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Lagrange theorem for polygroups

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Abstract: So far, isomorphism theorems in hyperstructure were proved for different structures of polygroups, hyperrings and etc. In this paper, the polygroups properties is studied with the introduction of a suitable equivalence relation. We show that the above relation is strongly regular. Our main purpose in the paper is investigating Lagrang theorem and other expressing of isomorphism theorems for polygroups.

Keywords: polygroup, Lagrange theorem, isomorphism theorems, fundamental relation, heart of hypergroup

1 Introduction

The theory of algebraic hyperstructures which is a generalization of ordinary algebraic structures was first introduced by Marty [9]. Since then, many researchers have studied the theory of hyperstructures and developed it. Moreover, the applications of this theory in other fields such as geometry, graphs and hypergraphs, lattices, automata, cryptography, codes, etc has been extensively studied, see ([1], [2], [3], [4], [10]).

Lagrange theorem, Isomorphism theorem and essential theorem of group products are some of the most important items in ordinary groups theory. The results of Lagrang theorem are applied in different parts of groups theory specially sylow and free abelian groups. In [6], Davvaz introduced some relations in polygroups specially the follow relation on polygroup H. Let N be a normal subpolygroup of polygroup H, then

$$a \equiv b(modN) \Longleftrightarrow ab^{-1} \cap N \neq \emptyset.$$

The above relation can be employed for investigation of isomorphism theorem with similar define for kernel of homomorphism about polygroups.

In the first part we will express the necessary topics for using in the main part of paper. In the Main Result part we wil obtain Lagrange theorem for hypergroups. For this we use a suitable equivalence relation on a normal subpolygroup from a polygroup. Finally we will prove isomorphism theorem of polygroups based on fundamental relation on a polygroup.

2 Priliminaries

Every $\cdot : H \times H \longrightarrow P^*(H)$ is called a hyperoperation or join operation. $(H, \cdot)$ is called hypergroupoid and a hypergroup is a structure $(H, \cdot)$ that satisfies two axioms:

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– (semihypergroup) $a(bc) = (ab)c$ for all $a, b, c \in H$,
– (quasihypergroup) $aH = H = Ha$ for all $a \in H$.

Let H be a hypergroup and K a nonempty subset of H. Then K is a subhypergroup of H if itself is a hypergroup under hyperoperation restricted to K. Hence it is clear that a subset K of H is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on H (See [5]).

A hypergroup is called a polygroup if

1. There exist $e \in H$ such that $ex = x = xe$ for all $x \in H$,
2. for all $x \in H$ there exists an unique element, say $x^{'} \in H$ such that $e \in xx^{'} \cap x^{'}x$ (we denote $x^{'}$ by x^{-1}),
3. for all $x, y, z \in H, z \in xy \Longrightarrow x \in zy^{-1} \Longrightarrow y \in x^{-1}z$.

A nonempty subset N of a polygroup $(H, \cdot)$ is called a subpolygroup if $(N, \cdot)$ is itself a polygroup. In this case we write $N <_p H$. A subpolygroup N is called normal in H if $xNx^-1 \subseteq N$, for all $x \in H$. In this case we write $N \trianglelefteq_p H$.

Lemma 1. [11] Let $N <_p H$. Then

1. for all $n \in N$, $Nn = nN = N$,
2. $NN = N$,
3. $(a^{-1})^{-1} = a$.

Let $(H, \cdot)$ Be a semihypergroup and R be an equivalence relation on H. If A and B are nonempty subsets of H, then

– $A\overline{R}B$ means that for all $a \in A$ there exist $b \in B$ such that aRb and for all $b^{'} \in B$ there exist $a^{'} \in A$such that $a^{'}Rb^{'}$;
– $A\overline{\overline{R}}B$ means that for all $a \in A$and $b \in B$ we have aRb.

Definition 1. [7] The equivalence relation R is called

1. Regular, if for all x of H, from aRb, it follows that $(ax)\overline{R}(bx)$ and $(xa)\overline{R}(xb)$;
2. Strongly regular, if for all x of H, from aRb, it follows that $(ax)\overline{\overline{R}}(bx)$ and $(xa)\overline{\overline{R}}(xb)$.

Proposition 1. [7] If $(H, \cdot)$ is a hypergroup and R is an equivalence relation on H, then R is rgular if and only if $(H/R, \otimes)$ is a hypergroup such that $\bar{x} \otimes \bar{y} = \{\bar{z} | z \in xy\}$ for all $\bar{x}, \bar{y} \in H/R$.

Definition 2. [7] Let $(H, \cdot)$ be a semihypergroup and n be a nonzero natural number. We say that

$$x\beta_n \text{ if there exists } a_1, a_2, \cdots, a_n \text{ in } H, \text{ such that } \{x, y\} \subseteq \textstyle\prod_{i=1}^{n} a_i.$$

Let $\beta = \bigcup_{n \geq 1} \beta_n$. Clearly, the relation β is reflexive and symmetric. Denote by β^* is transitive closure of β.

Proposition 2. [7] β^* is the smallest strongly regular relation on H.

Definition 3. [8] Let $(H_1, \cdot) and (H_2, *)$ be two hypergroupoids. A map $f : H_1 \longrightarrow H_2$, is called

1. a homomorphism if for all x, y of H, we have $f(xy) \subseteq f(x) * f(y)$;
2. a good homomorphism if for all x, y of H, we have $f(xy) = f(x) * f(y)$.

Definition 4. [7] Let $(H, \cdot)$ is a hypergroup and consider canonical projection $\varphi_H : H \longrightarrow H/\beta^*$. The heart of H is the set $\omega_H = \{x \in H | \varphi_H(x) = e\}$, where e is the identity of the group $(H/\beta^*, \otimes)$.

Lemma 2. citeDB1 If $(H,\cdot)$ is a hypergroup, then the relation β is an equivalence relation on H.

Definition 5. Let H, H' be hypergroups and let $f : H \longrightarrow H'$ be a homomorphism. The kernel of f is the set $K(f) = \{x \in H | f(x) \in \omega_{H'}\}$.

We define image of f, the set $im(f) = \{f(x) \in H'/\beta^* | x \in H\}$.

Lemma 3. [7] The kernel $K(f)$ of a hypergroup homomorphism $f : H \longrightarrow H'$ is a normal subhypergroup of H.

3 Main Result

Let N be a subpolygroup of polygroup H. We define the follow relation on H:

$$a \sim_N b \Longleftrightarrow ab^{-1} \subseteq N \quad \text{for all} \quad a,b \in H. \tag{1}$$

Proposition 3. If $N \trianglelefteq_p H$, then $\sim_N$ is an equivalence relation.

Proof. Let $N \trianglelefteq_p H$, then for all $x \in H$ we have,

$$xx^{-1} \subseteq xex^{-1} \subseteq xNx^{-1} \subseteq N.$$

Therefore $x \sim_N x$, so $\sim_N$ is reflexive. The relation $\sim_N$ is symmetric because for all $x,y \in H$, if $x \sim_N$ then $xy^{-1} \subseteq N$. Since $xy^{-1} \neq \emptyset$, there exist $n \in N$ such that $n \in xy^{-1}$. Therefore at definition of polygroup, $y \in n^{-1}x$, so $yx^{-1} \subseteq n^{-1}xx^{-1} \subseteq N$. Hence $y \sim_N x$. For all $x,y,z \in H$, if $x \sim_N y$ and $y \sim_N z$, then $xy^{-1} \subseteq N$ and $yz^{-1} \subseteq N$. We have

$$xz^{-1} \subseteq xez^{-1} \subseteq xy^{-1}yz \subseteq xy^{-1}yz \subseteq N.$$

Hence $x \sim_N z$. This show $\sim_N$ is transitive relation.

If we define relation $_N\sim$ on H as $a_N \sim b \Longleftrightarrow a^{-1}b \subseteq N$ for all $a,b \in H$, then the above proposition is correct too.

Theorem 1. Let $x,y \in H$ and $N \trianglelefteq_p H$, then $Nx = Ny$ if and only if $x \sim_N y$. similarly $xN = yN$ if and only if $x_N \sim y$.

Proof. Let for $N \trianglelefteq_p H$, $Nx = Ny$. Then $x \in Ny$ and so there exist $n \in N$ such that $x \in ny$. This show $xy^{-1} \subseteq N$.
If $xy^{-1} \subseteq N$, $x \in xy^{-1}y \subseteq Ny$. Therefore there exist $n \in N$ that $x \in ny$. Hence $y \in n^{-1}x$. The proof is completed.

Corollary 1. Let N be a normal subpolygroup of H, then

1. H is union of right (left) cosets N in H.
2. Two right(left) cosets N in H are either disjoint or equal.

Proof. The proof consequence of theorem.

Corollary 2. The relation $\sim_N$ is strongly regular.

Proof. If for some $a,b \in H$, $a \sim_N b$. Then let $n \in ab^{-1}$, so $a \in nb$. Now we choose $x \in H$ arbitrary. Let $y \in ax$ and $z \in bx$. We have $y \in nbx$ and $b \in zx^{-1}$ that it conclude $b^{-1}y \subseteq Nx$ and $z^{-1}b \subseteq Nx^{-1}$. Then

$$yz^{-1} \subseteq (bb^{-1})yz^{-1}(bb^{-1}) \subseteq b(b^{-1}yz^{-1}b)b^{-1} \subseteq b(NxNx^{-1})b^{-1} \subseteq bNb^{-1} \subseteq N.$$

Therefore $y \sim_N z$. This show $ax\overline{\overline{\sim_N}}bx$. Similarly, we can show, for all $x \in H$, if $a \sim_N b$ then $xa\overline{\overline{\sim_N}}xb$.

Corollary 3. The relation β is inclusive in $\sim_N$.

Proof. It is clear by 2 and 2.

Definition 6. Let N be a normal subpolygroup of a polygroup H. The index of N in H, denoted $[H:N]_p$, is the cardinal number of the set of distinct right (resp, left) coset of N in H.

Theorem 2. Let M,N are two normal subpolygroups of H, with $M <_p N$. Then $[H:M]_p = [H:N]_p[N:M]_p$. If any two of these indices are finite, then so is the third.

Proof. By corollary 1, $H = \bigcup_{i\in I} Na_i$ with $a_i \in G$, $|I| = [H:N]_p$ and the cosets Na_i mutally disjoint (that is, $Na_i = Na_j \Longleftrightarrow i = j$). Similarly $N = \bigcup_{j\in J} Mb_j$ with $b_j \in N$, $|J| = [N:M]_p$ and the cosets Mb_j are mutually disjoint. Therefore $H = \bigcup_{i\in I} Na_i = \bigcup_{i\in I}(\bigcup_{j\in J} Mb_j)a_i = \bigcup_{(i,j)\in I\times J} Mb_jMa_i$. It suffices to show that the cosets Mb_ja_i are mutually disjoint. For then by corollary we must have $[H:M]_p = |I\times J|$, whence $[H:M]_p = |I\times J| = |I||J| = [H:N]_p[N:M]_p$. (Hungerford) Let $Mb_ja_i = Mb_ra_t$, then $b_ja_i \subseteq Mb_ra_t$, so there exist $m \in M$ that $b_ja_i = mb_ra_t$. Since $m, b_r, b_j \in N$, for all $n \in N$, $nb_ja_i \subseteq Na_i$ and $nb_ja_i \subseteq nmb_ra_t \subseteq Na_t$. We have $Na_i \cap Na_t \neq \emptyset$, hence $Na_i = Na_t$ and $i = t$. In other hand, for all $m' \in M$, there exist $m'' \in M$ such that

$$m'b_j \subseteq m''b_ra_ia_i^{-1} \subseteq m''b_rM \subseteq m''Mb_r \subseteq Mb_r$$

Since, $m'b_j \neq \emptyset$, $Mb_j \cap Mb_r \neq \emptyset$. Thus $Mb_j = Mb_r$ and $r = j$. Therefore, the cosets Mb_ja_i are mutually disjoint. The last statement of the theorem is obvious.

Corollary 4. (*Lagrange Theorem for polygroups*) If N is a normal subpolygroup of a polygroup H, then $|H| = [H:N]_p|N|$. In particular if H is finite, the oreder of each normal subpolygroup, divides $|H|$.

Proof. Apply the theorem 2 with $M =< e >$ for the first statement. The second is clear.

Theorem 3. Let N is a normal subpolygroup of H and H/N the set of all cosets N in H, then H/N is a polygroup with order $[H:N]_p$ by hyper operation $(Na)(Nb) = \{nz | z \in ab\}$.

Proof. Hypergroupoid H/N with the above hyperooperation is semihypergroup, because for all, $Na, Nb, Nc \in H/N$, we have

$$(Na)(Nb)Nc = \bigcup_{z\in ab} NzNC = \bigcup_{w\in(ab)c} Nw = \bigcup_{w\in a(bc)} Nw = \bigcup_{z\in bc} NaNz = Na(NbNc).$$

The following conditions are satisfied:

- For all $Na \in H/N$, there exist unique element Ne, such that $NaNe = Na = NeNa$;

- For all $Na \in H/N$, there exist unique element Na^{-1}, such that $Ne \in NaNa^{-1} \cap Na^{-1}Na$.

For all $Na, Nb, Nc \in H/N$ we have

$$Na \in NbNc \Rightarrow a \in bc \Rightarrow b \in ac^{-1} \Rightarrow Nb \in NaNc^{-1}, c \in b^{-1}a \Rightarrow Nc \in Nb^{-1}Na.$$

H/N is a quasihypergroup, because for all $Na, Nb \in H/N$, $Na \in NbNb^{-1}Na \cap Nab^{-1}Nb$.

The above conditions show that H/N is polygroup.

Theorem 4. *(First isomorphism theorem)* Let $f : G \longrightarrow H$ be a good homomorphism on polygroups, then f induces isomorphism $G/ker(f) \cong im(f)$ on polygroups.

Proof. Let $K = ker(f)$. We define $\varphi : G/K \longrightarrow im(f)$ as $\varphi(Kx) = f(x)$. φ is well-define, because for all $x,y \in G$, $Kx = Ky$, that is $xy^{-1} \subseteq K$. Then for $z \in xy^{-1}$, we have $f(z)\beta e$ and $x \in zy$. Since f is good homomorphism, $f(x) \in f(zy) = f(z)f(y)$. We cocclude $f(x)\beta f(y)$. For all $Kx, Ky \in G/K$, we have

$$\varphi(KxKy) = \varphi(\{Kz|z \in xy\}) = \{f(z)|z \in xy\} = f(xy) = f(x)f(y) = \varphi(Kx)\varphi(Ky).$$

φ is injective, because if $\varphi(Kx) = \varphi(Ky)$, then $f(x)\beta f(y)$. By , $f(xy^{-1})\overline{\overline{\beta}} f(e)$. This show for all $\alpha \in xy^{-1}$, $f(\alpha)\beta f(e)$. Hence $\alpha \in K$ and $xy^{-1} \subseteq K$. Thus $Kx = Ky$. It is clear that φ be onto.

Corollary 5. Let N,M subpolygroup of polygroup H, that $N \trianglelefteq_p H$. Then

1. MN is a polygroup.
2. $N \cap M$ is normal subpolygroup of M.
3. N is normal subpolygroup of NM.
4. *(Second isomorphism theorem)* $M/M \cap N \cong NM/N$.

Proof. (1) Let $x,y \in MN$. Then, there exist $m_1,m_2 \in M$, $n_1,n_2 \in N$ such that $x \in m_1n_1$ and $y \in m_2n_2$. Thus $xy \subseteq m_1n_1m_2n_2$. Since $N \trianglelefteq_p H$, there is $n_1^{'} \in N$ that $n_1m_2 = m_2n_1^{'}$. This show $xy \in P^*(MN)$.
For all $x \in MN$, there exist $e_H \in MN$, such that $e_Hx = x = xe_H$.
For all $x \in MN$, there are $m \in M$, $n \in N$, such that $x \in mn$. Therefore $n^{-1} \in x^{-1}m$, so $x^{-1} \in n^{-1}m^{-1}$. By normality N in H, $x^{-1} \in MN$. Other conditions of polygroup is induced from H.
(2) It is straight forward.
(3) For all $n \in N$ and $x \in NM$, there exist $n^{'} \in N$, $m \in M$ such that $x \in n^{'}m$. So $x^{-1} \in m^{-1}(n^{'})^{-1}$ and we have

$$xnx^{-1} \subseteq (n^{'}m)n(m^{-1}(n^{'})^{-1}) = n^{'}(mnm^{-1})(n^{'})^{-1} \subseteq N.$$

(4) We define $\varphi : M \longrightarrow NM/N$ with $\varphi(m) = Nm$. It is clear that, φ is well-define and epimorphism.
Show $ker(\varphi) = M \cap N$.
If $m \in ker(\varphi)$, $\varphi(m) = Nm \in \omega_{NM/N}$, so $Nm\beta N$. By 2 $Nm \sim_{<e_{NM/N}>} N$, thus $Nm = NmN \subseteq N$. Hence $m \in M \cap N$. Vice-versa if $m \in M \cap N$, then

$$\varphi(m) = Nm = N = \varphi(e) \in \omega_{NM/N}.$$

Thus $m \in ker(\varphi)$. This complete the proof by first isomorphism theorem.

Corollary 6. *(Third isomorphism theorem)* Let H be a polygroup and N,M two normal subpolygroups, such that $M <_p N$, then $N/M \trianglelefteq_p H/M$ and

$$\frac{H/M}{N/M} \cong H/N$$

Proof. Let Mx be an arbitrary of H/M. Then for all $Mn \in N/M$ we have

$$MxMn = \{Mz|z \in xn\} = \{Mz|z \in n^{'}x\} = Mn^{'}Mx.$$

This show that $N/M \trianglelefteq_p H/M$.

We define $\varphi : H/M \longrightarrow H/N$ with $\varphi(Mx) = Nx$ for all $Mx \in H/M$. Easily, it can be showed as φ is a epimorphism. It sufficient to show, $ker(\varphi) = N/M$.

Let $Mx \in ker(\varphi)$, then $Nx\beta N$. By 2, $Nx \sim_{<e_{H/N}>} N$, so $Nx = NxN \subseteq N$. This show $x \in N$. Vice-versa, let $Mn \in N/M$, then we have

$$\varphi(Mn) = Nn = N = \varphi(M) \in \omega_{H/N}.$$

Thus $Mn \in ker(\varphi)$. This complete the proof.

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