converges to some point of U.
(iv) If there exists a nonnegative real number M such that $d_s(x,y,z) \leq M$ for all $x,y,z \in U$, then (U,d_s,K) is called a bounded strong b_2-metric space. Otherwise we consider it unbounded.

Example 3.7. *Let (U,d_s,K) be a strong b_2-metric space and $\bar{d}_s : U \times U \times U \to \mathbb{R}^+$ be a mapping defined by $\bar{d}_s(x,y,z) = \min\{d_s(x,y,z),1\}$. Then $(U,\bar{d}_s,K)$ is a bounded strong b_2-metric space.*

Lemma 3.8. *In a strong 2-metric space, if there exists a limit of a sequence, then it is unique.*

Proof. The proof is easily obtained from Definition 3.1 and Definition 3.6. □

Definition 3.9. *Let (U,d_s,K) be a strong b_2-metric space, $x \in U$ and $\varepsilon > 0$. Then the subset $B_{d_s}(x,\varepsilon) = \{y \in X : sup_{z\in X} d_s(x,y,z) < \varepsilon\}$ of U is said to be an open ε-ball centered at x with radius ε.*
A subset A in a strong b_2-metric space (U,d_s,K) is said to be an open set if for all $x \in A$ there is an open ε-ball $B_{d_s}(x,\varepsilon)$ such that $B_{d_s}(x,\varepsilon) \subseteq A$ and A is called a closed set if $U \setminus A$ is open. The subset A is called bounded if A is contained some open ε-ball $B_{d_s}(x,\varepsilon)$ of U.
Let $N \subseteq U$ and $x \in N$. If there is an open set G such that $x \in G \subseteq N$, then N is called a neighborhood of x. The set of all neighborhoods of x is denoted by $\mathscr{N}_x$. A collection $\mathfrak{B}_x \subseteq \mathscr{N}_x$ is called local base for a point x if all neighborhoods of x contains a member of $\mathfrak{B}_x$.

Lemma 3.10. *Every open ε-ball $B_{d_s}(x,\varepsilon)$ centered at $x \in U$ with radius $\varepsilon > 0$, is an open set in a strong b_2-metric space (U,d_s,K).*

Proof. Let us consider the open ε-ball $B_{d_s}(x,\varepsilon)$ at center $x \in U$ with radius $\varepsilon > 0$. If $y \in B_{d_s}(x,\varepsilon)$, then $sup_{z\in U} d_s(x,y,z) < \varepsilon$. Let $r^* = \frac{1}{2K}(\varepsilon - sup_{z\in U} d_s(x,y,z))$ for $K \geq 1$.
We claim that $B_{d_s}(y,\varepsilon^*) \subseteq B_{d_s}(x,\varepsilon)$. Let $t \in B_{d_s}(y,\varepsilon^*)$. Now,

$$t \in B_{d_s}(y,\varepsilon^*) \Rightarrow sup_{z\in U} d_s(y,t,z) < \varepsilon^*$$

Hence, $d_s(y,t,x_0) < \varepsilon^*$ for all $x_0 \in U$. Since d_s is a strong b_2-metric on U, from the condition (S2M4), we have

$$\begin{aligned} d_s(x,t,x_0) &\leq d_s(x,t,y)+d_s(x,y,x_0)+Kd_s(y,t,x_0) \\ &\leq Kd_s(x,t,y)+d_s(x,y,x_0)+K_sd(y,t,x_0) \end{aligned}$$

for $K \geq 1$. Therefore,

$$\begin{aligned} sup_{z\in U} d_s(x,t,z) &\leq K\varepsilon^* + sup_{z\in U} d_s(x,y,z)+K\varepsilon^*) \\ &< 2K\frac{1}{2K}(\varepsilon - sup_{z\in U} d_s(x,y,z)) + sup_{z\in U} d_s(x,y,z) = \varepsilon \end{aligned}$$

which implies that $t \in B_{d_s}(x,\varepsilon)$. Hence $B_{d_s}(x,\varepsilon)$ is an open set in U. □

Theorem 3.11. *Let (U,d_s,K) be a strong b_2-metric space. Then the collection of all open ε-balls*

$$\mathfrak{B} = \{B_{d_s}(x,\varepsilon) : x \in U, \varepsilon > 0\}$$

is a base for a topology on U. This topology is called the strong b_2-metric topology induced by d_s and denoted by τ_{d_s}.

Proof. It is clear that $x \in B_{d_s}(x,\varepsilon)$ for all $x \in U$ and $\varepsilon > 0$. Let $x,y \in U$, $\varepsilon_1,\varepsilon_2 > 0$ and $a \in B_{d_s}(x,\varepsilon_1) \cap B_{d_s}(y,\varepsilon_2)$. Let us choose $\varepsilon < \frac{1}{2K} min\{\varepsilon_1 - sup_{z\in U} d_s(x,a,z), \varepsilon_2 - sup_{z\in U} d_s(y,a,z)\}$ and show that $B_{d_s}(a,\varepsilon) \subseteq B_{d_s}(x,\varepsilon_1) \cap B_{d_s}(y,\varepsilon_2)$. If $b \in B_{d_s}(a,\varepsilon)$, then $sup_{z\in U} d_s(a,b,z) < \varepsilon$. Hence, $d_s(a,b,x_0) < \varepsilon$ for all $x_0 \in U$. Since d_s is a strong b_2-metric on U, from the condition (S2M4) we have

$$\begin{aligned} d_s(x,b,x_0) &\leq d_s(x,b,a)+d_s(x,a,x_0)+Kd_s(a,b,x_0) \\ &\leq Kd_s(x,b,a)+d_s(x,a,x_0)+Kd_s(a,b,x_0) \end{aligned}$$

for $K \geq 1$. Therefore,

$$\begin{aligned} sup_{z\in U} d_s(x,b,z) &\leq 2K\varepsilon + sup_{z\in U} d_s(x,a,z)) \\ &< 2K\frac{1}{2K}(\varepsilon_1 - sup_{z\in U} d_s(x,a,z) + sup_{z\in U} d_s(x,a,z) = \varepsilon \end{aligned}$$

which implies that $b \in B_{d_s}(x,\varepsilon_1)$. With the same process, we can show that $b \in B_{d_s}(y,\varepsilon_2)$ and the proof is completed. □

Corollary 3.12. *Every strong b_2-metric topological space (U,τ_{d_s}) is first countable.*

Proof. Since $\{B_{d_s}(x,\frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at $x \in U$, then we obtain that (U,τ_{d_s}) is first countable.

□

Corollary 3.13. *Every strong b_2-metric topological space (U,τ_{d_s}) is a normal (reguler, Hausdorff) space.*

Proof. Since each strong b_2-metric space (U,d_s) is a b_2-metric space and d_s is continuous in each variables, from Thereom 2.8, 2.9 and 2.10, we obtain that the strong b_2-metric topological space (U,τ_{d_s}) is a normal (reguler, Hausdorff) space. □

Theorem 3.14. *Let (U,d_s,K) be a strong b_2-metric space and τ_{d_s} be the topology induced by the strong b_2-metric d_s. Then for a sequence $(x_n) \subseteq U$, $x_n \to x$ in (U,d_s,K) if and only if $x_n \to x$ in (U,τ_{d_s}).*

Proof. Suppose that $x_n \to x$ in (U,d_s,K). Let G be an open set and $x \in G$. Then there exists $\varepsilon > 0$ such that $B_{d_s}(x,\varepsilon) \subseteq G$. Since $x_n \to x$ in (U,d_s,K), there is a natural number n_0 such that $d_s(x_n,x,z) < \frac{\varepsilon}{2}$ for all $z \in U$ whenever $n \geq n_0$. Then we have $\sup_{z\in U} d_s(x_n,x,z) \leq \frac{\varepsilon}{2} < \varepsilon$, that is $x_n \in B_{d_s}(x,\varepsilon)$. So we have $x_n \in G$, it means that is $x_n \to x$ in (U,τ_{d_s}).
Conversely, let $\varepsilon > 0$. Since $B_{d_s}(x,\varepsilon)$ is an open set contains $x \in U$, there exists a natural number n_0 such that $x_n \in B_{d_s}(x,\varepsilon)$ whenever $n \geq n_0$. It means that $\sup_{z\in U} d_s(x_n,x,z) < \varepsilon$ and we have $d_s(x_n,x,z) < \varepsilon$ for all $z \in U$ whenever $n \geq n_0$. Hence, we obtain that $x_n \to x$ in (U,d_s,K). □

Proposition 3.15. *Let (U,d_s,K) be a bounded strong b_2-metric space and $D_{d_s} : U \times U \to \mathbb{R}^+$ be a mapping defined by $D_{d_s}(x,y) = \sup_{z\in X} d_s(x,y,z)$. Then*
(i) D_{d_s} is a strong b-metric on U with the parameter K_s where $K_s = K+1$.
(ii) $x_n \to x$ in (U,D_{d_s},K_s) if and only if $x_n \to x$ in (U,d_s,K).
(iii) The topology τ_{d_s} induced by the bounded strong b_2-metric d is coincident with the topology $\tau_{D_{d_s}}$ induced by the strong b-metric d_s.

Corollary 3.16. *Every strong b_2-metric topological space (U,τ_{d_s}) is metrizable.*

Proof. If τ_{d_s} is a strong b_2-metric topology on U, then there exists a bounded strong b_2-metric ${d_s}^*$ such that $\tau_{d_s} = \tau_{{d_s}^*}$. From Proposition 3.15, we obtain that $\tau_{D_{{d_s}^*}}$ is a strong b-metric topology and $\tau_{{d_s}^*} = \tau_{D_{{d_s}^*}}$. Since $(U,\tau_{D_{{d_s}^*}})$ is metrizable, from Proposition 1.5, and $\tau_{d_s} = \tau_{D_{{d_s}^*}}$, (U,τ_{d_s}) is a metrizable space. □

Theorem 3.17. *If (U,d_s,K) is a complete strong b_2-metric space, then (U,τ_{d_s}) is of the second category. i.e., U cannot be expressed as a countable union of nowhere dense sets with respect to τ_{d_s}.*

Proof. Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is an arbitrary collection of nowhere dense subsets of U and $U = \bigcup_{n\in\mathbb{N}} A_n$. From this implication, we have $U = \bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}} \overline{A_n} \subseteq U$ which means that $U = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \overline{A_n}$. Let G be an open set in U. Since A_1 is nowhere dense, there is a point $x_1 \in G$ such that $x_1 \notin A_1$. From here, we have that $G \setminus \overline{A_1}$ is an open set and $x_1 \in G \setminus \overline{A_1}$. So, there is an $\varepsilon_1 > 0$ such that $\overline{B(x_1,\varepsilon_1)} \subseteq G \setminus \overline{A_1}$. Write $B_1 = B(x_1,\varepsilon_1)$. Again, since B_1 is open and A_2 is nowhere dense, there is a point $x_2 \in B_1 \setminus \overline{A_2}$. From this, we have that $B_1 \setminus \overline{A_2}$ is an open set and $x_2 \in B_1 \setminus A_2$. Therefore, there is an $\varepsilon_2 < \frac{1}{2}$ such that $\overline{B(x_2,\varepsilon_2)} \subseteq B1 \setminus \overline{A_2}$. Proceeding with the same way, we can obtain a sequence of closed sets $\{\overline{B_n}\}$ such that $\overline{B_{n+1}} \subseteq \overline{B_n}$ for all $n \in \mathbb{N}$ with radius $0 < \varepsilon_n < \frac{1}{n}$. Since $\overline{B_n}$ is decreasing, $x_m \in \overline{B_m} \subseteq \overline{B_n}$ for all $m \geq n$. Hence, we have $d_s(x_m,x_n,z) \to 0$ as $n,m \to \infty$ for all $z \in U$. This deduced that (x_n) is a Cauchy sequence in U. Since (U,d_s) is complete, then there is a point $x \in U$ such that $x_n \to x$ as $n \to \infty$. Now, we claim that $x \in \bigcap_{n\in\mathbb{N}} \overline{B_n}$. To show this assertion, assume that $x_k \neq x$ for some $k \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Let H be an open set in U such that $x \in H$. Then, there is a natural number n_1 such that $x_k \in H$ whenever $k \geq n_1$. Hence, we have $x_k \in \overline{B_n} \cap H \setminus \{x\}$ whenever $k \geq max\{n,n_1\}$. This shows that $x \in \bigcap_{n\in\mathbb{N}} \overline{B_n}$. Since $\overline{A_n} \cap \overline{B_n} = \emptyset$ for all $n \in \mathbb{N}$ and this means that $x \notin \overline{A_n}$ which contradicts to $U = \bigcup_{n\in\mathbb{N}} \overline{A_n}$. □

4. Conclusion

In mathematics, the notion of a metric is a generalization of the concept of distance which is a numerical measurement of how far apart objects or points are. Thus many different types of "distances" can be calculated with the metrics. Topological space is a generalization of metric spaces in which the idea of closeness is described in terms of relationships between sets rather than in terms of distance. Due to this relation, the research of whether a given (or an induced) topological space is metrizable turns out to be a very rich and interesting area that has motivated a great deal of study over the years. In the last years, the structures (such as 2-metric, G-metric, b-metric, b_2-metric etc.) which are extensions of metric spaces obtained with weakening the metric axioms is studied extensively. One of the most popular research areas in this structures is to induce a topology and study its properties. In the present paper, we consider the topological space induced by a given b_2-metric via ε-balls and study some properties of this topological space. Then, we show with the examples that some known properties for ε-balls in metric spaces have not existed in b_2-metric spaces. Also we introduce the concept of strong b_2-metric spaces in which these known properties are provided. Finally, we show that every strong b_2-metric topological space is a normal space and metrizable.

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