DOI:10.25092/baunfbed.742270 **J. BAUN Inst. Sci. Technol., 22(2), 555-566, (2020)**

Fibonacci polygons

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Geliş Tarihi (Received Date): 18.09.2019
Kabul Tarihi (Accepted Date): 30.01.2020

Abstract

There are many results giving geometric meaning of some algebraic statement or vice versa. In this paper, we answer a question proposed by B. U. Alfred in the first volume of the Fibonacci Quarterly about the existence of Fibonacci quadrilaterals which are quadrilaterals with edge lengths being successive Fibonacci numbers. We give a negative answer to this question in the case where the quadrilaterals are special convex quadrilaterals having 2 successive right angles, and extend it to Fibonacci pentagons and in general Fibonacci n-gons where $n \geq 6$. We show that without such a condition, it is always possible to construct a Fibonacci quadrilateral.

Keywords: *Fibonacci number, polygons.*

Fibonacci çokgenleri

Öz

Bazı cebirsel durumların geometrik anlamını ya da tam tersini gösteren birçok sonuç vardır. Bu çalışmada, B. U. Alfred tarafından Fibonacci Quarterly' nin birinci cildinde sorulan bir soruyu yanıtlıyoruz. Öyle ki bu soru kenar uzunlukları ardışık Fibonacci sayıları olan dörtgenlerin var olup olmadığı hakkındadır. Bu soruya ardışık iki iç açısı dik açı olan konveks dörtgenler ve daha da genişleterek Fibonacci Beşgenleri ve $n \geq 6$ olmak üzere Fibonacci n-genleri için olumsuz cevap vermekteyiz. Böyle bir koşul olmaksızın daima bir Fibonacci dörtgeninin çizmenin mümkün olduğunu göstereceğiz.

Anahtar kelimeler: *Fibonacci sayısı, çokgenler.*

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1. Introduction

There are some special number sequences having many applications and perhaps the most popular number sequence is the Fibonacci numbers F_n. Leonardo of Pisa who is also called Fibonacci was the first person who mentioned these numbers in his book “Liber Abaci”. Leonardo of Pisa had lived in Pisa in 13th century. The first few Fibonacci numbers are

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, · · ·

Fibonacci numbers have emerged from the famous rabbit production problem in *Liber Abaci*. This number sequence has also got a recurrence relation

$F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.

There are a lot of papers in literature related to many different properties of Fibonacci numbers in every area of science. A recent problem which aims to connect number theory with geometry is to search for the relation between special numbers including Fibonacci numbers and geometric shapes usually polygons. Which geometric shapes could have some special numbers as edge lengths? Which polygons could have successive elements of a special number sequence as edge lengths? In [3], the authors dealt with the triangles with the edge lengths being Fibonacci numbers.
For brevity throughout the paper, we give the below definition:

Definition 1. An n-gon having edge lengths as successive Fibonacci numbers is called a Fibonacci n-gon.

In particular, a triangle, a quadrilateral and a pentagon with edge lengths being successive Fibonacci numbers will be called a Fibonacci triangle, Fibonacci quadrilateral and Fibonacci pentagon.

The following is the main theorem of this paper:

Theorem 1. (Main Theorem) There is no convex Fibonacci n-gon having $n-2$ successive right angles for $n \geq 4$.

Proof. We first show that the result is true for $n \geq 6$. Then in the following sections, we show that the result is true for $n = 4$ and $n = 5$, respectively.
The sum of internal angles of a convex n-gon is $(n-2).180^o$. As we assume that there are $n-2$ right angles, the sum of the remaining two internal angles of this polygon would be $(n-2).90^o$. But when $n \geq 6$, this sum of two internal angles would be greater than or equal to 360^o which is impossible in a convex polygon. So n must be less than or equal to 5. Therefore it remains to look for the convex Fibonacci quadrilaterals and Fibonacci pentagons. We shall do this in the following sections.

2) Fibonacci quadrilaterals

In this paper we search for convex Fibonacci quadrilaterals having two consecutive right angles and Fibonacci pentagons having three consecutive right angles. Our motivation to study this problem is the second question in [1] which can be stated as follows:

"Is a Fibonacci quadrilateral possible? Under what circumstances?"
Alfred also gave another relation between geometric shapes and Fibonacci numbers in [2]. There are many identities related to Fibonacci numbers and we shall use some of them to prove our claims.

Theorem 2. No four consecutive Fibonacci numbers can be the edge lengths of a perpendicular trapezoid.

Proof. Let F_n, F_{n+1}, F_{n+2} and F_{n+3} with $n > 0$ be the edge lengths of this perpendicular trapezoid. There are two main cases to prove.

Case 1. Let the oblique edge have the largest edge length. In this case, there are three combinations for the remaining edge lengths:

i) Let the edge lengths be as in Figure 1. In this case, by drawing a height of the trapezoid as in Figure 1, we will obtain a contradiction by means of the Pythagorean theorem. We shall show that the equality

$$(F_{n+2} - F_n)^2 + F_{n+1}^2 = F_{n+3}^2 \quad (1)$$

can never be achieved.

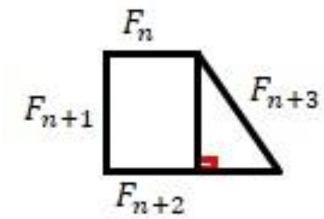

Figure 1.

Assume that Equation 1 is true, that is there is a perpendicular trapezoid as given in Figure 1. Then if we successively use the recurrence relation for Fibonacci numbers and make the necessary cancellations, we get

$$2F_{n+1}^2 + 4F_nF_{n+1} + F_n^2 = 0. \quad (2)$$

which is clearly a contradiction as the left hand side is always positive. So there is no perpendicular trapezoid with four successive Fibonacci numbers as its edge lengths as in Figure 1.

ii) Let the edge lengths of the trapezoid be as in Figure 2.

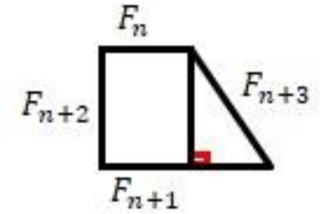

Figure 2.

As in the first subcase, we assume the existence of a perpendicular trapezoid as in Figure 2 and we try to obtain a contradiction. Then we have the relation

$$(F_{n+1} - F_n)^2 + F_{n+2}^2 = F_{n+3}^2 \tag{3}$$

and from this, we get

$$F_n^2 - 2F_{n+1}^2 = 4F_nF_{n+1} \tag{4}$$

which is a contradiction as the two sides have different signs. So there is no such trapezoid.

ii) Let the edge lengths of the trapezoid be as in Figure 3.

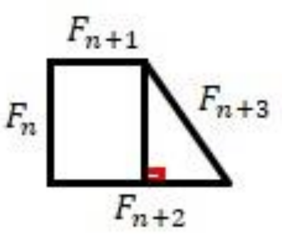

Figure 3.

If there exists such a trapezoid, then the equation

$$(F_{n+2} - F_{n+1})^2 + F_n^2 = F_{n+3}^2 \tag{5}$$

would be true. Then we can obtain the relation

$$F_n^2 - 4F_{n+1}^2 = 4F_nF_{n+1} \tag{6}$$

and this is a contradiction as both sides have different signs.

Case 2. If the oblique edge is one of the intermediate edges, then there are three subcases for the remaining edge lengths.

i) Let the edge lengths be as in Figure 4. Then we would have

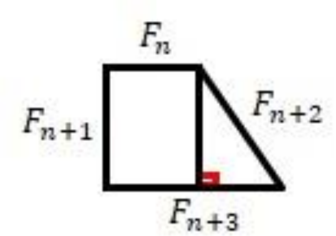

Figure 4.

$$(F_{n+3} - F_n)^2 + F_{n+1}^2 = F_{n+2}^2 \tag{7}$$

Assume that this equation is true. Then using the classical recurrence formula for Fibonacci numbers, we get

$$5F_{n+1}^2 = F_{n+2}^2 \quad (8)$$

which is a contradiction as Fibonacci numbers are integers.

ii) Let secondly the edge lengths be as in Figure 5. Then similarly we obtain

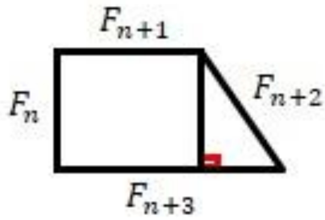

Figure 5.

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$$(F_{n+3} - F_{n+1})^2 + F_n^2 = F_{n+2}^2 \quad (9)$$

which gives

$$F_n^2 = 0$$

and this is once more a contradiction.

iii) Let now the edge lengths of the trapezoid be as in Figure 6. Hence we have the equation

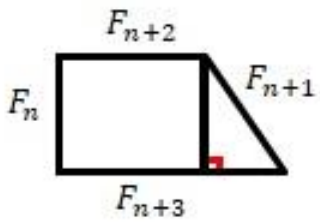

Figure 6.

$$(F_{n+3} - F_{n+2})^2 + F_n^2 = F_{n+1}^2 \quad (10)$$

giving $F_n = 0$. This is a contradiction. ■

Up to now, we dealt with the existence of the Fibonacci quadrilaterals having two consecutive right angles, in another words, with right Fibonacci trapezoids. If we omit this condition and look for a Fibonacci quadrilateral, we have the following result:

Theorem 3. Any four consecutive Fibonacci numbers can be the edge lengths of a quadrilateral.

Proof. We construct a quadrilateral with the edge lengths being successive Fibonacci numbers as follows. Start with a line piece $[AB]$ of length F_{n+3}. Draw a circle with center B and Radius F_{n+2} and a circle with center A and Radius F_n. As $F_{n+3} = F_{n+2} + F_{n+1}$, these two circles are tangent to each other. Let us say at T, see Figure 7, $F_{n+3} - (F_{n+2} + F_n) = F_{n-1}$.

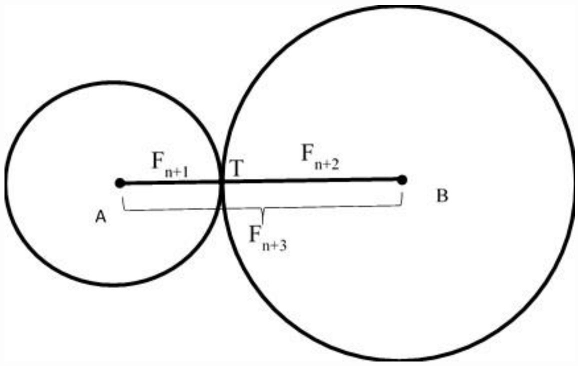

Figure 7.

Choose an arbitrary close point C to T on the larger circle and draw a circle with center C and radius F_n, see Figure 8.

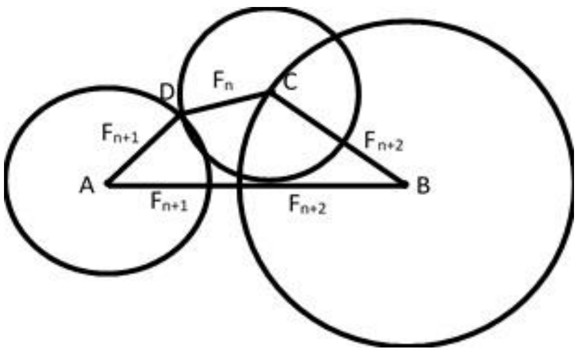

Figure 8.

One of the two intersection points, say D, of this new circle with the one centered at A gives us a convex 4–*gon* $ABCD$, with required edge lengths. ■

3) Fibonacci pentagons

We now extend our results obtained for quadrilaterals having two successive right angles in Section 2 to convex pentagons having three successive right angles.

Theorem 4. No consecutive Fibonacci numbers can be the edge lengths of a convex pentagon with three consecutive right angles.

Proof. We will give the proof in five main cases.

Case 1. If the length of the oblique edge is the smallest one, namely F_n, then there are three subcases for the remaining edge lengths.
i) Let us assume that there exists a pentagon with the edge lengths as shown in Figure 9.

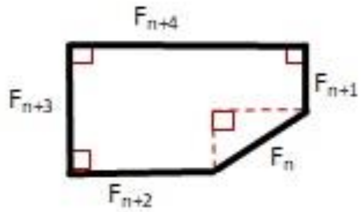

Figure 9.

Then in the right triangle, we have

$$(F_{n+4} - F_{n+2})^2 + (F_{n+3} - F_{n+1})^2 = F_n^2 \tag{11}$$

This relation becomes

$$F_{n+3}^2 + F_{n+2}^2 = F_n^2. \tag{12}$$

If we use the recurrence relation for Fibonacci numbers, we have

$$5F_{n+1}^2 + F_n^2 + 6F_nF_{n+1} = 0 \tag{13}$$

which is a contradiction as the left hand side is strictly positive.

ii) Secondly assume the existence of a pentagon with edge lenghts as in Figure 10.

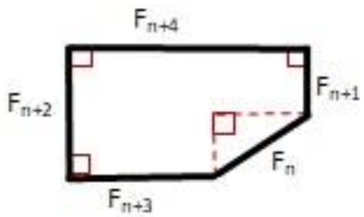

Figure 10.

Then we would have

$$(F_{n+4} - F_{n+3})^2 + (F_{n+2} - F_{n+1})^2 = F_n^2 \tag{14}$$

If we use the recurrence relation for Fibonacci numbers, then we reduce to

$$F_n^2 + F_{n+2}^2 = F_n^2 \tag{15}$$

and therefore we find $F_{n+2} = 0$ which is a contradiction.

iii) Let us assume the existence of a pentagon given in Figure 11.

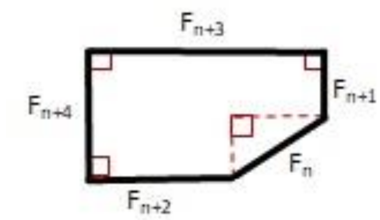

Figure 11.

Then we would have the equation

$$(F_{n+4} - F_{n+1})^2 + (F_{n+3} - F_{n+2})^2 = F_n^2 \tag{16}$$

which would reduce to $5F_{n+1}^2 + 3F_n^2 = 0$ which is a contradiction.

Case 2. Let the oblique edge be F_{n+1}. In this case, there are three subcases for the remaining edge lengths:

i) Consider the possibility of the pentagon given in Figure 12. Then

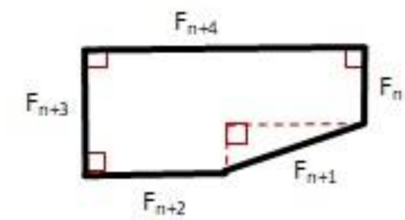

Figure 12.

$$(F_{n+4} - F_{n+2})^2 + (F_{n+3} - F_n)^2 = F_{n+1}^2 \tag{17}$$

and hence we get $3F_{n+1}^2 + F_{n+3}^2 = 0$ which is a contradiction.

ii) Let us now assume the existence of the pentagon in Figure 13

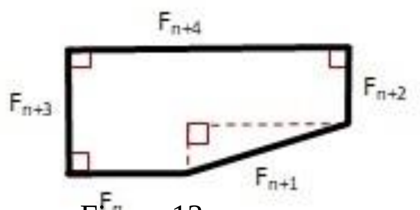

Figure 13.

Then we would have

$$(F_{n+4} - F_n)^2 + (F_{n+3} - F_{n+2})^2 = F_{n+1}^2 \tag{18}$$

hence we get $F_{n+4} = F_n$ and this is impossible.

iii) Thirdly, let's assume the existence of a pentagon with edge lengths given in Figure14.

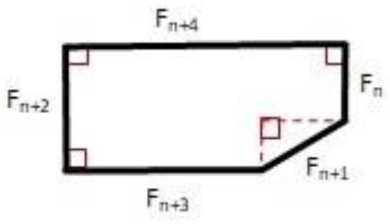

Figure 14.

Then we have
$$(F_{n+4} - F_{n+3})^2 + (F_{n+2} - F_n)^2 = F_{n+1}^2 \tag{19}$$

implying $F_{n+2} = 0$ which is a contradiction.

Case 3. In the final case, we may assume that the oblique edge is F_{n+2}. As before, there are three subcases for the remaining edge lengths.

First let us assume the existence of a pentagon as in Figure 15.

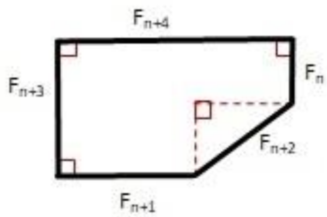

Figure 15.

Then we have the relation

$$(F_{n+4} - F_{n+1})^2 + (F_{n+3} - F_n)^2 = F_{n+2}^2 \tag{20}$$

which reduces to $3F_{n+2}^2 + 4F_{n+1}^2 = 0$ which is clearly a contradiction.

ii) Secondly assume the existence of the pentagon in Figure 16.

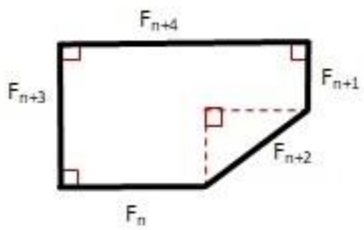

Figure 16.

Then

$$(F_{n+4} - F_n)^2 + (F_{n+3} - F_{n+1})^2 = F_{n+2}^2 \tag{21}$$

which reduces to $F_{n+4} = F_n$. This is a contradiction.

iii) Thirdly we assume that there exists a pentagon as in Figure 17.

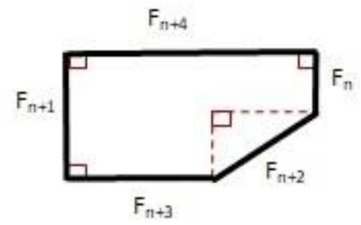

Figure 17.

In this case we have

$$(F_{n+4} - F_{n+3})^2 + (F_{n+1} - F_n)^2 = F_{n+2}^2 \tag{22}$$

which reduces to $F_{n+1} = F_n$ which is naturally a contradiction except for $n = 1$.

Case 4. Let the oblique edge be F_{n+3}. There are three subcases for remaining edge lengths.

i) First consider the pentagon in Figure 18.

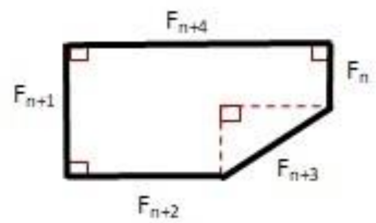

Figure 18.

Now we have

$$(F_{n+4} - F_{n+2})^2 + (F_{n+1} - F_n)^2 = F_{n+3}^2 \tag{23}$$

which, by the recurrence relation, reduces to $F_{n+1} = F_n$ which is impossible except for $n = 1$.

ii) Now consider the pentagon in Figure 19.

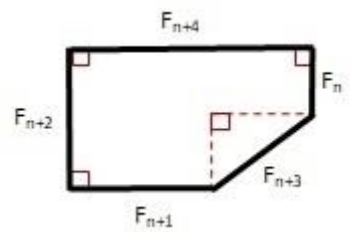

Figure 19.

Here we have

$$(F_{n+4} - F_{n+1})^2 + (F_{n+2} - F_n)^2 = F_{n+3}^2 \tag{24}$$

which is equivalent to

$$(2F_{n+2})^2 + F_{n+1}^2 = F_{n+2}^2 + 2F_{n+1}F_{n+2} + F_{n+1}^2 \tag{25}$$

hence we get $3F_{n+2} = 2F_{n+1}$ which is impossible.

iii) Now let us assume the existence of the pentagon in Figure 20. Then we have

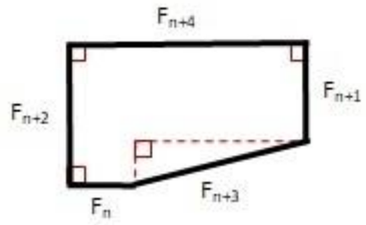

Figure 20.

$$(F_{n+4} - F_n)^2 + (F_{n+2} - F_{n+1})^2 = F_{n+3}^2 \tag{26}$$

which is equivalent to $5F_{n+1}^2 + 2F_nF_{n+1} + F_n^2 = 0$ which is contradiction.

Case 5. Let the oblique edge have the biggest edge length. There are two subcases here:

i) First we have a pentagon as Figure 21.

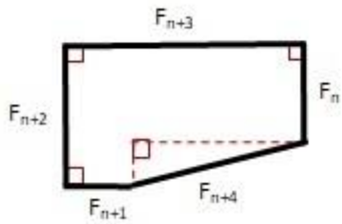

Figure 21.

and hence we write

$$(F_{n+3} - F_{n+1})^2 + (F_{n+2} - F_n)^2 = F_{n+4}^2 \tag{27}$$

if we use Fibonacci recurrence relation as before, we get $3F_{n+2} + 4F_{n+1} = 0$ which is impossible.

ii) Finally we have Figure 22,

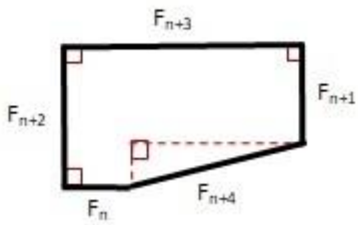

Figure 22.

here similarly to the above cases, we have

$$(F_{n+3} - F_n)^2 + (F_{n+2} - F_{n+1})^2 = F_{n+4}^2 \quad (28)$$

so $5F_{n+1}^2 + 12F_nF_{n+1} + 3F_n^2 = 0$ which is impossible. This completes the proof of the main theorem. ■

Acknowledgement. The authors would like to express their gratitude to Professor P.M. Voutier for his valuable comments and suggestions.

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