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Lipschitz Stability of inverse nodal problem for energy-dependent Sturm-Liouville equation

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Abstract: In this study, we solve the reconstruction and some stability problems for diffusion operator using nodal set of eigenfunctions. Moreover, we show that the space of all potential functions q is homeomorphic to the partition set of all asymptotically equivalent nodal sequences induced by an equivalence relation. To show this stability which is known Lipschitz stability, we have to construct two metric spaces and a map Φ_{dif} between these spaces. We find that Φ_{dif} is a homeomorphism when the corresponding metrics are magnified by the derivatives of q. Basically, this method is similar to [1] and [2] which is given for Sturm-Liouville and Hill operators, respectively and depends on the explicit asymptotic expansions of nodal points and nodal lengths.

Keywords: Inverse Nodal Problem, Diffusion Equation, Lipschitz Stability.

1 Introduction

Inverse spectral problems are about recovering operators by using their spectral characteristics as spectrum, norming constants and nodal points. Such problems have an important research area in mathematics and have many applications in natural sciences [3]. Inverse spectral problems are divided into two parts. One of them is inverse eigenvalue problem and the other one is inverse nodal problem. Inverse eigenvalue problems have been studied for along time by many authors [4], [5], [6], [7], [8], [9]. Inverse nodal problem was first posed and solved by McLaughlin [10] who showed that the knowledge of a dense set of nodal points of the eigenfunctions can alone determine the potential function of the Sturm-Liouville problem up to a constant. Independently, Shen and his coworkers studied the relation between the nodal points and the density function of the string equation [11]. Recently, many authors have studied inverse nodal problem for different operators [12], [13], [14], [15], [16], [17].

Consider the following diffusion operator L

$$Ly = y'' + \left[\lambda^2 - 2\lambda p(x) - q(x)\right] y = 0,\ x \in (0,\pi), \tag{1}$$

with two boundary conditions

$$y'(0) - hy(0) = 0, \tag{2}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{3}$$

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where λ is a spectral parameter and $h,H \in \mathbb{R}$; $q(x) \in W_2^1[0,\pi]$, $p(x) \in W_2^2[0,\pi]$ [18], [19]. Since L is determined by its potential functions, we identify L with $L_{p,q}$. (1.1)-(1.3) problems are called quadratic pencils of the Schrödinger operator (or diffusion operator). Some versions of this eigenvalue problem were studied in [20], [21], [22], [23], [24], [25], [26], [27].

Let λ_n be $n-$th eigenvalue of the given operator, $y(x,\lambda_n)$ be the eigenfunction corresponding to the eigenvalue λ_n and $0 < x_1^{(n)} < x_2^{(n)} < ... < x_{n-1}^{(n)} < \pi$ be the nodal points of the $n-$th eigenfunction $y(x,\lambda_n)$. In otherwords, $y(x_j^n,\lambda_n) = 0, j = 1,2,...,n-1$. Also let $I_j^{(n)} = [x_j^{(n)}, x_{j+1}^{(n)}]$ be the $j-$th nodal domain of the $n-$th eigenvalue and let $l_j^{(n)} = x_{j+1}^{(n)} - x_j^{(n)}$ be the nodal length. Define $x_0^n = 0, x_n^n = \pi$. We also define the function $j_n(x)$ on $(0,\pi)$ to be the largest index j such that $0 \le x_j^n \le x$. Thus, $j = j_n(x)$ if and only if $x \in [x_j^n, x_{j+1}^n)$ for fix x and n. Define $X = \{x_j^n\}, n \ge 0$, $j = 1,2,...n-1$. X is called the set of nodal points of (1.1)-(1.3).

Inverse nodal problem for diffusion operator is to determine potential functions and parameters in the boundary conditions. This type problems have been studied by many authors [28], [29], [30], [31], [32], [33], [34], [35], [36].

This study is organized as follows: in section 2, we mention some physical and spectral properties of diffusion operator. In section 3, we give a reconstruction formula for potential function and some important results for the problem (1.1)-(1.3). Finally, we solve Lipschitz and high order Lipschitz stability problems for diffusion operator in sections 4 and 5, respectively.

2 Some Physical and Spectral Properties of Diffusion Equation

Jaulent and Jean [37] stated the actual background of diffusion operators and discussed the inverse problem for the diffusion equation. Also, Gasymov and Guseinov studied the spectral theory of this operator [18].

The problem of describing the interactions between colliding particles is of fundemental interesting in physics. In many cases, a description can be carried out through a well known theoretical model. In particular, one is interested in collisions of two spinless particles, and it is supposed that the $s-$ wave binding energies and $s-$wave scattering matrix are exactly known from collision experiments. $s-$wave Schrödinger equation with a radial static potential V can be written as

$$y'' + [E - V(x)]\, y = 0,\ x \ge 0, \tag{4}$$

where the potential function depends on energy in some way and has the following form of energy dependence

$$V(x,E) = U(x) + 2\sqrt{E}Q(x). \tag{5}$$

$U(x)$ and $Q(x)$ are complex-valued functions. 4 reduces to the Klein-Gordon $s-$wave equation with the static potential $Q(x)$, for a particle of zero mass and the energy $\sqrt{E}$ with the additional condition $U(x) = -Q^2(x)$ [37].

The Klein Gordon equation is considered one of the most important mathematical models in quantum field theory. The equation appears in relativistic physics and is used to describe dispersive wave phenomena in general. The Klein-Gordon equation arise in physics in linear and nonlinear forms [38].

Now, we will consider Klein Gordon wave equation. After some straightforward computations, it turns diffusion equation which is given in 1. This form of Klein Gordon equation was first given in [39]. But here, we shall improve our understanding. Let consider following Klein Gordon wave equation

$$\left[\left(i\frac{\partial}{\partial t}-e\phi\right)^2-\left(\frac{1}{i}\nabla-e\overrightarrow{A}\right)^2\right]\Psi=m^2\Psi. \tag{6}$$

This equation represents a spinless particle of charge e and mass m in a scalar potential ϕ and vector potential $\overrightarrow{A}(r,t)$ where natural units $\hbar=c=1$ have been used. In case of $\overrightarrow{A}(r,t)=0$ and the scalar potential to be time independent, equation 6 is reduced the following form

$$\left[\nabla^2-\frac{\partial^2}{\partial t^2}-2ei\phi\frac{\partial}{\partial t}+e^2\phi^2\right]\Psi=m^2\Psi. \tag{7}$$

If we set $\Psi(r,t)=\varphi(r)e^{-iEt}$, equation 7 takes the form

$$\left[\nabla^2+(E-e\phi)^2\right]\varphi(r)=m^2\varphi(r), \tag{8}$$

where

$$\Psi_t(r,t)=-\varphi(r)iEe^{-iEt}$$
$$\Psi_{tt}(r,t)=-\varphi(r)E^2e^{-iEt}.$$

This equation is a well-known form of Klein-Gordon equation. The wave equation 8 with a spherically symmetric potential energy may be written in spherical coordinates

$$\left\{\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]+(E-e\phi)^2\right\}\varphi=m^2\varphi. \tag{9}$$

We first separate the radial and angular parts by substituting

$$\varphi(r,\theta,\phi)=R(r)Y(\theta,\phi),$$

into equation 9 and dividing through by φ, we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[(E-e\phi)^2-m^2\right]r^2=-\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]. \tag{10}$$

Since the left side of equation 10 depends only on r, and the right side depends only on θ and ϕ, both sides must be equal to a constant that we call $l(l+1)$ where l is orbital quantum number. Then, the equation 10 gives us a radial equation with the scalar potential $V=e\phi$ for the Klein Gordon equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[(E^2-m^2)+V(V-2E)-\frac{l(l+1)}{r^2}\right]R=0, \tag{11}$$

where $l=0,1,2,\ldots$ Substituting $R=\dfrac{\psi(r)}{r}$ in equation 11, we obtain

$$\psi''(r)+\left[K^2+V(V-2E)-\frac{l(l+1)}{r^2}\right]\psi(r)=0,$$

where $E^2=K^2+m^2$. In case of $l=m=0$, we get

$$\psi''(r)+\left[V^2-2KV\right]\psi(r)=-K^2\psi(r). \tag{12}$$

There are many important spectral properties of eigenvalues and eigenfunctions for the problem 1-3. We collect some of these properties in following lemmas.

Lemma 2.1. The eigenvalues of the operator $L_{p,q}$ are simple.

Proof: It can be easily proved by using similar way with in [40].

Lemma 2.2. The operator $L_{p,q}$ defined by 1-3 is in fact symmetric on space $L_2[0,\pi]$.

Proof: Let u and v be twice differentiable functions which satisfy the boundary conditions 2-3. This lemma can be esily proved by using integration by parts on these two functions u and v. Considering the definition of inner product on $L_2[0,\pi]$, we get

$$\begin{aligned}
&< L_{p,q}[u],v>=\int_0^\pi L_{p,q}[u]\,vdx=\int_0^\pi\left\{-u''+[q(x)+2\lambda p(x)]\,u\right\}vdx\\
&=hu(0)v(0)+v(\pi)Hu(\pi)-v(\pi)Hu(\pi)-hu(0)v(0)+\int_0^\pi\left\{-v''+[q(x)+2\lambda p(x)]\,v\right\}udx\\
&=< u,L_{p,q}[v]> .
\end{aligned}$$

Lemma 2.3. The eigenfunctions corresponding to different eigenvalues are orthogonal for the operator $L_{p,q}$.

Proof: Let ϕ_m and ϕ_n be the eigenfunctions corresponding to the eigenvalues λ_m and $\lambda_n,(\lambda_m\neq\lambda_n)$ respectively. If ϕ_k denotes the $k-$th eigenfunction, we can integrate the identity

$$-\phi_m''\phi_n+\phi_n''\phi_m+2p(x)\phi_m\phi_n(\lambda_m-\lambda_n)=\left(\lambda_m^2-\lambda_n^2\right)\phi_m\phi_n.$$

By using similar process to classical Sturm-Liouville problem, orthogonality of eigenfunctions corresponding to different eigenvalues can be showed easily.

Lemma 2.4. [18], [32] Let $\lambda_n, n\in\mathbb{Z}-\{0\}$ be the spectrum of the problem 1-3. It is well known that the sequence $\{\lambda_n : n=\pm1,\pm2,...\}$ satisfies the following classical asymptotic formula

$$\lambda_n=n+c_0+\frac{c_1}{n}+\frac{c_{1,n}}{n}, \tag{13}$$

where

$$c_0=\frac{1}{\pi}\int_0^{\pi}p(x)dx, c_1=\frac{1}{\pi}\left[h+H+\frac{1}{2}\int_0^{\pi}\left[q(x)+p^2(x)\right]dx\right], \sum_{n=1}^{\infty}|c_{1,n}|^2<\infty.$$

Let consider the equation 1 with the following initial conditions

$$y(0)=0,\ y'(0)=1. \tag{14}$$

We will denote by $\varphi(x,\lambda)$ the solution of 1 satisfying the initial condition 2 and by $\psi(x,\lambda)$ the solution of the same equation satisfying the initial conditions 14 [29].

Lemma 2.5. [29] The solutions of the problems 1-3 and 1, 3,14 have the following forms,

$$\varphi(x,\lambda)=\cos(\lambda x)-\frac{h}{\lambda}\sin(\lambda x)+\int_0^x\frac{\sin[\lambda(x-t)]}{\lambda}\left[q(t)+2\lambda p(t)\right]\varphi(t,\lambda)dt, \tag{15}$$

$$\psi(x,\lambda)=\frac{\sin(\lambda x)}{\lambda}+\int_0^x\frac{\sin[\lambda(x-t)]}{\lambda}\left[q(t)+2\lambda p(t)\right]\psi(t,\lambda)dt, \tag{16}$$

respectively.

Lemma 2.6. [29] Suppose that $q\in L_1[0,\pi]$and $p\in W_2^2[0,\pi]$. Then, the nodal points and nodal lengths have the following asymptotic forms, as $n\to\infty$

(a) For the problem 1-3,

$$x_j^n=\frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n}-\frac{h}{2\lambda_n^2}+\frac{1}{2\lambda_n^2}\int_0^{x_j^n}\left[1+\cos(2\lambda_n t)\right]\left[q(t)+2\lambda_n p(t)\right]dt+o\left(\frac{1}{\lambda_n^3}\right), \tag{17}$$

$$l_j^n=\frac{\pi}{\lambda_n}+\frac{1}{2\lambda_n^2}\int_{x_j^n}^{x_{j+1}^n}\left[1+\cos(2\lambda_n t)\right]\left[q(t)+2\lambda_n p(t)\right]dt+o\left(\frac{1}{\lambda_n^3}\right). \tag{18}$$

(b) For the problem 1, 3, 14,

$$x_j^n=\frac{j\pi}{\lambda_n}+\frac{1}{2\lambda_n^2}\int_0^{x_j^n}\left[1-\cos(2\lambda_n t)\right]\left[q(t)+2\lambda_n p(t)\right]dt+o\left(\frac{1}{\lambda_n^3}\right), \tag{19}$$

$$l_j^n=\frac{\pi}{\lambda_n}+\frac{1}{2\lambda_n^2}\int_{x_j^n}^{x_{j+1}^n}\left[1-\cos(2\lambda_n t)\right]\left[q(t)+2\lambda_n p(t)\right]dt+o\left(\frac{1}{\lambda_n^3}\right). \tag{20}$$

Lemma 2.7. [1] Suppose that $q \in L_1[0,\pi]$. Then, for almost every $x \in (0,\pi)$, with $j = j_n(x)$

$$\lim_{n\to\infty} \lambda_n \int_{x_j^n}^{x_{j+1}^n} q(t)dt = q(x).$$

Proof: It can be proved by using similar way with in [1].

The above equalities are still valid even if x_j^n and x_{j+1}^n are replaced by X_J^n and X_{J+1}^n ,respectively.

3 A Reconstruction Formula for Potential Function and some Important Results

In this section, we will give a reconstruction formula for the potential function of diffusion operator and some valuable results. Then, we shall give definitions of the metrics $d_{\Sigma_{dif}}$ and d_0. We will denote L_k^n (grid lengths) to go with X_k^n in the same way that l_k^n (nodal lengths) to go with x_k^n. Also, we denote the space Ω_{dif} as a collection of all diffusion operators $L_{p,q}$ and the space Σ_{dif} as a collection of all admissible double sequences of nodes such that corresponding functions are convergent in L_1. A pseudometric $d_{\Sigma_{dif}}$ will be defined on Σ_{dif}. Essentially, $d_{\Sigma_{dif}}\left(X,\overline{X}\right)$ is so close to

$$d_0\left(X,\overline{X}\right) = \overline{\lim_{n\to\infty}} n^2\pi \sum_{k=1}^{n-1} \left|L_k^n - \overline{L}_k^n\right| + n\int_0^\pi |p(x) - \overline{p}(x)|\,dx, \tag{21}$$

where $X,\overline{X} \in \Sigma_{dif}, L_k^n = X_{k+1}^n - X_k^n$ and $\overline{L}_k^n$ is defined similarly.

If we define $X \sim \overline{X}$ if and only if $d_{\Sigma_{dif}}(X,\overline{X}) = 0$, then $\sim$ is an equivalence relation on Σ_{dif} and $d_{\Sigma_{dif}}$ would be a metric for the partition set $\Sigma_{dif}^* = \Sigma_{dif}/\sim$. Let $\Sigma_{dif1} \subset \Sigma_{dif}$ be the subspace of all asymptotically equivalent nodal sequences and let $\Sigma_{dif1}^* = \Sigma_{dif1}/\sim$. Let Φ_{dif} be the homeomorphism between the spaces Ω_{dif} and Σ_{dif1}^*. We call Φ_{dif} as a nodal map for diffusion operator.

Theorem 3.1. Let $q \in L_1[0,\pi]$ and $p \in W_2^2[0,\pi]$. Define F_n by

$$F_n = n\left[\sum_{k=1}^{n-1} 2n^2 L_k^n - 2n\pi - 2p(x)\right],$$

for the problem 1-3. Then, under the condition $\int_0^\pi p(x)dx = 0$, F_n converges to q pointwisely almost everywhere and also in L_1 sense. Furthermore, pointwise convergence holds for all the continuity points of q.

Proof: We shall consider the asymptotic formulas for the nodal lengths of the problem 1-3. By Lemma 2.6, we have

$$l_j^n \frac{\lambda_n}{\pi} - 1 = \frac{1}{2\lambda_n\pi} \int_{x_j^n}^{x_{j+1}^n} [1+\cos(2\lambda_n t)]\,[q(t) + 2\lambda_n p(t)]\,dt + o\left(\frac{1}{\lambda_n^2}\right),$$

and, after some algebraic computations, we get

$$2\lambda_n^2\left(l_j^n\lambda_n-\pi\right)-2\lambda_n p(x)=\lambda_n\int_{x_j^n}^{x_{j+1}^n}q(t)dt+o(1).$$

If we consider the left side of this equality with the asymptotic formula of λ_n, it yields

$$\begin{aligned}2\lambda_n^2\left(l_j^n\lambda_n-\pi\right)-2\lambda_n p(x)&=\lambda_n\left(2\lambda_n^2 l_j^n-2\lambda_n\pi-2p(x)\right)\\&=\left[n+o\left(\frac{1}{n}\right)\right]\left\{2\left[n+o\left(\frac{1}{n}\right)\right]^2 l_j^n-2\left[n+o\left(\frac{1}{n}\right)\right]\pi-2p(x)\right\}\\&=n\left(2n^2 l_j^n-2n\pi-2p(x)\right)+o(1).\end{aligned}$$

Hence, to prove Theorem 3.1., it suffices to show Theorem 3.2. (b).

Remark: F_n can be obtained similarly for the problem 1, 3, 14.

Theorem 3.2. Suppose that $X\in\Sigma_{dif}$ is asymptotically nodal to $L_{p,q}\in\Omega_{dif}$. Then, we have

(a) For the problem (1.1)-(1.3),

$$h=\lim_{n\to\infty}2\lambda_n\pi\left(j-\frac{1}{2}-\frac{\lambda_n}{\pi}X_j^n\right).$$

(b) For almost every $x\in[0,\pi]$

$$q(x)=\lim_{n\to\infty}2\lambda_n\left[\lambda_n^2 l_j^n-\lambda_n\pi-p(x)\right],$$

where $q\in L_1[0,\pi]$.

Proof:

(a) From the Lemma 2.6, we know the following asymptotic formula for the problem 1-3,

$$X_j^n=\frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n}-\frac{h}{2\lambda_n^2}+\frac{1}{2\lambda_n^2}\int_0^{X_j^n}\left[1+\cos\left(2\lambda_n t\right)\right]\left[q(t)+2\lambda_n p(t)\right]dt+o\left(\frac{1}{\lambda_n^3}\right). \tag{22}$$

After some computations in (3.2), we get

$$\lambda_n\pi\left(j-\frac{1}{2}-\frac{\lambda_n}{\pi}X_j^n\right)=\frac{h}{2}-\frac{1}{2}\int_0^{X_j^n}\left[1+\cos\left(2\lambda_n t\right)\right]\left[q(t)+2\lambda_n p(t)\right]dt+o\left(\frac{1}{\lambda_n}\right).$$

Since X_j^n goes to zero as $n\to\infty$, we obtain

$$h=\lim_{n\to\infty}2\lambda_n\pi\left(j-\frac{1}{2}-\frac{\lambda_n}{\pi}X_j^n\right).$$

(b) It can be proved by using similar procedure with in [29].

Theorem 3.3. [1] Given $L_{p,q}$ in Ω_{dif}, the set of nodal points $\{x_k^n\}$ is also asymptotically nodal to $L_{p,q}$ itself.

Proof: We can easily prove by using similar way with in [1].

4 Lipschitz Stability of Inverse Nodal Problem for Diffusion Operator

In this section, we solve a Lipschitz stability problem for diffusion operator. Lipschitz stability is about a continuity between two metric spaces. To show this continuity, we will use a homeomorphism between these two metric spaces. These type stability problems were studied by many authors [1], [2], [41], [42], [43].

Definition 4.1. Let $\mathbb{N}' = \mathbb{N} - \{1,2\}$.

(i) $\Omega_{dif} = \{q \in L_1(0,\pi) : q$ is the potential function of the diffusion equation$\}$

Σ_{dif} = The collection of the all double sequences defined as

$$X = \left\{X_k^n : k = 1,2,...,n-2, n \in \mathbb{N}'\right\}, 0 < X_1^n < X_2^n < ... < X_{n-1}^n < \pi,$$

for each $n \in \mathbb{N}$. We will show Ω_{dif} and Σ_{dif} are homeomorphic to each other. Hence, when $\overline{X}$ is the nodal set associated with $(\overline{p},\overline{q})$ and $\overline{X}$ is close to X in Σ_{dif}, then $(\overline{p},\overline{q})$ is close to (p,q).

(ii) Let $X \in \Sigma_{dif}$ and define $X = \{X_k^n\}$ where $L_k^n = X_{k+1}^n - X_k^n$ and $I_k^n = \left(X_k^n, X_{k+1}^n\right)$. We say X is quasinodal to some $q \in \Omega_{dif}$ if X is an admissible sequence of nodes and satisfies (I) and (II) below:

(I) X has the following asymptotics uniformly for k, as $n \to \infty$

$$X_k^n = \frac{\left(k - \frac{1}{2}\right)\pi}{n} + O\left(\frac{1}{n^2}\right), k = 1,2,...,n$$

and the sequence

$$F_n = n\left[\sum_{k=1}^{n-1} 2n^2 L_k^n - 2n\pi - 2p(x)\right],$$

converges to q in L_1 for the problem 1-3.

(II) X has the following asymptotics uniformly for k, as $n \to \infty$

$$X_k^n = \frac{k\pi}{n} + O\left(\frac{1}{n^2}\right), k = 1,2,...,n.$$

Similarly, for this case F_n converges to q in L_1 for the problem 1, 3, 14.

Definition 4.2. Suppose that $X,\overline{X} \in \Sigma_{dif}$ with L_k^n and $\overline{L}_k^n$ as their respective grid lengths. Let

$$S_n\left(X,\overline{X}\right) = n^2\pi \sum_{k=1}^{n-1} \left|L_k^n - \overline{L}_k^n\right| + n\int_0^\pi |p(x)-\overline{p}(x)|\,dx. \tag{23}$$

and

$$d_0\left(X,\overline{X}\right) = \overline{\lim_{n\to\infty}} S_n\left(X,\overline{X}\right) \text{ and } d_{\Sigma_{dif}}\left(X,\overline{X}\right) = \overline{\lim_{n\to\infty}} \frac{S_n\left(X,\overline{X}\right)}{1+S_n\left(X,\overline{X}\right)}.$$

where p is so close to $\overline{p}$. This definition was first made by [1] for Sturm-Liouvile operator. Since the function $f(x) = \dfrac{x}{1+x}$ is monotonic, we get

$$d_{\Sigma_{dif}}\left(X,\overline{X}\right) = \frac{d_0\left(X,\overline{X}\right)}{1+d_0\left(X,\overline{X}\right)} \in [0,\pi].$$

Conversely,

$$d_0\left(X,\overline{X}\right) = \frac{d_{\Sigma_{dif}}\left(X,\overline{X}\right)}{1-d_{\Sigma_{dif}}\left(X,\overline{X}\right)}.$$

This equalities can be obtained easily.

Lemma 4.1. Let $X,\overline{X} \in \Sigma_{dif}$.

(a) $d_{\Sigma_{dif}}$ is a pseudometric on Σ_{dif}.

(b) If X and $\overline{X}$ belong to different cases, then $d_{\Sigma_{dif}}\left(X,\overline{X}\right) = 1$.

(c) If X belongs to case (I) or case (II), then

$$L_k^n = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right), k = 1,2,...,n.$$

(d) Let X and $\overline{X}$ belong to same case.

i) The interval $\delta_{n,k}$ between the points X_k^n and $\overline{X}_k^n$ has length $O\left(\frac{1}{n^2}\right)$, as $n \to \infty$.

ii) For all $x \in (0,\pi)$, define $J_n(x) = \max\{k : X_k^{(n)} \le x\}$ so that $k = J_n(x)$ if and only if $x \in [X_J^n, X_{J+1}^n)$. Then, $\left|J_n(x) - \overline{J}_n(x)\right| \le 1$ for sufficiently large n.

Proof: It can be proved similar to [1], [2].

After following theorem, we can say that inverse nodal problem for diffusion operator is Lipschitz stable.

Theorem 4.1. The metric spaces $\left(\Omega_{dif}, \|.\|_1\right)$ and $\left(\Sigma_{dif1}/\sim, d_{\Sigma_{dif}}\right)$ are homeomorphic to each other. Here, $\sim$ is the equivalence relation induced by $d_{\Sigma_{dif}}$. Furthermore

$$\|q-\overline{q}\|_1 = \frac{2d_{\Sigma_{dif}}\left(X,\overline{X}\right)}{1-d_{\Sigma_{dif}}\left(X,\overline{X}\right)},$$

where $d_{\Sigma_{dif}}\left(X,\overline{X}\right) < 1$ and p is so close to $\overline{p}$.

Proof: In view of Lemma 4.1., we only need to consider when $X,\overline{X} \in \Sigma_{dif}$ belong to same case. Without loss of generality, let $X,\overline{X}$ belong to case I. We have to show

$$\|q-\overline{q}\|_1 = 2d_0\left(X,\overline{X}\right).$$

According to the Theorem 3.1., F_n and $\overline{F}_n$ convergence to q and $\overline{q}$, respectively where p is close to $\overline{p}$. If we use definition of norm on L_1 for the potential functions, we get

$$\|q-\overline{q}\|_1 \leq 2n^3\int_0^{\pi}\left|L^n_{J_n(x)}-\overline{L}^n_{\overline{J}_n(x)}\right|dx+2n\int_0^{\pi}|p(x)-\overline{p}(x)|\,dx+o(1)$$

and after some algebraic operations

$$\|q-\overline{q}\|_1 \leq 2n^3\int_0^{\pi}\left|L^n_{J_n(x)}-\overline{L}^n_{J_n(x)}\right|dx+2n^3\int_0^{\pi}\left|\overline{L}^n_{J_n(x)}-\overline{L}^n_{\overline{J}_n(x)}\right|dx+2n\int_0^{\pi}|p(x)-\overline{p}(x)|\,dx+o(1). \tag{24}$$

Here, the integrals in the second and first terms can be written as

$$\int_0^{\pi}\left|\overline{L}^n_{J_n(x)}-\overline{L}^n_{\overline{J}_n(x)}\right|dx = o\left(\frac{1}{n^3}\right),$$

and

$$\int_0^{\pi}\left|L^n_{J_n(x)}-\overline{L}^n_{J_n(x)}\right|dx = \frac{\pi}{n}\sum_{k=1}^{n-1}\left|L^n_k-\overline{L}^n_k\right|,$$

respectively. If we consider these equalities in 24, we get

$$\|q-\overline{q}\|_1 \leq 2n^3 o\left(\frac{1}{n^3}\right)+2n^3\left[\frac{\pi}{n}\sum_{k=1}^{n-1}\left|L^n_k-\overline{L}^n_k\right|\right]+2n\int_0^{\pi}|p(x)-\overline{p}(x)|\,dx+o(1),$$

and

$$\|q-\overline{q}\|_1 \leq 2n^2\pi\sum_{k=1}^{n-1}\left|L^n_k-\overline{L}^n_k\right|+2n\int_0^{\pi}|p(x)-\overline{p}(x)|\,dx+o(1). \tag{25}$$

Similarly, we can easily obtain

$$\|q-\overline{q}\|_1 \geq 2n^2\pi\sum_{k=1}^{n-1}\left|L^n_k-\overline{L}^n_k\right|+2n\int_0^{\pi}|p(x)-\overline{p}(x)|\,dx+o(1). \tag{26}$$

Considering 25 and 26 together, it yields

$$\|q-\overline{q}\|_1 = 2n^2\pi\sum_{k=1}^{n-1}\left|L_k^n-\overline{L}_k^n\right| + 2n\int_0^{\pi}|p(x)-\overline{p}(x)|\,dx.$$

The proof is complete after by taking limit as $n\to\infty$.

5 High order Lipschitz stability for diffusion operator

In this section, we will solve a high order Lipschitz stability problem for diffusion operator. For $n\in\mathbb{N}$,

$$\Omega_{dif}(N) = \left\{q\in L_1[0,\pi] : q\in C^{N+1}[0,\pi]\right\}.$$

It has been proved in [33] that $m-$th derivative of potential function $q, q^{(m)}$ for diffusion operator can be approximated by some difference quotient of nodal length $\delta^m l_j^n$ where δ^m is $m-$th order difference quotient operator defined as [1],

$$\delta a_j^{(n)} = \frac{a_{j+1}^{(n)}-a_j^{(n)}}{a_j^{(n)}} \text{ and } \delta^m a_j^{(n)} = \frac{\delta^{m-1}a_{j+1}^{(n)}-\delta^{m-1}a_j^{(n)}}{a_j^{(n)}}.$$

Note that these operators depend on the double sequence $\{a_i^{(n)}\}$.

Let $\Sigma_{dif}(N)$ be the set of asymptotically equivalent nodal sequences in $\Omega_{dif}(N)$. Let $d_{\Omega_{dif}(N)}$ and $D_{\Sigma_{dif}(N)}$ be some metrics on $\Omega_{dif}(N)$ and $\Sigma^*_{dif}(N)$ respectively magnified by the L_1 norms of derivatives of the potential functions. We find that the nodal map Φ_{dif} is still a homeomorphism under these strengthed metrics.

If we define $X\sim_N \overline{X}$ if and only if $D_{\Sigma_{dif}(N)}\left(X,\overline{X}\right)=0$, then $\sim_N$ is an equivalence relation on $\Sigma_{dif}(N)$. Hence $d_{\Omega_{dif}(N)}$ is a metric on $\Omega_{dif}(N)$.

Definition 5.1. Let $X,\overline{X}\in\Sigma_{dif}$ and $X_k^n=\overline{X}_k^n=\pi$ for $k\geq n$. For $m=1,2,...,N$, let

$$S_{m,n}\left(X,\overline{X}\right) = \lambda_n^{\frac{1}{2}}\sum_{k=1}^{n-m-2}\left|\delta^m L_k^n-\delta^m\overline{L}_k^n\right| + \lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m\overline{p}(x_{\overline{J}})-\delta^m p(x_J)\right|dx$$
$$+\lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\left|\overline{p}^{(m)}(x)-p^{(m)}(x)\right|dx. \tag{27}$$

Define

$$d_m\left(X,\overline{X}\right) = \overline{\lim_{n\to\infty}} S_{m,n}\left(X,\overline{X}\right) \text{ and } d_{\Sigma_{dif}(m)}\left(X,\overline{X}\right) = \overline{\lim_{n\to\infty}}\frac{S_{m,n}\left(X,\overline{X}\right)}{1+S_{m,n}\left(X,\overline{X}\right)}.$$

We have to use the following reconstruction formula for the function $q^{(m)}$ in Theorem 5.1. to prove Theorem 5.3.

Theorem 5.1. [33] Let q is real valued, $(n+1)-$th order continuous function from the class $L_1[0,\pi]$ for $N\geq 1$ in

(1.1), and let $j = j_n(x)$ for each $x \in [0,\pi]$. Then, as $n \to \infty$

$$q(x) = \lambda_n \left[2\lambda_n^2 l_j^n - 2\pi\lambda_n - 2p(x)\right] + O\left(\frac{1}{n}\right),$$

and, for all $m = 1,2,...,N$

$$q^{(m)}(x) = \frac{2\lambda_n^{3/2}}{\pi}\delta^m l_j^n - 2\lambda_n\delta^m p(x_j) - 2\lambda_n p^{(m)}(x) + O(1).$$

Theorem 5.2. The metric spaces $(\Omega_{dif}(N), d_{\Omega_{dif}(N)})$ and $(\Sigma_{dif1}(N)/\sim_N, D_{\Sigma_{dif}(N)})$ are homeomorphic to each other, where $\sim_N$ is the equivalence relation induced by $D_{\Sigma_{dif}(N)}$.

Proof: In view of Lemma 4.1., we only need to consider $X,\overline{X} \in \Sigma_{dif}$ belong to same case. We shall show that $\left\|q^{(m)} - \overline{q}^{(m)}\right\| = 2d_m(X,\overline{X})$. Hence, to prove Theorem 5.2., it suffices to prove Theorem 5.3.

After Theorem 5.3., we can say that the inverse nodal problem for the diffusion operator is high order Lipschitz stable.

Theorem 5.3. Suppose that q and $\overline{q}$ are both Nth order continuous functions from the class $L_1[0,\pi]$; $L_{p,q},\overline{L}_{\overline{p},\overline{q}}$ belong to same case where p is so close to $\overline{p}$. Let X and $\overline{X}$ be the corresponding asymptotically equivalent nodal sequences. Then, for all $m = 1,2,...,N$

$$\left\|q^{(m)} - \overline{q}^{\,(m)}\right\| = 2d_m(X,\overline{X}).$$

Proof: By Theorem 5.1 and Lemma 2.6, we can write

$$q^{(m)} - \overline{q}^{(m)} = \frac{2\lambda_n^{3/2}}{\pi}\left[\delta^m L_J^n - \delta^m \overline{L}_{\overline{J}}^n\right] + 2\lambda_n\left[\delta^m\overline{p}(x_{\overline{J}}) - \delta^m p(x_J)\right] + 2\lambda_n\left[\overline{p}^{(m)}(x) - p^{(m)}(x)\right] + o(1),$$

and

$$\left|q^{(m)} - \overline{q}^{(m)}\right| = \left|\frac{2\lambda_n^{3/2}}{\pi}\left[\delta^m L_J^n - \delta^m \overline{L}_{\overline{J}}^n\right] + 2\lambda_n\left[\delta^m\overline{p}(x_{\overline{J}}) - \delta^m p(x_J)\right] + 2\lambda_n\left[\overline{p}^{(m)}(x) - p^{(m)}(x)\right] + o(1)\right|.$$

Then, by using the definition of norm on L_1 space, we obtain

$$\left\|q^{(m)} - \overline{q}^{(m)}\right\|_1 = \int_{X_1^n}^{X_{n-m-1}^n}\left|q^{(m)}(x) - \overline{q}^{(m)}(x)\right|dx,$$

and

$$\left\|q^{(m)} - \overline{q}^{(m)}\right\|_1 = \int_{X_1^n}^{X_{n-m-1}^n}\left|\frac{2\lambda_n^{3/2}}{\pi}\left[\delta^m L_J^n - \delta^m \overline{L}_{\overline{J}}^n\right] + 2\lambda_n\left[\delta^m\overline{p}(x_{\overline{J}}) - \delta^m p(x_J)\right] + 2\lambda_n\left[\overline{p}^{(m)}(x) - p^{(m)}(x)\right] + o(1)\right|dx \qquad (28)$$

when n is sufficiently large. By using the property of triangle inequality in (5.2), we get

$$\left\|q^{(m)}-\overline{q}^{(m)}\right\|_1 \le \frac{2\lambda_n^{3/2}}{\pi}\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m L_J^n-\delta^m \overline{L}_{\overline{J}}^n\right|dx+2\lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m\overline{p}(x_{\overline{J}})-\delta^m p(x_J)\right|dx$$
$$+2\lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\left|\overline{p}^{(m)}(x)-p^{(m)}(x)\right|dx+o(1),$$

and after some computations,

$$\left\|q^{(m)}-\overline{q}^{(m)}\right\|_1 \le \frac{2\lambda_n^{3/2}}{\pi}\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m L_J^n-\delta^m \overline{L}_J^n\right|dx+\frac{2\lambda_n^{3/2}}{\pi}\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m \overline{L}_J^n-\delta^m \overline{L}_{\overline{J}}^n\right|dx$$
$$+2\lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\delta^m\overline{p}(x_{\overline{J}})-\delta^m p(x_J)dx+2\lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\left|\overline{p}^{(m)}(x)-p^{(m)}(x)\right|dx+o(1).$$

Now, by using [[44], Lemma 2.1] and Lemma 4.1., we obtain

$$\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m L_J^n-\delta^m \overline{L}_J^n\right|dx=\sum_{k=1}^{n-m-2}\left|\delta^m L_k^n-\delta^m \overline{L}_k^n\right|\left|L_k^n\right|,$$

and

$$\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m \overline{L}_J^n-\delta^m \overline{L}_{\overline{J}}^n\right|dx=O\left(\frac{1}{n^4}\right).$$

Moreover, $L_k^n=\dfrac{\pi}{n}+O\left(\frac{1}{n^2}\right)$. Thus,

$$\frac{2\lambda_n^{3/2}}{\pi}\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m L_J^n-\delta^m \overline{L}_{\overline{J}}^n\right|dx \le \frac{2\lambda_n^{3/2}}{\pi}O\left(\frac{1}{n^4}\right)+\frac{2\lambda_n^{3/2}}{\pi}\sum_{k=1}^{n-m-2}\left|\delta^m L_k^n-\delta^m \overline{L}_k^n\right|\left|\frac{\pi}{n}+O\left(\frac{1}{n^2}\right)\right|$$
$$=o\left(\frac{1}{n^{5/2}}\right)+2n^{1/2}\sum_{k=1}^{n-m-2}\left|\delta^m L_k^n-\delta^m \overline{L}_k^n\right|+\frac{2}{\pi}n^{3/2}nO(n^{-2})o\left(\frac{1}{n^3}\right)$$
$$=2n^{1/2}\sum_{k=1}^{n-m-2}\left|\delta^m L_k^n-\delta^m \overline{L}_k^n\right|+o\left(\frac{1}{n^{5/2}}\right).$$

Therefore,

$$\left\|q^{(m)}-\overline{q}^{(m)}\right\|_1 \le 2n^{1/2}\sum_{k=1}^{n-m-2}\left|\delta^m L_k^n-\delta^m \overline{L}_k^n\right|+2\lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\left|\delta^m\overline{p}(x_{\overline{J}})-\delta^m p(x_J)\right|dx$$
$$+2\lambda_n\int_{X_1^n}^{X_{n-m-1}^n}\left|\overline{p}^{(m)}(x)-p^{(m)}(x)\right|dx+o(1). \tag{29}$$

If we take limit as $n \to \infty$ and use the metric definition, we obtain

$$\left\| q^{(m)} - \overline{q}^{(m)} \right\| \leq 2d_m(X, \overline{X}). \tag{30}$$

Using similar procedure,

$$\left\| q^{(m)} - \overline{q}^{(m)} \right\| \geq 2d_m(X, \overline{X}). \tag{31}$$

Then, considering (5.4) and (5.5) together, we get

$$\left\| q^{(m)} - \overline{q}^{(m)} \right\| = 2d_m(X, \overline{X}).$$

This completes the proof.

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