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Compact and noncompact structures of the nonlinearly dispersive *GNLS*(*m*,*n*,*k*,*l*) equation

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Abstract: In this paper, we establish exact-special solutions of the generalized nonlinear dispersion *GNLS*(m,n,k,l) equation. We use the ansatz method for acquiring the compactons, solitary patterns, solitons and other types of solutions.

Keywords: Generalized nonlinear dispersion, Ansatz method, solitary patterns,

1 Introduction

The mathematical modeling of scientific events usually is expressed by nonlinear evolution equations. So, it is crucial to reach general solutions, which give some behaviors about the character and the structure of the equations for researchers, of these corresponding nonlinear equations. Many effective methods have been improved to provide much information for physicians and engineers to present the actual physical characters of solutions [1]. We recall that most of these methods use the wave variable transformation to reduce the nonlinear PDE to ODE in order to acquire the solution. Severals are the generalized Miura [2], Darboux [3], Cole-Hopf [4], Hirota's dependent variable [5], the Backlund [6] transformations and sine-cosine [7], homogeneous balance (HB) [8], similarity reduction [9], automated tanh-function [10], Exp-function [11], $(G^{'}/G)$-expansion [12] methods and there are many papers about the basis of compactons [13–32].

De Angelis studied [14] the cubic quintic nonlinear Schrödinger equation;

$$ip_x + p_{tt} + \gamma p|p|^2 + \alpha p|p|^4 = 0, \tag{1}$$

Yan [15] studied NLS(m,n) equation;

$$ip_t + \left(p|p|^{m-1}\right)_{xx} + \mu p|p|^{n-1} = 0, \tag{2}$$

and obtained envelope compactons and solitary pattern solutions.

Agrawal and Zeng studied [16],[17] w.r.t. the GNLS equation;

$$ip_t + Ap_{xx} + Bp|p|^2 + iCp_{xxx} + iD\left(p|p|^2\right)_x = 0, \tag{3}$$

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Hirota introduced [18] another GNLS equation in the following form

$$ip_t + Ap_{xx} + Bp|p|^2 + iCp_{xxx} + iD|p|^2 p_x = 0. \tag{4}$$

Yan introduced and studied [19] the GNLS equation with nonlinear dispersion (called GNLS(*m,n,k,l*) equation) and the equation given by

$$ip_t + A\left(p|p|^{m-1}\right)_{xx} + Bp|p|^{n-1} + iC\left(p|p|^{k-1}\right)_{xxx} + iD\left(p|p|^{l-1}\right)_x = 0, \tag{5}$$

where A, B, C and D are arbitrary constants.

For this paper our aim is to apply the ansatz method [20–23] to Eq. (5) in light of the above-mentioned studies and then we give some explanations about the solutions and conclusions in the last section.

2. Ansatz method

For initially, the solution of the equation is considered

$$\left(\frac{dv}{dy}\right)^2 = A_0 - A_1 v^2 \tag{6}$$

where $A_0 \neq 0$ and $A_1 \neq 0$ are constants. When $A_1 > 0$, Eq. (6) has the solutions:

$$\begin{aligned} v_1 &= \pm\sqrt{\tfrac{A_0}{A_1}}\sin\left[\sqrt{A_1}\,(y+a)\right], \\ v_2 &= \pm\sqrt{\tfrac{A_0}{A_1}}\cos\left[\sqrt{A_1}\,(y+a)\right], \end{aligned} \tag{7}$$

where a is constant. If $A_1 < 0$ then the Eq. (6) has the solutions:

$$\begin{aligned} v_3 &= \pm\sqrt{\tfrac{A_0}{A_1}}\sinh\left[\sqrt{A_1}\,(y+a)\right], \\ v_4 &= \pm i\sqrt{\tfrac{A_0}{A_1}}\cosh\left[\sqrt{A_1}\,(y+a)\right], \end{aligned} \tag{8}$$

where $i = \sqrt{-1}$. For secondly, if the solutions of the other equation is considered,

$$\left(\frac{dv}{dy}\right)^2 = v^2\left(A_2 + A_3 v^2\right) \tag{9}$$

where $A_2 \neq 0$ and $A_3 \neq 0$ are constants. When $A_2 < 0$, Eq. (9) has the solutions

$$\begin{aligned} v_5 &= \pm\sqrt{\tfrac{A_2}{A_3}}\sec\left[\sqrt{-A_2}y\right], \\ v_6 &= \pm\sqrt{\tfrac{A_2}{A_3}}\csc\left[\sqrt{-A_2}y\right], \end{aligned} \tag{10}$$

If $A_2 > 0$ in Eq. (9), then the solutions are

$$\begin{aligned} v_7 &= \pm\sqrt{\tfrac{A_2}{A_3}}\operatorname{sech}\left[\sqrt{-A_2}y\right], \\ v_6 &= \pm i\sqrt{\tfrac{A_2}{A_3}}\operatorname{csch}\left[\sqrt{-A_2}y\right], \end{aligned} \tag{11}$$

2.1 Nonlinear Dispersive *GNLS(m,n,k,l)* Equation

If we take the solution with transform $u(x,t) = \phi(\xi)\exp(i\sigma t)$with

$\xi = c(x - \lambda t)$, $(c, \lambda \text{ and } \sigma \text{ are constants})$ we have the following ODE:

$$-\sigma\phi + Ac^2(\phi^m)'' + B\phi^n + i\left[-\lambda c\phi' + Cc^3\left(\phi^k\right)''' + Dc\left(\phi^l\right)'\right] = 0 \tag{12}$$

By separating real and imaginary parts of the above equation and then by integration of imaginary part we have

$$-\sigma\phi + Ac^2(\phi^m)'' + B\phi^n = 0 \tag{13}$$

$$-\lambda\phi + Cc^2\left(\phi^k\right)'' + D\phi^l = 0 \tag{14}$$

To reach compactons, solitary patterns, solitons and periodic solutions of nonlinearly dispersive GNLS equation firstly assume the following constraints:

$$\lambda = \sigma, \quad A = C, \quad B = D, \quad m = k \text{ and } n = l \tag{15}$$

then by the following transformation

$$\frac{d\phi^m}{d\xi} = z, \quad \frac{d^2\phi^m}{d\xi^2} = z\frac{dz}{d\phi^m} \tag{16}$$

substituting (16) into (13) leads to the following equation,

$$\frac{Ac^2}{2}\left(n\phi^{\frac{n-3}{2}}\frac{d\phi}{d\xi}\right)^2 = \frac{n\sigma}{n+1} - \frac{nB}{m+n}\phi^{n-1}. \tag{17}$$

Letting $\phi = w^{\frac{2}{n-1}}$, we have

$$\phi = w^{\frac{2}{n-1}} \Rightarrow d\phi = \frac{2}{n-1}\phi^{\frac{2}{n-1}-1}dw \tag{18}$$

which changes Eq. (13) to

$$\frac{2Ak^2n}{(m-1)^2}\left(w^{\frac{n-m}{m-1}}\frac{dw}{d\xi}\right)^2 = \frac{\sigma}{n+1} - \frac{n}{m+n}w^2. \tag{19}$$

Case I. If we take $n = 2m-1$ in Eq. (19), then we get:

$$p_1(x,t) = \exp[i\sigma t]\left[\pm\frac{1}{A_4}\left(A_5 - A_4{}^2k^2(x-\lambda t)^2 \pm 2A_4{}^2k^2(x-\lambda t)^2A_6 - A_4{}^2A_6{}^2\right)\right]^{\frac{1}{m-1}} \tag{20}$$

where

$$A_4 = \frac{B(m-1)^2}{2Ak^2(3m-1)(2m-1)} \text{ and } A_5 = \frac{(m-1)^2}{4Ak^2m(2m-1)}$$

.

Case II. If $m = n$ we know that Eq. (19) becomes

$$\left(\frac{dw}{d\xi}\right)^2 = \frac{\sigma(m-1)^2}{2Ak^2m(m+1)} - \frac{B(m-1)^2}{4Ak^2m^2}w^2. \tag{21}$$

in which we assume that $m \neq -1$ and $A, B, k, \lambda \neq 0$.

If $AB > 0$, it becomes from Eqs. (6) and (21) that

$$p_2(x,t) = \exp[i\tau t]\left(\frac{2\sigma m}{B(m+1)}\sin^2\left[\frac{m-1}{2m|k|}\sqrt{\frac{B}{A}}(c(x-\lambda t)+a)\right]\right)^{\frac{1}{m-1}}, \tag{22}$$

$$p_3(x,t) = \exp[i\tau t]\left(\frac{2\sigma m}{B(m+1)}\cos^2\left[\frac{m-1}{2m|k|}\sqrt{\frac{B}{A}}(c(x-\lambda t)+a)\right]\right)^{\frac{1}{m-1}}, \tag{23}$$

If $AB < 0$, it becomes from Eqs. (7) and (21) that

$$p_4(x,t) = \exp[i\tau t]\left(-\frac{2\sigma m}{B(m+1)}\sinh^2\left[\frac{m-1}{2m|c|}\sqrt{-\frac{B}{A}}(c(x-\lambda t)+a)\right]\right)^{\frac{1}{m-1}}, \tag{24}$$

$$p_5(x,t) = \exp[i\tau t]\left(\frac{2\sigma m}{B(m+1)}\cosh^2\left[\frac{m-1}{2m|c|}\sqrt{-\frac{B}{A}}(c(x-\lambda t)+a)\right]\right)^{\frac{1}{m-1}}. \tag{25}$$

Theorem 1. The nonlinear dispersion GNLS equation has solutions under the constraint (15) described as follows.

1. When $AB > 0$ and $m > 1$

$$p(x,t) = \begin{cases} \exp[i\tau t]\left(\frac{2\sigma m}{B(m+1)}\cos^2\left[\frac{m-1}{2m|c|}\sqrt{\frac{B}{A}}(c(x-\lambda t)+a)\right]\right)^{\frac{1}{m-1}}, & \left|\frac{m-1}{2m|c|}\sqrt{\frac{B}{A}}(c(x-\lambda t))\right| \leq \frac{\pi}{2} \\ 0, \quad \text{Otherwise} & \end{cases} \tag{26}$$

is the solitary wave solution(s.w.s) with compact support of GNLS equation.

2. When $AB > 0$ and $m > 1$

$$p(x,t) = \begin{cases} \exp[i\tau t]\left(\frac{2\sigma m}{B(m+1)}\sin^2\left[\frac{m-1}{2m|c|}\sqrt{\frac{B}{A}}c(x-\lambda t)\right]\right)^{\frac{1}{m-1}}, & 0 \leq \frac{m-1}{2m|c|}\sqrt{\frac{B}{A}}c(x-\lambda t) \leq \pi \\ 0, \quad \text{Otherwise} & \end{cases} \tag{27}$$

is a compacton solution.

3. When $AB < 0$ and $m < 1$

$$p(x,t) = \exp[i\tau t]\left(\frac{B(m+1)}{2\sigma m}\csc^2\left[\frac{m-1}{2m|c|}\sqrt{\frac{B}{A}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}} \tag{28}$$

is a s.w.s for GNLS Eq. for $0 < (x-\lambda t) < \frac{2m|c|}{m-1}\sqrt{\frac{A}{B}}\pi$.

4. When $AB < 0$and $m < 1$

$$p(x,t) = \exp[i\tau t]\left(\frac{B(m+1)}{2\sigma m}\sec^2\left[\frac{m-1}{2m|c|}\sqrt{\frac{B}{A}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}} \tag{29}$$

is a s.w.s for the GNLS equation for $|x-\lambda t| < \frac{2m|c|}{m-1}\sqrt{\frac{A}{B}}\pi$.

Theorem 2. The nonlinear dispersion GNLS equation has the following solutions with *m*=*n* under the constraint (19).

1.When $B<0$, $\sigma m(m+1)>0$

$$p(x,t)=\exp[i\tau t]\left(-\frac{2\sigma m}{B(m+1)}\sinh^2\left[\frac{m-1}{2m|c|}\sqrt{-\frac{B}{A}}c(x-\lambda t)\right]\right)^{\frac{1}{m-1}}, \tag{30}$$

is a solitary pattern solution (s.p.s).

1.When $B<0$, $\sigma m(m+1)<0$

$$p(x,t)=\exp[i\tau t]\left(\frac{2\sigma m}{B(m+1)}\cosh^2\left[\frac{m-1}{2m|c|}\sqrt{-\frac{B}{A}}c(x-\lambda t)\right]\right)^{\frac{1}{m-1}}, \tag{31}$$

is a s.p.s, when $m<1$, Eq. (31) is a bounded solution.

1.*m*=*n* <1 solutions (30) and (31) become s.w.s

$$p(x,t)=\exp[i\tau t]\left(-\frac{2\sigma m}{B(m+1)}\operatorname{csch}^2\left[\frac{m-1}{2m|c|}\sqrt{-\frac{B}{A}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}}, \tag{32}$$

which is the singular s.w.s. The solution improves a singularity at a finite point, i.e., for any fixed $t=t_0$, there is blow up of the solution [24], [25].

$$p(x,t)=\exp[i\tau t]\left(\frac{2\sigma m}{B(m+1)}\operatorname{sech}^2\left[\frac{m-1}{2m|c|}\sqrt{-\frac{B}{A}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}}, \tag{33}$$

which is the bell-shaped s.w.s.

Case III. If $n=1$ then the Eq. (19) becomes

$$\left(\frac{dw}{d\xi}\right)^2=w^2\left(\frac{\sigma(m-1)^2}{4Ac^2m}+\frac{-B(m-1)^2}{2Ac^2(m+1)}w^2\right). \tag{34}$$

If $\sigma Am<0$ and $m>1$ it becomes from Eqs. (9) and (34) that

$$p_6(x,t)=\exp[i\tau t]\left(\frac{\sigma(m+1)}{2Bm}\sec^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}(c(x-\lambda t))\right]\right)^{\frac{1}{m-1}}, \tag{35}$$

$$p_7(x,t)=\exp[i\tau t]\left(\frac{\sigma(m+1)}{2Bm}\csc^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}(c(x-\lambda t))\right]\right)^{\frac{1}{m-1}}. \tag{36}$$

Case IV. If $\sigma Am>0$ and $m>1$ it becomes from Eqs. (10) and (34) that

$$p_8(x,t)=\exp[i\tau t]\left(\frac{\sigma(m+1)}{2Bm}\operatorname{sech}^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}(c(x-\lambda t))\right]\right)^{\frac{1}{m-1}}, \tag{37}$$

which is the bell-shaped s.w.s.

$$p_9(x,t) = \exp[i\tau t]\left(-\frac{\sigma(m+1)}{2Bm}\operatorname{csch}^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}(c(x-\lambda t))\right]\right)^{\frac{1}{m-1}}, \tag{38}$$

which is the singular s.w.s. The solution improves a singularity at a finite point, i.e., for any fixed $t = t_0$, there is blow up of the solution [24], [25].

Case V. If $\sigma Am < 0$ and $m > 1$

we have the following compacton solutions with the aid of Eqs. (37) and (38)

$$p_{10}(x,t) = \exp[i\tau t]\left(\frac{2Bm}{\sigma(m+1)}\cos^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}}, \tag{39}$$

$\left|(m-1)(x-\lambda t)\sqrt{-\frac{\sigma}{Am}}\right| \le \pi$ and $p = 0$ otherwise.

$$p_{11}(x,t) = \exp[i\tau t]\left(\frac{2Bm}{\sigma(m+1)}\sin^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}}, \tag{40}$$

for $0 \le \left|(m-1)(x-\lambda t)\sqrt{-\frac{\sigma}{Am}}\right| \le 2\pi$ and $p = 0$otherwise.

Case VI. If $\sigma Am > 0$ and $m < 1$

Using $\cosh(x) = \cos(ix)$ and $\sinh(x) = -\sin(ix)$ we have the following s.p.s

$$p_{12}(x,t) = \exp[i\tau t]\left(-\frac{2Bm}{\sigma(m+1)}\sinh^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}}, \tag{41}$$

$$p_{13}(x,t) = \exp[i\tau t]\left(\frac{2Bm}{\sigma(m+1)}\cosh^2\left[\frac{m-1}{2|c|}\sqrt{-\frac{\sigma}{Am}}c(x-\lambda t)\right]\right)^{\frac{1}{1-m}} \tag{42}$$

4. Conclusions

We used the ansatz method with the aid of reducing the order of the equation for finding some new exact solutions for GNLS(m, n, k, l) equation. We have acquired different types of exact solutions to the equation. The solutions obtained are expressed in terms of s.w, s.p, compacton, bell shaped and singular.

Conflict Interest

I declare that there is no conflict of interests regarding the publication of this article.

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