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On Slant Helices and Mannheim Curves in E^3

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Abstract
In this paper, the equation of the ones that provide the Mannheim curve feature in slant helices have been obtained and the intrinsic equation of Mannheim curves have been given for the first time.

Keywords: *Slant Helices; Mannheim Curves; Curvature; Torsion.*

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1. Introduction

In classical differential geometry, a general helix in the Euclidean 3-space, is a curve whose tangent vector makes a constant angle with a fixed direction in every point. A slant helix is defined by the property that its principal normal vector makes a constant angle with a fixed direction in a similar way. A regular smooth curve C in Euclidean 3-space E^3 is a Mannheim curve if there exist another regular smooth curve $\widehat{C}$, apart from C and a bijection $\Phi : C \to \widehat{C}$ such that the principal normal line at each point of C coincides with the binormal line of $\widehat{C}$ at the corresponding point under a Φ bijection. Then, $\widehat{C}$ is called a Mannheim mate curve of C.

In the recent times, there has been remarkable interest in the slant helix among geometers. For instance, Menninger [1] gave for the first time a comprehensive characterization of the slant helix in three-dimensional Euclidean space from the point of its curvature and torsion, and obtained an explicit arc-length parametrisation of its tangent vector.

The paper mentioned has guided us in studying on the slant helices that satisfy the definition of Mannheim curve.

2. Preliminaries

A curve is called a **slant helix** if its principal normal vector field makes a constant angle with a fixed line in space. The tuple $F = (T, N, B, \kappa, \tau)$ is defined as a **Frenet apparatus** or **Frenet system** and the pair (κ, τ) a **Frenet development** associated with the curve [1].

Theorem 2.1. *[1](Frenet Apparatus of Slant Helix) A regular C^2 space curve is a slant helix if it has a Frenet development satisfying $\kappa_{SH} = \dfrac{1}{m}\varphi' \cos\varphi$, $\tau_{SH} = \dfrac{1}{m}\varphi' \sin\varphi$ with a differentiable function $\varphi(s)$ and $m = \cot\theta \neq 0$.*

Given such a Frenet development let $\Omega(s) := \dfrac{\varphi(s)}{n}$, $\lambda_1 := 1 - n$, $\lambda_2 := 1 + n$ with $n = \cos\theta$. Then the tangent vector of the slant helix thus characterized can be parametrized as follows:

$$T_{SH} = \frac{1}{2}\begin{pmatrix} \lambda_1 \cos\lambda_2\Omega(s) + \lambda_2 \cos\lambda_1\Omega(s), \\ \lambda_1 \sin\lambda_2\Omega(s) + \lambda_2 \sin\lambda_1\Omega(s), \\ 2\dfrac{n}{m}\sin\Omega(s) \end{pmatrix}.$$

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Definition 2.1. Let E^3 be the 3-dimensional Euclidean space with the standard inner product. If there exits matching relationship between space curve C and $\widehat{C}$ such that at the matching point of the curves, the principal normal vector of C coincides with the binormal vector of $\widehat{C}$, then C is defined as a Mannheim curve, and $\widehat{C}$ is defined as a Mannheim mate curve of C. The pair $\{C, \widehat{C}\}$ is called a Mannheim pair [2].

It is well known that a regular, smooth curve C in E^3 is a Mannheim curve if and only if its curvature function κ and its torsion function τ satisfy the equality

$$\kappa = c(\kappa^2 + \tau^2),$$

on each point of C, where c is a positive constant.

According to the recent studies, the detailed discussions concerned with the Mannheim curves can be found in literature (see [2–7]). But, the intrinsic equation of Mannheim curves does not seem to have been given in the literature. In the following theorem, we give the intrinsic equation of Mannheim curves for the first time.

Theorem 2.2. *The intrinsic equation of a Mannheim curve $\alpha(u)$ can be given*

$$\kappa = \lambda(\sin\Phi + 1), \quad \tau = \lambda\cos\Phi,$$

where Φ is a parameter and $\lambda = \dfrac{\kappa^2 + \tau^2}{2\kappa}$ is a constant. The curvature and torsion functions can also be parametrized by the rational functions,

$$\kappa = \frac{\lambda(1+t)^2}{(1+t^2)}, \quad \tau = \lambda\left(\frac{1-t^2}{1+t^2}\right),$$

where $t = \tan\dfrac{\Phi}{2}$.

Proof. If $\tau \neq 0$ then the relation $\lambda = \dfrac{\kappa^2 + \tau^2}{2\kappa}$ can be rearranged to give

$$\tau^2 + (\kappa - \lambda)^2 = \lambda^2.$$

In a plane with κ and τ as coordinates this relation represent a circle of radius λ along κ axis. The trigonometric parameterisation of this circle gives the result.

The alternative, rational parameterisation is obtained using the tangent-half angle substitution with parameter $t = \tan(\dfrac{\Phi}{2})$.

As it can be clearly seen in the statement above, if Φ is constant then the curvature κ and torsion τ will also be constant and the curve $\alpha(u)$ will be a helix. □

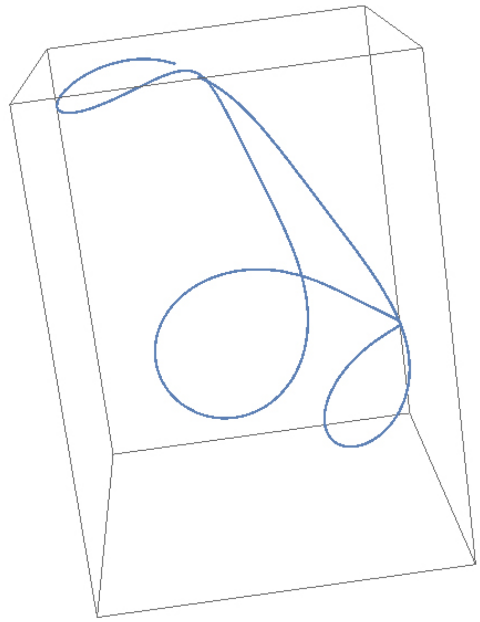

Figure 1. Mannheim curve.

Example 2.1. Let s be curve parameter of the curve α. If we choose $\Phi = s$ and $\kappa = \sin s + 1$, $\tau = \cos s$ the Mannheim curve can be seen in Figure 1.

We can give parametric equation of Mannheim curve in E^3 from Eisenhart's book [8] as follows:

Theorem 2.3. *[8, p. 51] Let C be a curve defined by*

$$X(u) = \begin{bmatrix} \lambda \int h(u) \sin u \, du, \\ \lambda \int h(u) \cos u \, du, \\ \lambda \int h(u) g(u) \, du \end{bmatrix}. \quad u \in U \subset \mathbb{R}.$$

Here $\mathbb{R}$ denotes the set of all real numbers, λ is a positive constant number, $g : U \to \mathbb{R}$ is any smooth function and $h(u) : U \to \mathbb{R}$ is given by

$$h(u) = \frac{\left\{1 + ((g(u))^2 + (\dot{g}(u))^2\right\}^3 + \left\{1 + (g(u))^2\right\}^3 \left\{\ddot{g}(u) + g(u)\right\}^2}{\left\{1 + (g(u))^2\right\}^{3/2} \left\{1 + (g(u))^2 + (\dot{g}(u))^2\right\}^{5/2}},$$

Here the dot $(\cdot)$ denotes the derivative with respect to u. Then the curvature function κ and the torsion function τ of C satisfy

$$\kappa(u) = \lambda \left\{(\kappa(u))^2 + (\tau(u))^2\right\},$$

on each point $X(u)$ of C.

A parametric representation of generalized Mannheim curves in E^4 is given in [5].

3. Slant helices and Mannheim curves

In this section we investigate the curves that satisfy both of the properties of Mannheim curve and slant helix. To do this, we use the slant helix characterization given by Menninger in Theorem 2.1 and we also find the characterization of the differentiable function $\varphi(s)$ given in Theorem 2.1.

Theorem 3.1. *Let a regular space curve C be a slant helix in Euclidean 3-space E^3. Then the curve C is a Mannheim curve if differentiable function $\varphi(s)$ satisfies the following equation*

$$\varphi(s) = 2\arctan u(s),$$

where $u(s)$ is a differentiable function and

$$\kappa_{SH} = \frac{1}{m}\varphi' \cos\varphi, \ \tau_{SH} = \frac{1}{m}\varphi' \sin\varphi.$$

Proof. Let C be a slant helix in E^3, then Theorem 2.1 gives the following equations

$$\kappa_{SH} = \frac{1}{m}\varphi' \cos\varphi, \quad \tau_{SH} = \frac{1}{m}\varphi' \sin\varphi,$$

with a differentiable function $\varphi(s)$ and $m = \cot\theta \neq 0$.

We know that C is a Mannheim curve if and only if κ and τ satisfy the equality

$$\kappa = c(\kappa^2 + \tau^2),$$

on each point of C; where c is a positive constant number. If we use $\kappa = \kappa_{SH}$ and $\tau = \tau_{SH}$ in the equality we obtain the following differential equation

$$\frac{1}{m^2}\varphi'^2(s) = c\frac{1}{m}\varphi'(s)\cos\varphi(s),$$

$$\cos\varphi(s) = \lambda\varphi'(s),$$

where $\lambda = cm$. By solving this equation we find

$$\varphi(s) = 2\arctan\left(\tanh(\lambda c_1 + \frac{s}{2\lambda})\right) = 2\arctan\left(\frac{-1 + e^{c+s/\lambda}}{1 + e^{c+s/\lambda}}\right),$$

$$\varphi(s) = 2\arctan u(s),$$

where $u = \dfrac{-1 + e^{c+s/\lambda}}{1 + e^{c+s/\lambda}}$ and c_1 is a constant number.

Corollary 3.1. *Let C be a space curve. If $\kappa = \frac{1}{m}\varphi' \cos\varphi, \quad \tau = \frac{1}{m}\varphi' \sin\varphi$, $m = \cot\theta \neq 0$ and $\varphi(s) = 2\arctan\left(\frac{-1 + e^{c+s/\lambda}}{1 + e^{c+s/\lambda}}\right)$ then parametric equation of a Slant-Mannheim curve can be given by*

$$X(s) = \frac{1}{2}\begin{pmatrix} \int(\lambda_1\cos\lambda_2\Omega(s) + \lambda_2\cos\lambda_1\Omega(s))ds, \\ \int(\lambda_1\sin\lambda_2\Omega(s) + \lambda_2\sin\lambda_1\Omega(s))ds, \\ \int 2\frac{n}{m}\sin\Omega(s)ds \end{pmatrix}.$$

Example 3.1. Let C be a slant helix. If we choose $\theta = \pi/3$ and $u = \dfrac{-1 + e^s}{1 + e^s}$ then we have

□

$$m = \cot\pi/3 = \frac{1}{\sqrt{3}}, \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{3}{2}, \quad n = \cos\pi/3 = \frac{1}{2}$$

and

$$\Omega(s) = \frac{\varphi(s)}{n} = \frac{2\arctan\left(\frac{-1 + e^s}{1 + e^s}\right)}{1/2} = 4\arctan\left(\frac{-1 + e^s}{1 + e^s}\right).$$

We have the following equation;

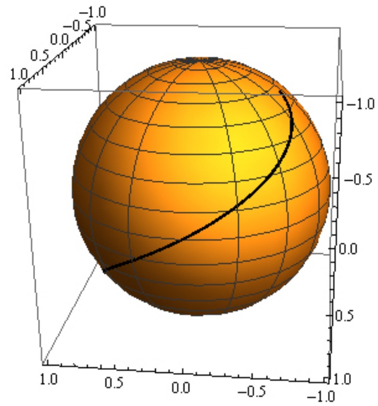

Figure 2. Tangent indicatrix of Slant helix C.

$$T=\frac{1}{2}\begin{pmatrix} \frac{1}{2}\cos\left(6\arctan(\frac{-1+e^s}{1+e^s})\right)+\frac{3}{2}\cos(2arc\tan(\frac{-1+e^s}{1+e^s})\Big), \\ \frac{1}{2}\sin\left(6\arctan(\frac{-1+e^s}{1+e^s})\right)+\frac{3}{2}\sin(2\arctan(\frac{-1+e^s}{1+e^s})\Big), \\ \sqrt{3}\sin\left(2\arctan(\frac{-1+e^s}{1+e^s})\right) \end{pmatrix}.$$

It can be seen in Figure 2 that spherical image the tangent indicatrix of Slant helix C is a spherical helix.

By integrating of T we obtain

$$C(u)=\begin{pmatrix} \frac{e^s(e^{2s}-1)}{(e^{2s}+1)^2}+\arctan e^s, \\ \frac{2e^{2s}(s-4)+e^{4s}s+s}{4(e^{2s}+1)^2}+\frac{2\log(e^{2s}+1)-3s}{4}, \\ \frac{\sqrt{3}}{4}\left(\log(e^{2s}+1)-s\right) \end{pmatrix}.$$

In Figure 3 it can be seen the Slant-Mannheim curve $C(u)$.

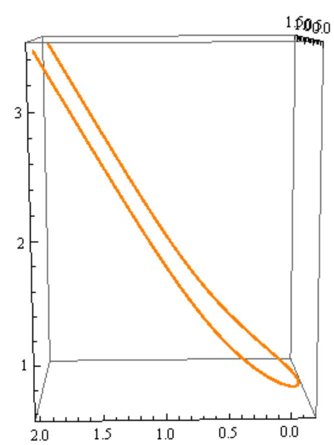

Figure 3. The Slant-Mannheim curve.

4. Conclusion

In this paper, we see that Mannheim curves can be plotted with intrinsic equations. Besides, we offer a general equation of curves that satisfy the properties of both slant and Mannheim curves.

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