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Generalization of $([e],[e]\vee[c])$-Ideals of BE-algebras

A. F. Ali[1], *S. Abdullah*[2] *and M. S. Kamran* [3] *and M. Aslam* [4]

[1] Department of Basic Sciences, Riphah Internaional University, Islamabad, Pakistan.
[2] Department of Mathematics, Quaid-e-Aazam University, Islamabad, Pakistan.
[3] Department of Basic Sciences, Riphah Internaional University, Islamabad Pakistan
[4] Department of Mathematics, King Khlid University Saudi Arabia.

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Abstract: In this paper, using N-structure, the notion of an N-ideal in a BE-algebra is introduced. To obtain a more general form of an N-ideal, a point N-structure which is (k-conditionally) employed in an N-structure is proposed. Using these notions, the concept of an $([e],[e]\vee[c_k])$-ideal is introduced and related properties are investigated. The notion $([e],[e]\vee[c_k])$-ideal is a generalization of $([e],[e]\vee[c])$-ideal. We derive some characterizations of $([e],[e]\vee[c_k])$-ideals of BE-algebras.

Keywords: BE-algebra, (Transitive, self distributive) BE-algebra, Ideal, N-ideal, $([e],[e]\vee[c_k])$-ideal.

1 Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A.

So far most of the generalization of the crisp set have been conducted on the unit interval $[0,1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets spread positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$.

Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool.

To attain such an object, Lee et al.[12] introduced a new function which is called a negative-valued function, and constructed N-structures. They applied N-structures to BCK/BCI-algebras, and discussed N-ideals in BCK/BCI-algebras. In 1966, Iseki and Imai [7] and Iseki [8] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, Kim and Kim [3] introduced the notion of a BE-algebra and filter of a BE-algebra, and investigated several properties. So and Ahn [6] introduced the notion of ideals in BE-algebras. They considered several descriptions of ideals in BE-algebras. They defined the upper set in BE-algebra and derived some relation between ideal theory and upper set in BE-algebras. Kim and Lee generalized the concept of So and Ahn, to defined extended upper set in BE-algebras [4]. They provided the relations between filters and extended upper set in BE-algebras. Ahn et al. introduced the concepts of self-distributive and transitive BE-algebras and also discussed some properties of filters in

* Corresponding author e-mail: saleemabdullah81@yahoo.com

commutative BE-algebras [1]. They discussed some properties of characterizations of generalized upper set $A_n(u,v)$ to the structure of ideals and filters in BE-algebras. The Congruence's and BE-relations on BE-algebra was defined by Yon et al. [2]. Recently, Saeid et al. studied filters in BE-algebras [10]. They defined positive filters, normal filters and implicative filters of BE-algebras. Also gave some relation among these types of filters in BE-algebras. They introduced some interesting results on these filters. Also gave some characterization of BE-algebras by these filters.

Kang, and Jun [9], introduced the notion of an N-ideal of BE-algebra. In [9], a point N-structure which is (Conditionally) employed in an N-structure is proposed. The concept of $([e],[e]\vee[c])$-ideals and discussed the related properties. The present author introduce the new notion called k-conditionally employed of point N-structure. We use this concept to N-ideal of BE-algebra to introduce a generalization of $([e],[e]\vee[c])$-ideals of BE-algebras.

In this paper, we introduce the notion of k-conditionally employed to N-structure. By applying of employed and k-conditionally employed to N-structure with point N-structure to introduce the concept of an $([e],[e]\vee[c_k])$-ideal in a BE-algebra and give some related properties. The aim of this paper is to introduce a new generalization of the concept of $([e],[e]\vee[c])$-ideal. We give some characterization of $([e],[e]\vee[c_k])$-ideals of BE-algebras by the level sets. We further discuss some interesting results of $([e],[e]\vee[c_k])$-ideals of BE-algebra.

2 Preliminaries

Definition 1. [3] Let $K(\tau)$ be a class of type $\tau=(2,0)$. A system $(X;*,1)\in K(\tau)$ define a **BE-Algebra** if the following axioms hold:
(V_1) $(\forall x\in X)$ $(\,x*x=1\,)$,
(V_2) $(\forall x\in X)$ $(\,x*1=1\,)$,
(V_3) $(\forall x\in X)$ $(\,1*x=x\,)$,
(V_4) $(\forall x,y,z\in X)$ $(\,x*(y*z)=y*(x*z)\,)$.

Definition 2. [6] A relation "$\leq$" on a BE-algebra X is defined by $(\forall x,y\in X)$ $(\,x\leq y\Leftrightarrow x*y=1\,)$.

Definition 3. [6] A BE-algebra X is called **Self-distributive** if $x*(y*z)=(x*y)*(x*z)$ for all $x,y,z\in X$.

Definition 4. [4] A BE-algebra $(X;*,1)$ is said to be **Transitive** if it satisfies: $(\forall x,y,z\in X)$ $(\,y*z\leq(x*y)*(x*z)\,)$.

Definition 5. [6] Let I a non-empty subset of an BE-algebra X then I is called an **Ideal** of X if;
(1) $(\forall x\in X,\, s\in I)$ $(\,x*s\in I\,)$,
(2) $(\forall x\in X,\, s,q\in I)$ $(\,(s*(q*x))*x\in I\,)$.

Lemma 1. [6] A non-empty subset I of X is an ideal of X if and only if it satisfies:
(1) $1\in I$,
(2) $(\forall x,z\in X)$ $(\,\forall y\in I\,)$ $(\,x*(y*z)\in I\Rightarrow x*z\in I\,)$.

3 N-ideals of a BE-algebra

Definition 6. [9] An element of $\tau(X,[-1,0])$ is called a **Negative-valued function** from X to $[-1,0]$ (briefly, N-function on X).

Definition 7. [9] An ordered pair (X,f) of X and an N-function f on X is called an N**-structure.**

Definition 8. [9] For any N-structure (X,f) the non-empty set

$$C(f;t) := \{x \in X \mid f(x) \le t\}$$

is called a **closed** (f,t)**-cut** of (X,f), where $t \in [-1,0]$.

Definition 9. [9] By an N-ideal of X we mean an N-structure (X,f) which satisfies the following condition: $(\forall t \in [-1,0])$ $(\ C(f;t) \in J(X) \cup \{\emptyset\}\ )$.where $J(X)$ is a set of all ideal of X.

Example 1. Let $X = \{1,\alpha,h,m,0\}$ be a set with a multiplication table given by;

$*$	1	α	h	m	0
1	1	α	h	m	0
α	1	1	α	m	m
h	1	1	1	m	m
m	1	α	h	1	α
0	1	1	α	1	1

Then $(X;*,1)$ is a BE-algebra. Consider an N-structure (X,f) in which t is defined by;

$$f(y) = \begin{cases} -0.7 \text{ if } \ y \in \{1,\alpha,h\} \\ -0.2 \text{ if } \ y \in \{m,0\} \end{cases}$$

Then

$$C(f;t) = \begin{cases} \{1,\alpha,h\} \text{ if } \ t \in [-0.7,0] \\ \emptyset \qquad\quad \text{ if } \ t \in [-1,-0.7) \end{cases}$$

Note that $\{1,\alpha,h\}$ is an ideals of BE-algebra X, and hence (X,f) is an N-ideal of X.

Lemma 2. Each N-ideal (X,f) of BE-algebra X satisfies the condition $(\forall x \in X)$ $(\ f(1) \le f(x)\ )$.

Proof. Since in BE-algebra we have $x*x = 1$, thus we have $f(1) = f(x*x) \le f(x)$ for all $x \in X$.

Proposition 1. Each N-ideal f of BE-algebra X satisfies the condition $(\forall x,y \in X)$ $(\ f((x*y)*y) \le f(x)\ )$.

Proof. Straightforward.

Proposition 2. Each N-ideal f of BE-algebra X satisfies the condition; $(\forall x,y \in X)$ $(\ f(y) \le \max\{f(x), f(x*y)\}\ )$.

Proof. It can be easily proved.

Corollary 1. If $x \le y$, then each N-ideal f of BE-algebra X satisfies the condition; $f(y) \le f(x)$.

Proof. Suppose $x \le y$ for all $x,y \in X$. Then $x*y = 1$, so

$$f(y) = f(1*y) = f((x*y)*y)$$

By proposition 1, $f((x*y)*y) \le f(x)$, hence $f(y) \le f(x)$.

4 $([e],[e]\vee[c_k])$-Ideals

In [9], Jun et. al introduced the concept of N-ideals in BE-algebras and generalized N-ideals in BE-algebra by using the concept of employed and conditionally employed. In this section, we give further generalization of conditionally employed which is called k-conditionally employed. We use this concept to N-ideals of BE-algberas to introduce the notion of $([e],[e]\vee[c_k])$-ideals of BE-algebras. The concept of $([e],[e]\vee[c_k])$-ideals of BE-algebras is a generalization of N-ideals and $([e],[e]\vee[c])$-ideals of BE-algebras.

Let f be an N-structure of of BE-algebra X in which f is given by;

$$f(y)=\begin{cases}0 \text{ if } y\neq x\\ t \ \text{ if } \ y=x\end{cases}$$

where $t\in[-1,0)$, In this case, f is represented by $\frac{x}{t}$. $(X,\frac{x}{t})$ is called **Point N-structure** [9]. A Point N-structure $(X,\frac{x}{t})$ is called **Employed** in an N-structure (X,f) of BE-algebra X if $f(x)\leq t$ for all $x\in X$, and $t\in[-1,0)$. It is represented as $(X,\frac{x}{t})[e](X,f)$ or $\frac{x}{t}[e]f$. A point N-structure $(X,\frac{x}{t})$ is called (k-**Conditionally**) **Employed** with an N-structure (X,f) if $f(x)+t+k+1<0$ for all $x\in X$, $t\in[-1,0)$ and $k\in[0,1)$. It is denoted by $(X,\frac{x}{t})[c_k](X,f)$ or $\frac{x}{t}[c_k]f$. To say that $(X,\frac{x}{t})([e]\vee[c_k])(X,f)$ (or briefly, $\frac{x}{t}([e]\vee[c_k])f$) we mean $(X,\frac{x}{t})[e](X,f)$ or $(X,\frac{x}{t})[c_k](X,f)$ (or briefly, $\frac{x}{t}[e]$ or $\frac{x}{t}[c_k]f$). To say that $\frac{x}{t}\overline{\alpha}f$ we mean $\frac{x}{t}\alpha f$ does not hold for $\alpha\in\{[e],[c_k],[e]\vee[c_k]\}$.

Definition 10. An N-structure (X,f) is called $([e],[e]\vee[c_k])$-ideal of X if it satisfied;
(1) $\frac{y}{t}[e]f\Rightarrow\frac{x*y}{t}([e]\vee[c_k])f$,
(2) $\frac{x}{t}[e]f$, $\frac{y}{r}[e]f\Rightarrow\frac{(x*(y*z))*z}{\max\{t,r\}}([e]\vee[c_k])f$ for all $x,y,z\in X$, where $t,r\in[-1,0)$ and $k\in[0,1)$.

Example 2. Let $X=\{1,\gamma,0,m,\omega\}$ be a set with a multiplication table given by;

$*$	1	γ	0	m	ω
1	1	γ	0	m	ω
γ	1	1	γ	m	m
0	1	1	1	m	m
m	1	γ	0	1	γ
ω	1	1	γ	1	1

Let (X,f) be an N-structure. Then f is defined an N-structure (X,f), as;

$$f=\begin{pmatrix}1 & \gamma & 0 & m & \omega\\ -0.9 & -0.8 & -0.7 & -0.9 & -0.8.\end{pmatrix}$$

For $k\in(0.4,1)$ Hence, f is an $([e],[e]\vee[c_k])$-ideal of X.

Theorem 1. For any N-structure (X,f), the following are equivalent:
1. (X,f) is a $([e],[e]\vee[c_k])$-ideal of X.
2. (X,f) satisfies the following inequalities:

2.1. $(\forall x,y\in X)$ ($f(x*y)\leq \max\{f(y),\frac{-k-1}{2}\}$),
2.2. $(\forall x,y,z\in X)$ ($f((x*(y*z))*z)\leq \max\{f(x),f(y),\frac{-k-1}{2}\}$). where $k\in(-1,0]$.

Proof. Let (X,f) be a $([e],[e]\vee[c_k])$-ideal of X. Suppose that $f(x*y)>\max\{f(y),\frac{-k-1}{2}\}$ for all $x,y\in X$. If we take $t_y:=\max\{f(y),\frac{-k-1}{2}\}$, $t_y\in[\frac{-k-1}{2},0]$, $\frac{y}{t_y}[e]f$ and $\frac{x*y}{t_y}[\overline{e}]f$. Also, $f(x*y)+t_y+k+1>2t_y+1\geq 0$, and so $\frac{x*y}{t_y}[\overline{c_k}]f$. This is a contradiction. Thus, $f(x*y)\leq\max\{f(y),\frac{-k-1}{2}\}$ for all $x,y\in X$. Also, suppose that $f((x*(y*z))*z)>\max\{f(x),f(y),\frac{-k-1}{2}\}$ for some $x,y,z\in X$. Take $t:=\max\{f(x),f(y),\frac{-k-1}{2}\}$. Then, $t\geq\frac{-k-1}{2}$, $\frac{x}{t}[e]f$ and $\frac{y}{t}[e]f$, but $\frac{x*(y*z))*z}{t}[\overline{e}]f$. Also, $f((x*(y*z)*z)+t+k+1>2t+k+1\geq 0$, i.e., $\frac{x*(y*z))*z}{t}[\overline{c_k}]f$. This is a contradiction, and hence $f((x*(y*z))*z)\leq\max\{f(x),f(y),\frac{-k-1}{2}\}$ for all $x,y,z\in X$.
Conversely, suppose that (X,f) satisfies (2.1) and (2.2). Let $\frac{y}{t}[e]f$ for all $y\in X$ and $t\in[-1,0)$. Then, $f(y)\leq t$. We shall prove that $\frac{(x*y)}{t}[e]\vee[c_k]f$. Since from (2.1) we have $f(x*y)\leq\max\{f(y),\frac{-k-1}{2}\}$. Then, if $f(y)>\frac{-k-1}{2}$, then $f(x*y)\leq\max\{f(y),\frac{-k-1}{2}\}=f(y)\leq t$. Thus, $\frac{(x*y)}{t}[e]f$ which is a contradiction. If $f(y)\leq\frac{-k-1}{2}$, which implies that $f(x*y)+t+k+1<2f(x*y)+k+1\leq 2\max\{f(y),\frac{-k-1}{2}\}+k+1=0$, i.e., $\frac{x*y}{t}[c_k]f$. Thus $\frac{x*y}{t}([e]\vee[c_k])f$. Let $\frac{x}{t}[e]f$ and $\frac{y}{r}[e]f$ for all $x,y,z\in X$ and $t,r\in[-1,0)$. Then $f(x)\leq t$ and $f(y)\leq r$. Suppose that $\frac{(x*y*z))*z}{\max\{t,r\}}[\overline{e}]f$, i.e., $f((x*(y*z))*z)>\max\{t,r\}$. If $\max\{f(x),f(y)\}>\frac{-k-1}{2}$, then

$$f((x*(y*z))*z)\leq\max\{f(x),f(y),\frac{-k-1}{2}\}=\max\{f(x),f(y)\}\leq\max\{t,r\}.$$

This is impossible, and so $\max\{f(x),f(y)\}\leq\frac{-k-1}{2}$. It follows that $f((x*(y*z))*z)+\max\{t,r\}+k+1<2f((x*(y*z))*z)+k+1\leq 2\max\{f(x),f(y),\frac{-k-1}{2}\}+k+1=0\Rightarrow\frac{(x*y*z))*z}{\max\{t,r\}}[\overline{c_k}]f$. Hence $\frac{(x*y*z))*z}{\max\{t,r\}}([e]\vee[c_k])f$, and therefore (X,f) is a $([e],[e]\vee[c_k])$-ideal of X.

If $(k=0)$, then the following holds.

Corollary 2. For any N-structure (X,f), the following are equivalent:
1. (X,f) is a $([e],[e]\vee[c])$-ideal of X.
2. (X,f) satisfies the following inequalities:
2.1. $(\forall x,y\in X)$ ($f(x*y)\leq\max\{f(y),-0.5\}$).
2.2. $(\forall x,y,z\in X)$ ($f((x*(y*z))*z)\leq\max\{f(x),f(y),\frac{-k-1}{2}\}$)

Theorem 2. Every $([e],[e]\vee[c_k])$-ideal (X,f) of an BE-algebra X satisfies the following inequalities:
(1) $(\forall x\in X)$.($f(1)\leq\max\{f(x),\frac{-k-1}{2}\}$),
(2) $(\forall x,y\in X)$ ($f((x*y)*y)\leq\max\{f(x),\frac{-k-1}{2}\}$). where $k\in(-1,0]$.

Proof. (1): By using (V_1) and theorem 1(2.1), we have

$$f(1)=f(x*x)\leq\max\{f(x),\frac{-k-1}{2}\}$$

for all $x\in X$.
(2): By using (V_3), we have $f((x*y)*y)=f((x*(1*y))*y)$ for all $x,y\in X$
Then by using theorem 1(2.2), we get
$f((x*(1*y))*y)\leq\max\{f(x),f(1),\frac{-k-1}{2}\}=\max\{f(x),\frac{-k-1}{2}\}$, because by (1) $f(1)\leq\max\{f(x),\frac{-k-1}{2}\}$ for all $x,y\in X$.
Hence, $f((x*y)*y)\leq\max\{f(x),\frac{-k-1}{2}\}$ for all $x,y\in X$.

If $(k=0)$, then the following holds.

Corollary 3. Every $([e],[e]\vee[c])$-ideal of a BE-algebra X satisfies the following inequalities:
(1) $(\forall x\in X).(\ f(1)\leq \max\{f(x),-0.5\}\ )$,
(2) $(\forall x,y\in X)\ (\ f((x*y)*y)\leq \max\{f(x),-0.5\}\ )$.

Corollary 4. Each $([e],[e]\vee[c_k])$-ideal (X,f) satisfies the following condition;
$(\forall x,y\in X)\ (\ x\leq y\Rightarrow f(y)\leq \max\{f(x),\frac{-k-1}{2}\}\ )$. where $k\in(-1,0]$.

Proof. Let $x\leq y$ for all $x,y\in X$. Then $x*y=1$,and so

$$f(y)=f(1*y)=f((x*y)*y)\leq \max\{f(x),\frac{-k-1}{2}\}$$

Hence, $f(y)\leq \max\{f(x),\frac{-k-1}{2}\}$.

If $(k=0)$, then the following holds.

Lemma 3. Each $([e],[e]\vee[c])$-ideal (X,f) satisfies the following condition;
$(\forall x,y\in X)\ (\ x\leq y\Rightarrow f(y)\leq \max\{f(x),-0.5\}\ )$.

Proposition 3. Let (X,f) be an N-structure such that
(1) $(\forall x\in X)\ (\ f(1)\leq \max\{f(x),\frac{-k-1}{2}\}\ )$,
(2) $(\forall x,y,z\in X)\ (\ f(x*z)\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}\ )$.
Then the following implication is valid.
$(\forall x,y\in X)\ (\ x\leq y\Rightarrow f(y)\leq \max\{f(x),\frac{-k-1}{2}\}\ )$, where $k\in(-1,0]$.

Proof. Suppose $x\leq y$ for all $x,y\in X$. Then $x*y=1$, and by using (1) we get

$$\begin{aligned}f(y)&=f(1*y)\leq \max\{f(1*(x*y)),f(x),\frac{-k-1}{2}\}\\&=\max\{f(1*1),f(x),\frac{-k-1}{2}\}\\&=\max\{f(1),f(x),\frac{-k-1}{2}\}\\&=\max\{f(x),\frac{-k-1}{2}\}\end{aligned}$$

Hence, $f(y)\leq \max\{f(x),\frac{-k-1}{2}\}$.

If $(k=0)$, then the following holds.

Lemma 4. Let (X,f) be an N-structure such that
(1) $(\forall x\in X)\ (\ f(1)\leq \max\{f(x),-0.5\}\ )$,
(2) $(\forall x,y,z\in X)\ (\ f(x*z)\leq \max\{f(x*(y*z)),f(y),-0.5\}\ )$.
Then the following implication is valid.
$(\forall x,y\in X)\ (\ x\leq y\Rightarrow f(y)\leq \max\{f(x),-0.5\}\ )$.

Theorem 3. Let (X,f) be an N-structure of transitive BE-algebra X. Then (X,f) is an $([e],[e]\vee[c_k])$-ideal of X if and only if it satisfies the following inequalities:
(1) $(\forall x\in X)\ (\ f(1)\leq \max\{f(x),\frac{-k-1}{2}\}\ )$,
(2) $(\forall x,y,z\in X)\ (\ f(x*z)\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}\ )$, where $k\in(-1,0]$.

Proof. Suppose that (X,f) is an $([e],[e]\vee[c])$-ideal of X. From theorem 2(1),it is easily seen that

$$f(1)\leq \max\{f(x),\frac{-k-1}{2}\}.$$

Since X is transitive,

$$((y*z)*z)*((x*(y*z))*(x*z))=1 \quad (\mathbf{G})$$

for all $x,y,z\in X$. By using (V_3) and $(\mathbf{G})$

$$f(x*z)=f(1*(x*z))=f((((y*z)*z)*((x*(y*z))*(x*z))*(x*z))$$

By using theorem 1(2.2), 2(2), we have

$$\begin{aligned}f((((y*z)*z)*((x*(y*z))*(x*z))*(x*z))&\leq \max\{f((y*z)*z),f(x*(y*z)),\frac{-k-1}{2}\}\\&=\max\{f(x*(y*z)),f((y*z)*z),\frac{-k-1}{2}\}\\&\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}\end{aligned}$$

Hence $f(x*z)\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}$ for all $x,y,z\in X$.
Conversely suppose that (X,f) satisfies (1) and (2). By using (2), (V_1), (V_2) and (1)

$$\begin{aligned}f(x*y)&\leq \max\{f(x*(y*y)),f(y),\frac{-k-1}{2}\}\\&=\max\{f(x*1),f(y),\frac{-k-1}{2}\}\\&=\max\{f(1),f(y),\frac{-k-1}{2}\}\\&=\max\{f(y),\frac{-k-1}{2}\}\end{aligned}$$

Also by using (2) and (1) we get

$$\begin{aligned}f((x*y)*y)&\leq \max\{f((x*y)*(x*y)),f(x),\frac{-k-1}{2}\}\\&=\max\{f(1),f(x),\frac{-k-1}{2}\}\\&=\max\{f(x),\frac{-k-1}{2}\}\end{aligned}$$

for all $x,y\in X$. Now, since $(y*z)*z\leq (x*(y*z))*(x*z)$ for all $x,y,z\in X$, it follows that from proposition 3, we have

$$f((x*(y*z))*(x*z))\leq \max\{f((y*z)*z),\frac{-k-1}{2}\}$$

So, from (2), we have

$$\begin{aligned}f((x*(y*z))*z)&\leq \max\{f((x*(y*z))*(x*z)),f(x),\frac{-k-1}{2}\}\\&\leq \max\{f((y*z)*z),f(x),\frac{-k-1}{2}\}\\&\leq \max\{f(x),f(y),\frac{-k-1}{2}\}\end{aligned}$$

for all $x,y,z\in X$. Using theorem 1, we conclude that (X,f) is a $([e],[e]\vee[c])$-ideal of X.

If $(k=0)$, then the following holds.

Corollary 5. Let (X,f) be an N-structure of transitive BE-algebra X. Then (X,f) is an $([e],[e]\vee[c])$-ideal of X if and only if it satisfies the following inequalities:
(1) $(\forall x\in X)\ (\ f(1)\leq \max\{f(x),-0.5\}\ )$,
(2) $(\forall x,y,z\in X)\ (\ f(x*z)\leq \max\{f(x*(y*z)),f(y),-0.5\}\ )$.

Theorem 4. Let X be a transitive BE-algebra. If (X,f) is a $([e],[e]\vee[c_k])$-ideal of X such that $f(1)>\frac{-k-1}{2}$, then (X,f) is an N-ideal of X, where $k\in(-1,0]$.

Proof. Suppose that (X,f) is a $([e],[e]\vee[c_k])$-ideal of X such that $\frac{-k-1}{2}<f(1)$. Then $\frac{-k-1}{2}<f(x)$ and so $\frac{-k-1}{2}<f(1)\leq f(x)$ for all $x\in X$ by theorem 3(1)

$$f(1)\leq \max\{f(x),\frac{-k-1}{2}\}$$

for all $x\in X$. It follows that from theorem 3(2),

$$\begin{aligned} f(x*z) &\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\} \\ &= \max\{f(x*(y*z)),f(y)\} \end{aligned}$$

for all $x,y,z\in X$. Hence (X,f) is an N-ideal of X.

If $(k=0)$, then the following holds.

Corollary 6. Let X be a transitive BE-algebra. If (X,f) is a $([e],[e]\vee[c])$-ideal of X such that $f(1)>-0.5$, then (X,f) is an N-ideal of X.

Theorem 5. If (X,f) is a $([e],[e]\vee[c_k])$-ideal of a transitive BE-algebra X. Show that

$$(\forall t\in[-1,\frac{-k-1}{2}))\ (Q(f;t)\in J(X)\cup\{\emptyset\})$$

where $Q(f;t):=\{x\in X\mid \frac{x}{t}[c_k]f\}$, $J(X)$ is a set of all ideal of X and $k\in(-0.5,0]$.

Proof. Suppose that $Q(f;t)\neq\emptyset$ for all $t\in[-1,\frac{-k-1}{2})$. Then there exists $x\in Q(f;t)$, and so $\frac{x}{t}[c]f$, i.e., $f(x)+t+k+1<0$. Using theorem 3(1), we have

$$\begin{aligned} f(1) &\leq \max\{f(x),\frac{-k-1}{2}\} \\ &= \begin{cases} \frac{-k-1}{2} & \text{if } f(x)\leq\frac{-k-1}{2} \\ f(x) & \text{if } f(x)>\frac{-k-1}{2} \end{cases} \\ &< -1-t-k \end{aligned}$$

which indicates that $1\in Q(f;t)$. Let $x*(y*z)\in Q(f;t)$ for all $x,y,z\in X$ here $y\in Q(f;t)$. Then $\frac{x*(y*z)}{t}[c_k]f$ and $\frac{y}{t}[c]f$, i.e., $f(x*(y*z))+t+k+1<0$ and $f(y)+t+k+1<0$. Using theorem 3(2), we get

$$f(x*z)\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}$$

Thus, if $\max\{f(x*(y*z)),f(y)\} > \frac{-k-1}{2}$, then

$$f(x*z) \leq \max\{f(x*(y*z)),f(y)\} < -1-t-k$$

If $\max\{f(x*(y*z)),f(y)\} \leq \frac{-k-1}{2}$, then $f(x*z) \leq \frac{-k-1}{2} < -1-t-k$. This show that $\frac{x*z}{t}[c_k]f$ i.e., $x*z \in Q(f;t)$. By using lemma 1, we have $Q(f;t)$ is an ideal of X.

If $(k=0)$, then the following holds.

Corollary 7. If (X,f) is a $([e],[e]\vee[c])$-ideal of a transitive BE-algebra X. Show that

$$(\forall t \in [-1,-0.5))\ (Q(f;t) \in J(X) \cup \{\emptyset\})$$

where $Q(f;t) := \{x \in X \mid \frac{x}{t}[c]f\}$, and $J(X)$ is a set of all ideal of X

Theorem 6. Let X be a transitive BE-algebra. Then the followings are equivalent:
(1) An N-structure (X,f) is a $([e],[e]\vee[c_k])$-ideal of X
(2) $(\forall t \in [-1,0))\ ([f]_t \in J(X) \cup \{\emptyset\})$
where $[f]_t := C(f;t) \cup \{x \in X \mid f(x)+t+k+1 \leq 0\}$, $J(X)$ is a set of all ideal of X, and $k \in (-1,0]$.

Proof. $(1) \Rightarrow (2)$: Suppose that (1) satisfies. Let $[f]_t \neq \emptyset$, here $t \in [-1,0)$. Then there exists $x \in [f]_t$, and so $f(x) \leq t$ or $f(x)+t+k+1 \leq 0$ for all $x \in X$ and $t \in [-1,0)$. If $f(x) \leq t$, then

$$\begin{aligned} f(1) &\leq \max\{f(x),\frac{-k-1}{2}\} \leq \max\{t,\frac{-k-1}{2}\} \\ &= \begin{cases} t & \text{if } t > \frac{-k-1}{2} \\ \frac{-k-1}{2} \leq -1-t-k & \text{if } t \leq \frac{-k-1}{2} \end{cases} \end{aligned}$$

By theorem 3(1). Hence $1 \in [f]_t$. If $f(x)+t+k+1 \leq 0$, then

$$\begin{aligned} f(1) &\leq \max\{f(x),\frac{-k-1}{2}\} \leq \max\{-1-t-k,\frac{-k-1}{2}\} \\ &= \begin{cases} -1-t-k & \text{if } t < \frac{-k-1}{2} \\ \frac{-k-1}{2} \leq t & \text{if } t \geq \frac{-k-1}{2} \end{cases} \end{aligned}$$

And so $1 \in [f]_t$. Let $x,y,z \in X$ be such that $y \in [f]_t$ and $x*(y*z) \in [f]_t$. Then $f(y) \leq t$ or $f(y)+t+k+1 \leq 0$, and $f(x*(y*z)) \leq t$ or $f(x*(y*z))+t+k+1 \leq 0$. Thus we let the four cases:
(a_1) $f(y) \leq t$ and $f(x*(y*z)) \leq t$,
(a_2) $f(y) \leq t$ and $f(x*(y*z))+t+k+1 \leq 0$,
(a_3) $f(y)+t+k+1 \leq 0$ and $f(x*(y*z)) \leq t$,
(a_4) $f(y)+t+k+1 \leq 0$ and $f(x*(y*z))+t+k+1 \leq 0$.

For case (a_1), theorem 3(2), implies that

$$f(x*z)\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}\leq \max\{t,\frac{-k-1}{2}\}$$
$$=\begin{cases}\frac{-k-1}{2} & \text{if } t<\frac{-k-1}{2}\\ t & \text{if } t\geq\frac{-k-1}{2}\end{cases}$$

so that $x*z\in C(f;t)$ or $f(x*z)+t+k\leq\frac{-k-1}{2}+\frac{-k-1}{2}+k=-1$. Thus $x*z\in[f]_t$. For case (a_2), we have

$$f(x*z)\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}\leq \max\{-1-t-k,t,\frac{-k-1}{2}\}$$
$$=\begin{cases}-1-t-k & \text{if } t<\frac{-k-1}{2}\\ t & \text{if } t\geq\frac{-k-1}{2}\end{cases}$$

Thus $x*z\in[f]_t$.
For case (a_3), the prove is same to case (a_2). For case (a_4) we have,

$$f(x*z)\leq \max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}\leq \max\{-1-t-k,\frac{-k-1}{2}\}$$
$$=\begin{cases}-1-t-k & \text{if } t<\frac{-k-1}{2}\\ \frac{-k-1}{2} & \text{if } t\geq\frac{-k-1}{2}\end{cases}$$

So that, $x*z\in[f]_t$. By using lemma 1, $[f]_t$ is an ideal of X.
(2) $\Rightarrow$ (1): Suppose that (2) hold. If $f(1)>\max\{f(y),\frac{-k-1}{2}\}$ for all $y\in X$, then $f(1)>t_y\geq\max\{f(y),\frac{-k-1}{2}\}$ for some $t_y\in[\frac{-k-1}{2},0)$. It follows that $x\in C(f;t_y)\subseteq[f]_{t_y}$ but $1\notin C(f;t_y)$. Also, $f(1)+t_y+k+1>2t_y+k+1\geq 0$. Hence $1\notin[f]_{t_y}$, which contradicts the supposition. So, $f(1)\leq\max\{f(y),\frac{-k-1}{2}\}$ for all $y\in X$. Suppose that for some $x,z\in X$, we have
$f(x*z)>\max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}$ (**D**)
Taking $t:=\max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}$ implies that $t\in[\frac{-k-1}{2},0)$, $x\in C(f;t)\subseteq[f]_t$, and $x*(x*z)\in C(f;t)\subseteq[f]_t$. Since $[f]_t$ is an ideal of X, we have $x*z\in[f]_t$, and so $f(x*z)\leq t$ or $f(x*z)+t+k+1\leq 0$. The inequality (**D**) induces $x*z\notin C(f;t)$, and $f(x*z)+t+k+1>2t+k+1\geq 0$. Thus $x*z\notin[f]_t$. It contradicts the supposition. Hence $f(x*z)\leq\max\{f(x*(y*z)),f(y),\frac{-k-1}{2}\}$ for all $x,y,z\in X$. Using theorem 3, we have, (X,f) is a $([e],[e]\vee[c_k])$-ideal of X.

If $(k=0)$, then the following holds.

Corollary 8. Let X be a transitive BE-algebra. Then the followings are equivalent:
(1) An N-structure (X,f) is a $([e],[e]\vee[c])$-ideal of X
(2) $(\forall t\in[-1,0))$ $([f]_t\in J(X)\cup\{\emptyset\})$
where $[f]_t:=C(f;t)\cup\{x\in X\mid f(x)+t+1\leq 0\}$, and $J(X)$ is a set of all ideal of X

5 Conclusion:

BE-algebra is a type of logical algebra like BCK/BCI/BCH-algebras. A BE-algebra is a another generalization of BCK/BCI/BCH-algebras. In this paper, we have investigated the concept of $([e],[e]\vee[c_k])$-ideal of a BE-algebra by using (k-conditionally) employed of N-structure with point N-structure. We also characterized transitive and distributive BE-algebra by $([e],[e]\vee[c_k])$-ideal. We also discussed their related properties and provide characterizations of $([e],[e]\vee[c_k])$-ideals.

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