New Trends in Mathematical Sciences
http://www.ntmsci.com

Polynomial Sets Generated By $e^{t}\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)$

Khursheed Ahmad[1] *and Naseem Ahmad Khan* [2]

[1] Department of Mathematics, Faculty of Science Aligarh Muslim University, Aligarh-202002, India
[2]Department of Applied MathematicsZ. H. College of Engineering and Technology Aligarh Muslim University, Aligarh-202002, India
[2]Department of Applied Mathematics Vivekananada College of Technology And Management, Aligarh-202002,U.P., India

Received: 22 April 2014, Revised: 12 December 2014, Accepted: 8 January 2015
Published online: 9 January 2015

Abstract: The present paper deal with three variables polynomial sets generated by functions of the form $e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)$. Its special case analogous to Laguerre polynomials have been discussed.

Keywords: Laguerre polynomials of three variables, hypergeometric function.

1 Introduction

Laguerre polynomials $L_n^{(\alpha)}(x)$ possess the generating relation [See Rainville [6] pp-130], $L_n^{(\alpha)}(x)$ is well known Laguerre Polynomials of one variable

$$e^t oF_1(-;1+\alpha;-xt) = \sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)t^n}{(1+\alpha)_n} \tag{1}$$

and $L_n^{(\alpha_1,\alpha_2)}(x_1,x_2)$ is Laguerre Polynomials of two variables due to S.F. Ragab [4] and Chatterjea [1] gave generating function of Laguerre polynomials of two variable $L_n^{(\alpha,\beta)}(x,y)$in the form

$$e^t oF_1(-;\alpha+1;-xt)oF_1(-;\beta+1;-yt) = \sum_{n=0}^{\infty}\frac{n!L_n^{(\alpha,\beta)}(x,y)t^n}{(\alpha+1)_n(\beta+1)_n} \tag{2}$$

One arrives at properties hold by $L_n^{(\alpha)}(x)$ (See Rainville [6], pp. 132-133). Motivated by (2) an attempt has been made to study three variables polynomials similar to one given in (2) and generated by functions of the form $e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)$.

2 Three-variable polynomial sets analogous to (2)

Let us consider the generating relation of the type

$$e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t) = \sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3)t^n \tag{3}$$

Let

$$F(t,x_1,x_2,x_3) = e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t) \tag{4}$$

Then

$$\frac{\partial F}{\partial x_1} = te^t\phi_1'\phi_2\phi_3 \tag{5}$$

* Corresponding author e-mail: naseemamr@rediffmail.com

$$\frac{\partial F}{\partial x_2} = te^t\phi_1\phi_2'\phi_3 \tag{6}$$

$$\frac{\partial F}{\partial x_3} = te^t\phi_1\phi_2\phi_3' \tag{7}$$

$$\frac{\partial F}{\partial t} = e^t\phi_1\phi_2\phi_3 + x_1e^t\phi_1'\phi_2\phi_3 + x_2e^t\phi_1\phi_2'\phi_3 + x_3e^t\phi_1\phi_2\phi_3' \tag{8}$$

Eliminating $\phi_1, \phi_1', \phi_2, \phi_2', \phi_3$ and ϕ_3' from the five equations (4), (5), (6), (7) and (8), we obtain

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)F - t\frac{\partial F}{\partial t} = -tF \tag{9}$$

Since

$$F = e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t) = \sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3)t^n$$

Equation (9) yields

$$\sum_{n=0}^{\infty}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)\sigma_n(x_1,x_2,x_3)t^n - \sum_{n=1}^{\infty}n\sigma_n(x_1,x_2,x_3)t^n$$

$$= -\sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3)t^{n+1}$$

$$= -\sum_{n=1}^{\infty}\sigma_{n-1}(x_1,x_2,x_3)t^n \tag{10}$$

from which the theorem follows.

Theorem 1. From

$$e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t) = \sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3)t^n \tag{11}$$

it follows that

$$\frac{\partial}{\partial x_1}\sigma_o(x_1,x_2,x_3) + \frac{\partial}{\partial x_2}\sigma_o(x_1,x_2,x_3) + \frac{\partial}{\partial x_3}\sigma_o(x_1,x_2,x_3) = 0 \tag{12}$$

and for $n \geq 1$

$$(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3})\sigma_n(x_1,x_2,x_3) - n\sigma_n(x_1,x_2,x_3) = -\sigma_{n-1}(x_1,x_2,x_3) \tag{13}$$

Next, let us assume that the functions ϕ_1, ϕ_2 and ϕ_3 in (11) have the formal power-series expansions.

$$\phi_1(u_1) = \sum_{k_1=0}^{\infty}\gamma_{k_1}u_1^{k_1};\gamma_0 \neq 0 \tag{14}$$

$$\phi_2(u_2) = \sum_{k_2=0}^{\infty}\delta_{k_2}u_2^{k_2};\delta_0 \neq 0 \tag{15}$$

and

$$\phi_3(u_3) = \sum_{k_3=0}^{\infty}\xi_{k_3}u_3^{k_3};\xi_0 \neq 0 \tag{16}$$

Then (11) yields

$$\sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3)t^n = \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)\left(\sum_{k_1=0}^{\infty}\gamma_{k_1}x_1^{k_1}t^{k_1}\right)\left(\sum_{k_2=0}^{\infty}\delta_{k_2}x_2^{k_2}t^{k_2}\right)\left(\sum_{k_3=0}^{\infty}\xi_{k_3}x_3^{k_3}t^{k_3}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2}\frac{\gamma_{k_1}\delta_{k_2}\xi_{k_3}x_1^{k_1}x_2^{k_2}x_3^{k_3}}{(n-k_1-k_2-k_3)!}\right)t^n \tag{17}$$

so that

$$\sigma_n(x_1,x_2,x_3) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2}\frac{\gamma_{k_1}\delta_{k_2}\xi_{k_3}x_1^{k_1}x_2^{k_2}x_3^{k_3}}{(n-k_1-k_2-k_3)!} \tag{18}$$

Now consider the sum

$$\sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,x_3)t^n = \sum_{n=0}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2}\frac{(c)_n\gamma_{k_1}\delta_{k_2}\xi_{k_3}x_1^{k_1}x_2^{k_2}x_3^{k_3}}{(n-k_1-k_2-k_3)!}t^n \tag{19}$$

$$= \sum_{n=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{((c)_{n+k_1+k_2+k_3}\gamma_{k_1}\delta_{k_2}\xi_{k_3}x_1^{k_1}x_2^{k_2}x_3^{k_3}t^{n+k_1+k_2+k_3}}{n!}$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(c)_{k_1+k_2+k_3}\gamma_{k_1}\delta_{k_2}\xi_{k_3}(x_1t)^{k_1}(x_2t)^{k_2}(x_3t)^{k_3}\times\sum_{n=0}^{\infty}\frac{(c+k_1+k_2+k_3)_n}{n!}t^n$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{(c)_{k_1+k_2+k_3}\gamma_{k_1}\delta_{k_2}\xi_{k_3}(x_1t)^{k_1}(x_2t)^{k_2}(x_3t)^{k_3}}{(1-t)^{c+k_1+k_2+k_3}} \tag{20}$$

$$\sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,x_3)t^n = (1-t)^{-c}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(c)_{k_1+k_2+k_3}\gamma_{k_1}\delta_{k_2}\xi_{k_3}\left(\frac{x_1t}{1-t}\right)^{k_1}$$

$$\times\left(\frac{x_2t}{1-t}\right)^{k_2}\left(\frac{x_3t}{1-t}\right)^{k_3} \tag{21}$$

We thus arrive at the following theorem:

Theorem 2. From

$$e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t) = \sum_{n=0}^{\infty}\sigma_n(x_1,x_2,x_3)t^n$$

$$\phi_1(u_1) = \sum_{k_1=0}^{\infty}\gamma_{k_1}u_1^{k_1},\quad \phi_2(u_2) = \sum_{k_2=0}^{\infty}\delta_{k_2}u_2^{k_2},\quad \phi_3(u_3) = \sum_{k_3=0}^{\infty}\xi_{k_3}u_3^{k_3}$$

$$;\gamma_0\neq 0,\delta_0\neq 0,\xi_0\neq 0$$

it follows that for arbitrary c.

$$(1-t)^{-c}G\left(\frac{x_1t}{1-t},\frac{x_2t}{1-t},\frac{x_3t}{1-t}\right) = \sum_{n=0}^{\infty}(c)_n\sigma_n(x_1,x_2,x_3)t^n \tag{22}$$

in which

$$G(u_1,u_2,u_3) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(c)_{k_1+k_2+k_3}\gamma_{k_1}\delta_{k_2}\xi_{k_3}u_1^{k_1}u_2^{k_2}u_3^{k_3} \tag{23}$$

3 Applications of Theorems 1 & 2

The role of theorem 2 is as follows: If a set $\sigma_n(x_1,x_2,x_3)$ has a generating function of the form $e^t\phi_1(x_1t)\phi_2(x_2t)\phi_3(x_3t)$, Theorem 2 yields for $\sigma_n(x_1,x_2,x_3)$ another generating function of the form exhibited in (22). For instance, if $\phi_1(u_1),\phi_2(u_2)$ and $\phi_3(u_3)$ are specified pFq the theorem gives for $\sigma_n(x_1,x_2,x_3)$ a class (c arbitrary) of generating functions involving three variables hypergeometric functions.

Let us now apply Theorems 1 and 2 to Laguerre Polynomials of three variables $L_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)$ due to Khan, M. A. and Shukla, A. K. [3] defined by explicitly

$$L_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)=\frac{(\alpha_1+1)_n(\alpha_2+1)_n(\alpha_3+1)_n}{(n!)^3}$$

$$\times\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2}\frac{(-n)_{k_1+k_2+k_3}x_1^{k_1}x_2^{k_2}x_3^{k_3}}{(\alpha_1+1)_{k_1}(\alpha_2+1)_{k_2}(\alpha_3+1)_{k_3}k_1!k_2!k_3!} \tag{24}$$

The generating function is given by

$$e^t oF_1(-;\alpha_1+1;-x_1t)oF_1(-;\alpha_2+1;-x_2t)oF_1(-;\alpha_3+1;-x_3t)$$

$$=\sum_{n=0}^{\infty}\frac{(n!)^2L_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)t^n}{(\alpha_1+1)_n(\alpha_2+1)_n(\alpha_3+1)_n} \tag{25}$$

We use theorem 1 to conclude that $L_o^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)$ is a constant and for $n\geq 1$,

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}\right)L_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)-nL_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)$$

$$=-\frac{(\alpha_1+n)(\alpha_2+n)(\alpha_3+n)}{n}L_{n-1}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3) \tag{26}$$

In applying theorem 2 to Laguerre polynomials of three variables $L_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)$, we note that

$$\sigma_n(x_1,x_2,x_3)=\frac{(n!)^2L_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)}{(\alpha_1+1)_n(\alpha_2+1)_n(\alpha_3+1)_n}$$

and that

$$\phi_1(u_1)=oF_1(-;1+\alpha_1;-u_1)=\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}u_1^{k_1}}{k_1!(1+\alpha_1)_{k_1}} \tag{27}$$

$$\phi_2(u_2)=oF_1(-;1+\alpha_2;-u_2)=\sum_{k_2=0}^{\infty}\frac{(-1)^{k_2}u_2^{k_2}}{k_2!(1+\alpha_2)_{k_2}} \tag{28}$$

$$\phi_3(u_3)=oF_1(-;1+\alpha_3;-u_3)=\sum_{k_3=0}^{\infty}\frac{(-1)^{k_3}u_3^{k_3}}{k_3!(1+\alpha_3)_{k_3}} \tag{29}$$

Then

$$\gamma_{k_1}=\frac{(-1)^{k_1}}{k_1!(1+\alpha_1)_{k_1}},\delta_{k_2}=\frac{(-1)^{k_2}}{k_2!(1+\alpha_2)_{k_2}},\xi_{k_3}=\frac{(-1)^{k_3}}{k_3!(1+\alpha_3)_{k_3}} \tag{30}$$

and

$$G(u_1,u_2,u_3)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(c)_{k_1+k_2+k_3}\gamma_{k_1}\delta_{k_2}\xi_{k_3}u_1^{k_1}u_2^{k_2}u_3^{k_3}$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{(c)_{k_1+k_2+k_3}(-1)^{k_1+k_2+k_3}u_1^{k_1}u_2^{k_2}u_3^{k_3}}{k_1!k_2!k_3!(1+\alpha_1)_{k_1}(1+\alpha_2)_{k_2}(1+\alpha_3)_{k_3}}$$

$$=\psi_2^{(3)}[c;1+\alpha_1,1+\alpha_2;1+\alpha_3;-u_1,-u_2,-u_3] \tag{31}$$

Therefore Theorem 2 yields

$$(1-t)^{-c}\psi_2^{(3)}\left[c;1+\alpha_1,1+\alpha_2,1+\alpha_3;\frac{-x_1t}{1-t},\frac{-x_2t}{1-t},\frac{-x_3t}{1-t}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(n!)^2 (c)_n L_n^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3) t^n}{(1+\alpha_1)_n (1+\alpha_2)_n (1+\alpha_3)_n} \tag{32}$$

where $\psi_2^{(3)}$ is given in the form [6, p. 62 (11)]

$$\psi_2^{(3)}[a;b,c,d;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}}{(b)_m (c)_n (d)_p} \frac{x^m y^n z^p}{m!n!p!} \tag{33}$$

References

[1] Chatterjea, S. K., *A note on Laguerre polynomials of two variables*, Bull. Cal. Math. Soc., 83 (1991) 263.

[2] Khan, M. A. and Alidad, B., *Polynomial sets generated by $e^t\phi(xt)\psi(yt)$*, Proyecciones Journal of Mathematics, 29 (3) (2010), 201-207.

[3] Khan, M. A. and Shukla, A. K., *On Laguerre polynomials of three variables*, Proyecciones Journal of Mathematics 17 (2) (1998) 227-235.

[4] Ragab, S. F., *On Laguerre Polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$*, Bull, Cal. Math. Soc., 83 (1991), 253-262

[5] Rainville, E. D., *Certain generating functions and associated polynomials*, Amer. Math. Monthly, 52, (1945), 239-250

[6] Rainville, E. D., *Special Functions*, Macmillan, New York; Reprinted by Chelsea Publ. Co., Bronx, New York (1971).

[7] Srivastave, H. M. and Manocha, H. L., *A treatise on generating functions*, Ellis Horwood Limited Chichester, Halsfed Press, a division of John Wiley and Sons, New York (1984).