New Trends in Mathematical Sciences
http://www.ntmsci.com

A new type of interval valued fuzzy normal subgroups of groups

Saleem Abdullah [1] *and Muhammad Naeem* [2]

[1] Department of Mathematics, Quaid-i-Azam University 45320 Islamabad 44000, Pakistan.
[2] Deanship of Preparatory Year, Umm ul Qurra University, Makkah, Saudi Arabia.

Received: 7 December 2014, Revised: 12 December 2014, Accepted: 22 December 2014
Published online: 25 December 2014

Abstract: In this paper we are using the notions of *not belonging* $(\overline{\in})$ and *non quasi-k-coincidence* $\left(\overline{q_k}\right)$ of an interval valued fuzzy point with an interval valued fuzzy set, we define the concepts of interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroups and interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy cosets which is a generalization of fuzzy normal subgroups, fuzzy coset, interval valued fuzzy normal subgroups, interval valued fuzzy coset, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy normal subgroups and interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy cosets. We give some characterizations of an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroup and interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy coset, and deal with several related properties. The important achievement of the study with an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroup and interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy cosets is the generalization of that the notions of fuzzy normal subgroups, fuzzy coset, interval valued fuzzy normal subgroups, interval valued fuzzy coset, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy normal subgroups and interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy cosets. We prove that the set of all interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy cosets of G is a group, where the multiplication is defined by $\overleftarrow{\lambda}_x\cdot\overleftarrow{\lambda}_y=\overleftarrow{\lambda}_{xy}$ for all $x,y\in G$. If $\widetilde{\mu}:F\to D[0,1]$ is defined by $\widetilde{\mu}\left(\overleftarrow{\lambda}_x\right)=\widetilde{\lambda}(x)$ for all $x\in G$. Then $\widetilde{\mu}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroup of F.

Keywords: Group, normal subgroup, coset, interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroup, interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy coset.

1 Introduction

The basic notion of fuzzy set was introduced by L.A. Zadeh [24] of 1965. Rosenfeld was the first who used the concept of fuzzy set in algebra. He introduced the concept of fuzzy subgroup of a group in 1971 [19]. The fuzzy algebra play a vital role in Mathematics with wide applications in control engineering, computer sciences, information sciences, coding theory. Murali [17] gave the definition of a fuzzy point belonging to fuzzy subset under a natural equivalence. Bhakat and Das give more generalized concept of Rosenfeld's fuzzy subgroups and they introduced the notion of $(\in,\in\vee q)$-fuzzy subgroups of a group in [7] by using the concept of "*belongingness*" $(\in)$ and "*quasi-coincidence*" (q) of fuzzy point and fuzzy set, which was introduced by Pu and Liu [18]. Liu [15] introduced the concept of fuzzy normality of a fuzzy subgroup in 1982. A detail study of the fuzzy normal subgroups was introduced by Mukherjee and Bhattacharya [16]. Fuzzy normal subgroup was more studied by Bhakat [5,6], Bhakat and Das [7], and Yuan et al. [23]. In particular, an $(\in,\in\vee q)$-fuzzy subgroup is an imperative and beneficial generalization of Rosenfeld's fuzzy subgroup. In [21], Shabir et al. introduced the notion of (α,β)-fuzzy, ideals of regular semigroups. M. Aslam et al defined (α,β)-fuzzy Γ-ideals in Γ-LA-semigroup and characterized different classes of Γ-LA-semigroup by the properties of (α,β)-fuzzy Γ-ideals. There some extensions of fuzzy set. Interval valued fuzzy set is one of them which was

* Corresponding author e-mail: saleemabdullah@yahoo.com

introduced by Zadeh [25]. In [9], Biswas introduced the concept of interval valued fuzzy subgroup of groups and he studied some different properties of interval valued fuzzy subgroup. Latha et al., introduced the notion of interval valued (α,β)-fuzzy subgroups [14]. For more detail see [4,11,26,27]Jun, in [12] introduced the generalized concept of quasi-coincident (q) by quasi-k-coincident (q_k) of a fuzzy point with a fuzzy set, and he defined $(\in,q_k)$- fuzzy subalgebra and $(\in,\in\vee q_k)$- fuzzy subalgebra in BCK/BCI-algebra. Jun et al. initiated the concept of $(\in,\in\vee q_k)$-fuzzy subgroups, $(\in,\in\vee q_k)$-fuzzy normal subgroups, $(\in,\in\vee q_k)$-fuzzy cosets and generalized the concept of level subsets[13]. In [22], Shabir et al. adobted the notion of $(\in,\in\vee q_k)$-fuzzy ideals in semigroups and they characterized regular and intra-regular semigroups by the properties of $(\in,\in\vee q_k)$-fuzzy ideals. Recently, Abdullah et al., proposed the concept of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroups and $(\overline{\in},\overline{\in}\vee\overline{q_{\tilde{k}}})$-fuzzy cosets [2].

In this paper we are using the notions of *not belonging* $(\overline{\in})$ and *non quasi-k-coincidence* $(\overline{q_{\tilde{k}}})$ of an interval valued fuzzy point with an interval valued fuzzy set, we define the concepts of interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroups and interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy cosets which is a generalization of fuzzy normal subgroups, fuzzy coset, interval valued fuzzy normal subgroups, interval valued fuzzy coset, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy normal subgroups and interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy cosets. We give some characterizations of an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroup and interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy coset, and deal with several related properties. The important achievement of the study with an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroup and interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy cosets is the generalization of that the notions of fuzzy normal subgroups, fuzzy coset, interval valued fuzzy normal subgroups, interval valued fuzzy coset, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy normal subgroups and interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy cosets. We prove that the set of all interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy cosets of G is a group, where the multiplication is defined by $\overleftarrow{\lambda}_x\cdot\overleftarrow{\lambda}_y=\overleftarrow{\lambda}_{xy}$ for all $x,y\in G$. If $\widetilde{\mu}:F\to D[0,1]$ is defined by $\widetilde{\mu}\left(\overleftarrow{\lambda}_x\right)=\widetilde{\lambda}(x)$ for all $x\in G$. Then $\widetilde{\mu}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy normal subgroup of F.

2 Preliminaries

In this section, we will introduced the basic concept of definitions of previous literature.

If $A\subseteq G$, then the interval valued characteristic function C_A of A is a function from A into $\{0,1\}$, defined by

$$\widetilde{C_A}(x)=\begin{cases}\widetilde{1} & \text{if } x\in A\\ \widetilde{0} & \text{if } x\notin A\end{cases}$$

A fuzzy subset of a universe X is a function λ from X into the unit closed interval $[0,1]$, that is, $\lambda:X\to[0,1]$. For any two fuzzy subsets λ and μ of G, $\lambda\le\mu$ means that, for all $x\in G$, $\lambda(x)\le\mu(x)$. The symbols $\lambda\wedge\mu$, and $\lambda\vee\mu$ will mean the following fuzzy subsets of G.

$$(\lambda\wedge\mu)(x)=\lambda(x)\wedge\mu(x)$$
$$(\lambda\vee\mu)(x)=\lambda(x)\vee\mu(x)$$

for all $x \in X$. More generally, if $\{\lambda_i : i \in \Omega\}$ is a family of fuzzy subsets of G, then $\wedge_{i\in\Omega}\lambda_i$ and $\vee_{i\in\Omega}\lambda_i$ are defined by

$$(\wedge_{i\in\Omega}\lambda_i)(x) = \wedge_{i\in\Omega}(\lambda_i(x))$$
$$(\vee_{i\in\Omega}\lambda_i)(x) = \vee_{i\in\Omega}(\lambda_i(x))$$

and are called the intersection and the union of the family $\{\lambda_i : i \in \Omega\}$ of fuzzy subsets of G, respectively.

An interval valued fuzzy subset of a universe X is a function $\widetilde{\lambda}$ from X into $D[0,1]$, that is, $\lambda : X \to D[0,1]$, where $D[0,1]$ mean subinterval of $[0,1]$. For any two interval valued fuzzy subsets $\widetilde{\lambda}$ and $\widetilde{\mu}$ of G, $\widetilde{\lambda} \leq \widetilde{\mu}$ means that, for all $x \in G$, $\widetilde{\lambda}(x) \leq \widetilde{\mu}(x)$. The symbols $\widetilde{\lambda} \wedge \widetilde{\mu}$, and $\widetilde{\lambda} \vee \widetilde{\mu}$ will mean the following fuzzy subsets of G.

$$\left(\widetilde{\lambda} \wedge \widetilde{\mu}\right)(x) = \widetilde{\lambda}(x) \wedge \widetilde{\mu}(x)$$
$$\left(\widetilde{\lambda} \vee \widetilde{\mu}\right)(x) = \widetilde{\lambda}(x) \vee \widetilde{\mu}(x)$$

for all $x \in X$.

Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be two interval valued fuzzy subsets of G. Then the product of $\widetilde{\lambda}$, $\widetilde{\mu}$ is denoted by $\widetilde{\lambda} \circ \widetilde{\mu}$ and defined by

$$\left(\widetilde{\lambda} \circ \widetilde{\mu}\right)(x) = \begin{cases} \vee_{x=yz}\left\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z)\right\}, & \text{if } x = yz, \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

Definition 1. An interval valued fuzzy subset $\widetilde{\lambda}$ of X of the form

$$\widetilde{\lambda}(y) = \begin{cases} \widetilde{t} & if \ \ y = x \\ \widetilde{0} & if \ \ y \neq x \end{cases}$$

is called an interval valued fuzzy point with support x and value $\widetilde{t}$, where $\widetilde{t} \in D(0,1]$ and is denoted by $x_{\widetilde{t}}$ or $[x;\widetilde{t}]$. An interval valued fuzzy point $x_{\widetilde{t}}$ is said to be not belong to (resp, not quasi-coincident with) an interval valued fuzzy subset $\widetilde{\lambda}$, written as $x_{\widetilde{t}}\overline{\in}\widetilde{\lambda}$(resp, $x_{\widetilde{t}}\overline{q}\widetilde{\lambda}$) if $\widetilde{\lambda}(x) < \widetilde{t}$ (resp, $\widetilde{\lambda}(x) + \widetilde{t} \leq 1$). If $x_{\widetilde{t}}\overline{\in}\widetilde{\lambda}$ and (resp, $x_{\widetilde{t}}\overline{q}\widetilde{\lambda}$), then we write $x_{\widetilde{t}}\overline{\in \wedge q}\widetilde{\lambda}$ (resp, $x_{\widetilde{t}}\overline{\in \vee q}\widetilde{\lambda}$). And $x_{\widetilde{t}}$ be not quasi-k-coincident with a fuzzy subset $\widetilde{\lambda}$, written as $x_{\widetilde{t}}\overline{q_k}\widetilde{\lambda}$ if $\widetilde{\lambda}(x) + \widetilde{t} + \widetilde{k} \leq \widetilde{1}$ or $\widetilde{\lambda}(x) + \widetilde{t} \leq \widetilde{1} - \widetilde{k}$, and $x_{\widetilde{t}}\overline{\in \vee q_k}\widetilde{\lambda}$ if $x_{\widetilde{t}}\overline{\in}\widetilde{\lambda}$ or $x_{\widetilde{t}}\overline{q_k}\widetilde{\lambda}$, here $\widetilde{t} \in D(0,1]$ and $\widetilde{k} \in D[0,1)$ [12].

Definition 2. [9]An interval valued fuzzy subset $\widetilde{\lambda}$ of G is called an interval valued fuzzy subgroup of G if for all $x, y \in G$,
(i) $\widetilde{\lambda}(xy) \geq \min\left(\widetilde{\lambda}(x), \widetilde{\lambda}(y)\right)$ and (ii) $\widetilde{\lambda}\left(x^{-1}\right) \geq \widetilde{\lambda}(x)$.
or $\widetilde{\lambda}\left(x^{-1}y\right) \geq \min\left(\widetilde{\lambda}\left(x^{-1}\right), \widetilde{\lambda}(y)\right)$.

Definition 3. [22] Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of G. We give the definition of the upper part $\widetilde{\lambda}^{+}$ as follows, $\widetilde{\lambda}^{+}(x) = \widetilde{\lambda}(x) \vee \frac{1-\widetilde{k}}{2}$.

Definition 4. [9] An interval valued fuzzy subset $\widetilde{\lambda}$ of G is said to be an interval valued fuzzy normal subgroup of G if it is an interval valued fuzzy subgroup of G and holds: $\widetilde{\lambda}\left(y^{-1}xy\right) \geq \widetilde{\lambda}(x)$ $(\forall\, x, y \in G)$ $\left(\widetilde{t} \in D(0,1]\right)$.

Definition 5. [9] Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be two interval valued fuzzy subgroups of G. Then, $\widetilde{\lambda}$ is called an interval valued fuzzy conjugate of $\widetilde{\mu}$ if for $x \in G$, we have $\widetilde{\lambda}(y) = \widetilde{\lambda}\left(x^{-1}yx\right)$ for all $x \in G$.

Definition 6. [14] An interval valued fuzzy subset $\widetilde{\lambda}$ of a group G is called an interval valued $(\in, \in \vee q)$-fuzzy subgroup of G if for all $x, y \in G$,

(i) $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{0.5}\}$ and (ii) $\widetilde{\lambda}\left(x^{-1}\right) \geq \min\{\widetilde{\lambda}(x), \widetilde{0.5}\}$

or $\widetilde{\lambda}\left(x^{-1}y\right) \geq \min\left(\widetilde{\lambda}\left(x^{-1}\right), \widetilde{\lambda}(y), \widetilde{0.5}\right)$ for all $x^{-1}, y \in G$.

Definition 7. [14] For a interval valued fuzzy subgroup $\widetilde{\lambda}$ of G the normalizer of $\widetilde{\lambda}$, denoted by $N\left(\widetilde{\lambda}\right)$ and is defined by $N\left(\widetilde{\lambda}\right) = \{y \in G; \widetilde{\lambda}(y^{-1}xy) \geq \min\{\widetilde{\lambda}(x), 0.5\} \ \forall \, x \in G\}$.

3 $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-Fuzzy Normal Subgroups

In what follows, let G denote a group with e as the identity element, and $\widetilde{k}$ an arbitrary element of $D[0,1)$ unless otherwise specified.

Definition 8. An interval valued fuzzy subset $\widetilde{\lambda}$ of a group G is said to be an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup of G if for all $x, y \in G$ and $\widetilde{t}, \widetilde{r} \in D(0,1]$ and $\widetilde{k} \in D[0,1)$.

(i) $\max\{\widetilde{\lambda}(xy), \frac{\widetilde{1-k}}{2}\} \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}$ and

(ii) $\max\{\widetilde{\lambda}(x^{-1}), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(x)$

or (i) $(xy)_{M\{\widetilde{t},r\}} \overline{\in} \widetilde{\lambda}$ implies $x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$ or $y_{\widetilde{r}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$ and

(ii) $x_{\widetilde{t}}^{-1} \overline{\in} \widetilde{\lambda}$ implies $x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$.

Definition 9. An interval valued fuzzy subset $\widetilde{\lambda}$ of a group G is called an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup of G if for all $x, y \in G$ and $\widetilde{t}, \widetilde{r} \in D(0,1]$ and $\widetilde{k} \in D[0,1)$.

(i) $\max\{\widetilde{\lambda}(xy), \frac{\widetilde{1-k}}{2}\} \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}$ and

(ii) $\max\{\widetilde{\lambda}(x^{-1}), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(x)$

or (i) $(xy)_{m\{\widetilde{t},\widetilde{r}\}} \overline{\in} \widetilde{\lambda}$ implies $x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$ or $y_{\widetilde{r}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$ and

(ii) $x_{\widetilde{t}}^{-1} \overline{\in} \widetilde{\lambda}$ implies $x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$.

If $\widetilde{k} = \widetilde{0}$, then $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G.

Definition 10. An interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup $\widetilde{\lambda}$ of G is called an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G if for any $x, y \in G$ and $\widetilde{t} \in D(0,1]$ and $\widetilde{k} \in D[0,1)$, $\left(y^{-1}xy\right)_{\widetilde{t}} \overline{\in} \widetilde{\lambda}$ implies $x_{\widetilde{t}} \overline{\in} \vee \overline{q_k} \widetilde{\lambda}$.

An interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G with $\widetilde{k} = \widetilde{0}$ is become an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy normal subgroup of G. Every interval valued $(\overline{\in}, \overline{\in})$-fuzzy normal subgroup is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G.

Theorem 1. Let $\widetilde{\lambda}$ be an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup of G. Then, the following assertion are equivalent:

(1) $\widetilde{\lambda}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup.

(2) $\forall \, x, y \in G$, $\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(x)$.

(3) $\forall \, x, y \in G$, $\max\{\widetilde{\lambda}(xy), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(yx)$.

(4) $\forall \, x, y \in G$, $\max\{\widetilde{\lambda}(x^{-1}y^{-1}xy), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(x)$.

Proof. $(1) \Rightarrow (2)$ Let us take $x, y \in G$ such that $\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} < \widetilde{\lambda}(x)$. Then, we have

$$\max\{\widetilde{\lambda}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} < \widetilde{t} \leq \widetilde{\lambda}(x)$$

for some $\widetilde{t} \in (\frac{\widetilde{1}-\widetilde{k}}{2}, \widetilde{1}]$. It follows that $x_{\widetilde{t}} \in \widetilde{\lambda}$.

In case of $\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} = \widetilde{\lambda}(y^{-1}xy)$, that is $\frac{\widetilde{1}-\widetilde{k}}{2} < \widetilde{\lambda}(y^{-1}xy)$ implies $\widetilde{\lambda}(y^{-1}xy) < \widetilde{t}$ implies $(y^{-1}xy)_{\widetilde{t}} \overline{\in} \widetilde{\lambda}$. Moreover, we have

$$\begin{aligned} &\widetilde{\lambda}(x) \geq \widetilde{t} > \widetilde{\lambda}(y^{-1}xy) > \tfrac{\widetilde{1}-\widetilde{k}}{2} \\ &\widetilde{\lambda}(x) + \widetilde{t} + \widetilde{k} > \widetilde{1} \Rightarrow x_{\widetilde{t}} q_{\widetilde{k}} \widetilde{\lambda}. \end{aligned}$$

As $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $x_{\widetilde{t}} q_{\widetilde{k}} \widetilde{\lambda}$ implies $x_{\widetilde{t}} \in \wedge q_{\widetilde{k}} \widetilde{\lambda}$, which is not true because $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

In this case we have $\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} = \frac{\widetilde{1}-\widetilde{k}}{2}$. Then,

$$\begin{aligned} &\tfrac{\widetilde{1}-\widetilde{k}}{2} < \widetilde{t} \leq \widetilde{\lambda}(x) \Longrightarrow x_{\widetilde{t}} \in \widetilde{\lambda} \\ &\widetilde{\lambda}(x) + \widetilde{t} + k > \tfrac{\widetilde{1}-\widetilde{k}}{2} + \tfrac{\widetilde{1}-\widetilde{k}}{2} + \widetilde{k} = \widetilde{1} \end{aligned}$$

this imply $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{1} - \widetilde{k} \Longrightarrow x_{\widetilde{t}} q_k \widetilde{\lambda}$. So $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $x_{\widetilde{t}} q_{\widetilde{k}} \widetilde{\lambda} \Rightarrow x_{\widetilde{t}} \in \wedge q_{\widetilde{k}} \widetilde{\lambda}$. Thus, we have $\widetilde{\lambda}$ is not an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G, which is a contradiction, Hence, for all $x, y \in G$, we have

$$\max\{\widetilde{\lambda}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(x).$$

$(2) \Rightarrow (1)$ Let $x, y \in G$ and $(y^{-1}xy)_{\widetilde{t}} \overline{\in} \widetilde{\lambda} \Longrightarrow \widetilde{\lambda}(y^{-1}xy) < \widetilde{t}$ for some $\widetilde{t} \in (\frac{\widetilde{1}-\widetilde{k}}{2}, 1]$. Then, we have

$$\widetilde{t} > \max\{\widetilde{\lambda}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(x) \Rightarrow \widetilde{t} > \widetilde{\lambda}(x).$$

Thus, $x_{\widetilde{t}} \overline{\in} \widetilde{\lambda} \Rightarrow x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$. Therefore, $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup.

$(2) \Leftrightarrow (3)$ By using (2) we get $\widetilde{\lambda}(xy) = \widetilde{\lambda}\left(y^{-1}yxy\right)$. Then, we have

$$\begin{aligned} \max\{\widetilde{\lambda}(xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} &= \max\{\widetilde{\lambda}(y^{-1}yxy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &= \max\{\widetilde{\lambda}(y^{-1}(yx)y, \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &\geq \widetilde{\lambda}(yx). \end{aligned}$$

Thus, $\max\{\widetilde{\lambda}(xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(yx)$ for all $x, y \in G$. Now using (3), we have

$$\begin{aligned} \max\{\widetilde{\lambda}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} &= \max\{\widetilde{\lambda}((y^{-1}x)y), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &\geq \widetilde{\lambda}(yy^{-1}x) = \widetilde{\lambda}(x). \end{aligned}$$

Thus, we have $\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(x)$ for all $x, y \in G$.

$(3) \Rightarrow (4)$ Since $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G. So for all $x, y \in G$ we have $x^{-1}, y^{-1} \in G$, Thus,

$$\begin{aligned}
\max\{\widetilde{\lambda}(x^{-1}y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} &= \widetilde{\lambda}(x^{-1}y^{-1}xy) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&= \begin{pmatrix} \widetilde{\lambda}\left(x^{-1}y^{-1}xy\right) \vee \\ \tfrac{\widetilde{1}-\widetilde{k}}{2} \end{pmatrix} \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&\geq \left(\widetilde{\lambda}\left(x^{-1}\right) \wedge \widetilde{\lambda}\left(y^{-1}xy\right)\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&\geq \left(\widetilde{\lambda}\left(x^{-1}\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right) \wedge \\
&\quad \left(\widetilde{\lambda}\left(y^{-1}xy\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right) \\
&\geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(x) \\
&\geq \widetilde{\lambda}(x).
\end{aligned}$$

Hence, $\max\{\widetilde{\lambda}(x^{-1}y^{-1}xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(x)$ for all $x, x^{-1}, y, y^{-1} \in G$.

$(4) \Rightarrow (2)$ Let $x, y \in G$. Then, we have

$$\begin{aligned}
\max\{\widetilde{\lambda}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} &= \widetilde{\lambda}(xx^{-1}y^{-1}xy) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&= \begin{pmatrix} \widetilde{\lambda}\left(xx^{-1}y^{-1}xy\right) \\ \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \end{pmatrix} \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&\geq \left(\widetilde{\lambda}(x) \wedge \widetilde{\lambda}\left(x^{-1}y^{-1}xy\right)\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&\geq \left(\widetilde{\lambda}(x) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right) \wedge \\
&\quad \left(\widetilde{\lambda}\left(x^{-1}y^{-1}xy\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right) \\
&\geq \left(\widetilde{\lambda}(x) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right) \wedge \widetilde{\lambda}(x) \\
&\geq \widetilde{\lambda}(x).
\end{aligned}$$

Hence, $\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(x)$ for all $x, y \in G$. This completes the proof.

if we put $k = 0$ in Theorem 1, then we get.

Corollary 1. *Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subgroup of G. Then the following conditions are equivalent:*

(a) *$\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy normal subgroup.*

(b) $\max\{\widetilde{\lambda}(y^{-1}xy), 0.5\} \geq \widetilde{\lambda}(x)$, *for all $x, y \in G$.*

(c) $\max\{\widetilde{\lambda}(xy), 0.5\} \geq \widetilde{\lambda}(yx)$, *for all $x, y \in G$.*

(d) $\max\{\widetilde{\lambda}(x^{-1}y^{-1}xy), 0.5\} \geq \widetilde{\lambda}(x)$, *for all $x, y \in G$*

Proof. Proof follows from Theorem 1.

Theorem 2. Let $\widetilde{\lambda}$ be any interval valued fuzzy subset of G. Then, we have $(1) \Leftrightarrow (2)$.

(1) $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup. of G.

(2) $\left(\widetilde{t} \in (\frac{\widetilde{1}-\widetilde{k}}{2}, \widetilde{1}]\right)$ $(\widetilde{\lambda}_{\widetilde{t}} \neq \emptyset$, $\widetilde{\lambda}_{\widetilde{t}}$ is a normal subgroup of $G)$.

Proof. $(1) \Rightarrow (2)$ Since $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G, so for all $\left(\widetilde{t} \in (\frac{\widetilde{1}-\widetilde{k}}{2}, \widetilde{1}]\right)$ $\widetilde{\lambda}_{\widetilde{t}} \neq \emptyset$, $\widetilde{\lambda}_{\widetilde{t}}$ is a subgroup of G by [1, Theorem 3.4]. Next we have to show that $\widetilde{\lambda}_{\widetilde{t}}$ is a normal subgroup. Let $x \in \widetilde{\lambda}_{\widetilde{t}}$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}$ and

$y \in G$ for $\widetilde{t} \in (\frac{\widetilde{1-k}}{2}, 1]$. Since $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup, so for all $x, y \in G$,

$\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(x) \geq \widetilde{t}$
$\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{t}$,

thus

$$\widetilde{\lambda}(y^{-1}xy) \geq \widetilde{t} \Longrightarrow y^{-1}xy \in \widetilde{\lambda}_{\widetilde{t}}$$

for all $y \in G$. Hence, $\widetilde{\lambda}_{\widetilde{t}}$ is a normal subgroup of G.

Conversely, suppose that $\left(\widetilde{t} \in (\frac{\widetilde{1-k}}{2}, 1]\right)$, $\widetilde{\lambda}_{\widetilde{t}} \neq \emptyset$, $\widetilde{\lambda}_{\widetilde{t}}$ is a normal subgroup of G. By using [1, Theorem 3.4] we say that $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G. Now we use the contradiction method to prove our required result. Suppose that $\widetilde{\lambda}$ is not an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup, so there exists some $x, y \in G$ such that $\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} < \widetilde{\lambda}(x)$. Then,

$$\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} < \widetilde{t} \leq \widetilde{\lambda}(x)$$

for some $\widetilde{t} \in (\frac{\widetilde{1-k}}{2}, 1]$. It follows that

$$x \in \widetilde{\lambda}_{\widetilde{t}} \text{ and } \max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} < \widetilde{t}$$

then

$\widetilde{\lambda}(y^{-1}xy) < \frac{\widetilde{1-k}}{2} < \widetilde{t}$ or
$\frac{\widetilde{1-k}}{2} < \widetilde{\lambda}(y^{-1}xy) < \widetilde{t}$.

Hence in both cases

$$\widetilde{\lambda}(y^{-1}xy) < \widetilde{t} \Rightarrow y^{-1}xy \notin \widetilde{\lambda}_{\widetilde{t}}.$$

Thus, for $x \in \widetilde{\lambda}_{\widetilde{t}}$ and $y \in G$ we have, $y^{-1}xy \notin \widetilde{\lambda}_{\widetilde{t}}$. Since $\widetilde{\lambda}_{\widetilde{t}}$ is a normal subgroup of G, which contradicts to our supposition. Therefore, for all $x, y \in G$, we have

$$\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(x).$$

Hence, $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

If we take $k = 0$ in Theorem 2, then we get the following corollary.

Corollary 2. Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of G. Then, we have $(1) \Leftrightarrow (2)$.

(1) $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy normal subgroup.

(2) $\widetilde{t} \in (0.5, 1]$, $(\widetilde{\lambda}_{\widetilde{t}} \neq \emptyset$, $\widetilde{\lambda}_{\widetilde{t}}$ is a normal subgroup of $G)$.

Proof. Proof follows from Theorem 2.

Theorem 3. Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G. Then, $(\forall \widetilde{t} \in (0, \frac{\widetilde{1-k}}{2}]$ and $k \in [0, 1))$, $\underline{Q_k}\left(\widetilde{\lambda}, \widetilde{t}\right) = \{x \in G \mid x\underline{q_{\widetilde{k}}}\widetilde{\lambda}\ \} \neq \emptyset$ is a normal subgroup of G.

Proof. As $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G, so $\underline{Q_{\widetilde{k}}}\left(\widetilde{\lambda}, \widetilde{t}\right)$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G [1, Theorem 3.5]. Next we prove that $\underline{Q_{\widetilde{k}}}\left(\widetilde{\lambda}, \widetilde{t}\right)$ is a normal subgroup. Since $\widetilde{\lambda}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup, so for any $x, y \in G$, we have

$$\max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1-k}}{2}\} \geq \widetilde{\lambda}(x).$$

Let $x \in \underline{Q_{\widetilde{k}}}\left(\widetilde{\lambda}, \widetilde{t}\right)$ and $y \in G$, that is for some $\widetilde{t} \in (0, \frac{\widetilde{1}-\widetilde{k}}{2}]$, so the only possibility is that $x_{\widetilde{t}}\underline{q_{\widetilde{k}}}\widetilde{\lambda}$. Then, $\widetilde{\lambda}\,(x) + \widetilde{t} \geq \widetilde{1} - \widetilde{k} \Rightarrow \widetilde{\lambda}\,(x) \geq \widetilde{1} - \widetilde{k} - \widetilde{t}$. So

$$\begin{aligned} \max\{\widetilde{\lambda}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} &\geq \widetilde{\lambda}(x) \geq \widetilde{1} - \widetilde{k} - \widetilde{t} \\ \widetilde{\lambda}(y^{-1}xy) &\geq \widetilde{1} - \widetilde{k} - \widetilde{t} \\ \widetilde{\lambda}(y^{-1}xy) + \widetilde{t} &\geq \widetilde{1} - \widetilde{k} \\ &\geq (y^{-1}xy)_{\widetilde{t}}\underline{q_{\widetilde{k}}}\widetilde{\lambda} \\ (y^{-1}xy) &\in \underline{Q_{\widetilde{k}}}\left(\widetilde{\lambda}, \widetilde{t}\right). \end{aligned}$$

Thus, $\underline{Q_k}\left(\widetilde{\lambda}, \widetilde{t}\right)$ is a normal subgroup of G.

If we put $\widetilde{k} = \widetilde{0}$ in previous Theorem, then we have the following corollary.

Corollary 3. Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy normal subgroup of G. Then, for all $(\widetilde{t} \in (\widetilde{0}, \widetilde{0.5}])$, $(\underline{Q}\left(\widetilde{\lambda}, \widetilde{t}\right) = \{x \in G \mid x\underline{q}\widetilde{\lambda}\ \} \neq \emptyset$ is a normal subgroup of $G)$.

Proof. Similar as in the above Theorem 3.

Theorem 4. Let N be a non-empty subset of G. Then, N is a normal subgroup of G if and only if the interval valued characteristic function $\widetilde{C_N}$ is an $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal of G.

Proof. As we know that N is a subgroup of a group G, so $\widetilde{C_N}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup of G [1, Theorem 3.7]. Next we prove the normality condition. Suppose that $x \notin N$ implies $\widetilde{C_N}(x) = 0$. Then, we have

$$\max\{\widetilde{C_N}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{C_N}(x)$$

for any $y \in G$.

If $x \in N \Longrightarrow y^{-1}xy \in N$ for all $x \in G$. Since N is a normal subgroup of G. So

$$\begin{aligned} \widetilde{C_N}\,(x) = \widetilde{1} \Longrightarrow \widetilde{C_N}\left(y^{-1}xy\right) &= \widetilde{1} \\ &= \max\{\widetilde{C_N}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &\geq \widetilde{C_N}(x). \end{aligned}$$

Hence, $\widetilde{C_N}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G.

Conversely, suppose that characteristic function $\widetilde{C_N}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G. To prove N is a normal subgroup of G.

Since $\widetilde{C_N}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup, so N is a subgroup of G [1, Theorem 3.7]. Suppose $x \in N$, then $\widetilde{C_N}\,(x) = \widetilde{1}$, since for any $y \in G$, so

$$\begin{aligned} \max\{\widetilde{C_N}(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} &\geq \widetilde{C_N}(x) = \widetilde{1} \\ \widetilde{C_N}(y^{-1}xy) &= \widetilde{1} \\ y^{-1}xy &\in N. \end{aligned}$$

Hence, N is a normal subgroup of G.

Theorem 5. Let I be a normal subgroup of G. Define a fuzzy subset $\widetilde{\lambda}$ of G as follows,
(i) $(\forall x \in G \backslash I)(\widetilde{\lambda}(x) = \widetilde{1})$,
(ii) $(\forall x \in I)(\widetilde{\lambda}(x) \leq \frac{\widetilde{1}-\widetilde{k}}{2})$.
Then, $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

Proof. Let I be a subgroup of G.Then, $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G [1, Theorem 3.8]. Now for normality, let $x, y \in G$ and $\widetilde{t} \in D(0,1]$ such that $(y^{-1}xy)_{\widetilde{t}} \overline{\in} \widetilde{\lambda}$. Then, $\widetilde{\lambda}\left(y^{-1}xy\right) < \widetilde{t}$ this imply $\widetilde{\lambda}\left(y^{-1}xy\right) \neq \widetilde{1}$ implies that $\widetilde{\lambda}\left(y^{-1}xy\right) < \frac{\widetilde{1}-\widetilde{k}}{2}$, so $y^{-1}xy \notin I$. Thus, $x \notin I$, since I is a normal subgroup, so $\widetilde{\lambda}(x) < \frac{\widetilde{1}-\widetilde{k}}{2}$. If $\widetilde{t} > \frac{\widetilde{1}-\widetilde{k}}{2}$, then $\widetilde{\lambda}(x) < \frac{\widetilde{1}-\widetilde{k}}{2} < \widetilde{t}$. Thus, we have

$$\widetilde{\lambda}(x) < \widetilde{t} \Rightarrow x_{\widetilde{t}} \overline{\in} \widetilde{\lambda} \Rightarrow x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}.$$

If $\widetilde{t} \leq \frac{\widetilde{1}-\widetilde{k}}{2}$, then $\widetilde{\lambda}(x) < \widetilde{t} \leq \frac{\widetilde{1}-\widetilde{k}}{2}$ Thus, we have $\widetilde{\lambda}(x) + \widetilde{t} + \widetilde{k} < \frac{\widetilde{1}-\widetilde{k}}{2} + \frac{\widetilde{1}-\widetilde{k}}{2} + \widetilde{k} = \widetilde{1}$, therefore $\widetilde{\lambda}(x) + \widetilde{t} + \widetilde{k} < \widetilde{1}$. Hence, $x_{\widetilde{t}} \overline{q_{\widetilde{k}}} \widetilde{\lambda}$. Therefore $x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}$. Hence, $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

Theorem 6. Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G. Then, $\widetilde{\lambda}^{+}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

Proof. Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G. Then, $\widetilde{\lambda}^{+}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G [1, Theorem 3.11-A].

Now to prove $\widetilde{\lambda} \vee \frac{\widetilde{1}-\widetilde{k}}{2}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup. Since $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup, so for all $x, y \in G$ we have

$$\begin{aligned}
&\max\{\left(\widetilde{\lambda} \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right)(y^{-1}xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \left(\left(\widetilde{\lambda} \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right)\left(y^{-1}xy\right)\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&= \left(\widetilde{\lambda}\left(y^{-1}xy\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}\right) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2} \\
&\geq \widetilde{\lambda}(x) \vee \tfrac{\widetilde{1}-\widetilde{k}}{2}.
\end{aligned}$$

Hence, $\widetilde{\lambda} \vee \frac{\widetilde{1}-\widetilde{k}}{2}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

4 $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-Fuzzy Cosets

In this section we define interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy cosets and give some characterizations of interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy cosets.

Definition 11. Let $\widetilde{\lambda}$ be a interval valued fuzzy subgroup of G. For any $x \in G$ the fuzzy subset $\widetilde{\lambda}_x^l : G \to D[0,1]$, $y \to \widetilde{\lambda}\left(yx^{-1}\right)$ i.e., $\widetilde{\lambda}_x^l(y) = \widetilde{\lambda}\left(yx^{-1}\right)$ resp, $\widetilde{\lambda}_x^r : G \to D[0,1], y \to \widetilde{\lambda}\left(x^{-1}y\right)$ *i.e.*, $\widetilde{\lambda}_x^l(y) = \widetilde{\lambda}\left(x^{-1}y\right)$ is called the fuzzy left (resp. right) coset of G determined by x and $\widetilde{\lambda}$.

Definition 12. Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G for any $x \in G$, $\overleftarrow{\widetilde{\lambda}}_x \left(\text{resp}, \overrightarrow{\widetilde{\lambda}}_x\right) : G \to D[0,1]$ defined by $\overleftarrow{\widetilde{\lambda}}_x(y) = \max\{\widetilde{\lambda}_x^l(y), \frac{\widetilde{1}-\widetilde{k}}{2}\}$ $\left(\text{resp}, \overrightarrow{\widetilde{\lambda}}_x(y) = \max\{\widetilde{\lambda}_x^r(y), \frac{\widetilde{1}-\widetilde{k}}{2}\}\right)$ is called the interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy left (resp. right) coset of G determined by x and $\widetilde{\lambda}$.

Theorem 7. Let $\widetilde{\lambda}$ be an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup of G. Then, $\widetilde{\lambda}$ is an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G if and only if $\overleftarrow{\widetilde{\lambda}_x}=\overrightarrow{\widetilde{\lambda}_x}$ for all $x\in G$.

Proof. Let $\widetilde{\lambda}$ be an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G. Let $x\in G$. Then, for any $g\in G$ we have

$$\begin{aligned}\overleftarrow{\widetilde{\lambda}_x}(g) &= \max\{\widetilde{\lambda}_x^l(g),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\\ &= \max\{\widetilde{\lambda}\left(gx^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\\ &\leq \max\left\{\begin{array}{c}\max\{\widetilde{\lambda}\left(x^{-1}g\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\},\\ \tfrac{\widetilde{1}-\widetilde{k}}{2}\end{array}\right\}\\ &\leq \max\{\widetilde{\lambda}\left(x^{-1}g\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\\ &\leq \max\{\widetilde{\lambda}_x^r(g),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}=\overrightarrow{\widetilde{\lambda}_x}(g).\end{aligned}$$

So, $\overleftarrow{\widetilde{\lambda}_x}(g)\leq\overrightarrow{\widetilde{\lambda}_x}(g)$.

Similarly,

$$\overrightarrow{\widetilde{\lambda}_x}(g)\leq\overleftarrow{\widetilde{\lambda}_x}(g)$$

and so $\overleftarrow{\widetilde{\lambda}_x}(g)=\overrightarrow{\widetilde{\lambda}_x}(g)$ for any $g\in G$. Hence, $\overleftarrow{\widetilde{\lambda}_x}=\overrightarrow{\widetilde{\lambda}_x}$ for all $x\in G$.

Conversely, assume that $\overleftarrow{\widetilde{\lambda}_x}=\overrightarrow{\widetilde{\lambda}_x}$ for all $x\in G$. Then, for any $g\in G$ such that $\overleftarrow{\widetilde{\lambda}_x}(g)=\overrightarrow{\widetilde{\lambda}_x}(g)$

$$\max\{\widetilde{\lambda}_x^l(g),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}=\max\{\widetilde{\lambda}_x^r(g),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$

that is

$$\max\{\widetilde{\lambda}\left(gx^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}=\max\{\widetilde{\lambda}\left(x^{-1}g\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$

for all $g\in G$. Taking $g=xyx$ implies that

$$\begin{aligned}\max\{\widetilde{\lambda}\left(xyxx^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\} &= \max\{\widetilde{\lambda}\left(x^{-1}xyx\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\\ \max\{\widetilde{\lambda}(xy),\tfrac{\widetilde{1}-\widetilde{k}}{2}\} &= \max\{\widetilde{\lambda}(yx),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\end{aligned}$$

this imply $\max\{\widetilde{\lambda}(xy),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\geq\widetilde{\lambda}(yx)$. And also

$$\max\{\widetilde{\lambda}(yx),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\geq\widetilde{\lambda}(xy).$$

Using Theorem 1 we conclude that $\widetilde{\lambda}$ is an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup

Theorem 8. Let $\widetilde{\lambda}$ be an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup and the interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy coset of G determined by x and $\widetilde{\lambda}$ is denoted by $\overleftarrow{\widetilde{\lambda}_x}$. Let F be the set of all interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy cosets of $\widetilde{\lambda}$ in G. Then, F is a group if the multiplication is defined by $\overleftarrow{\widetilde{\lambda}_x}\cdot\overleftarrow{\widetilde{\lambda}_y}=\overleftarrow{\widetilde{\lambda}_{xy}}$ for all $x,y\in G$. Let $\widetilde{\mu}:F\to D[0,1]$ be defined by $\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_x}\right)=\widetilde{\lambda}(x)$ for all $x\in G$. Then, $\widetilde{\mu}$ is an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of $\widetilde{\lambda}$.

Proof. We first show that the given composition is well defined. Let $\overleftarrow{\lambda}_x = \overleftarrow{\lambda}_y$ and $\overleftarrow{\lambda}_z = \overleftarrow{\lambda}_w$. So for any $g \in G$,

$$\overleftarrow{\lambda}_x(g) = \overleftarrow{\lambda}_y(g) \text{ and } \overleftarrow{\lambda}_z(g) = \overleftarrow{\lambda}_w(g)$$

$\max\{\widetilde{\lambda}\left(gx^{-1}\right), \frac{\widetilde{1}-\widetilde{k}}{2}\} = \max\{\widetilde{\lambda}\left(gy^{-1}\right), \frac{\widetilde{1}-\widetilde{k}}{2}\}$ and
$\max\{\widetilde{\lambda}\left(gz^{-1}\right), \frac{\widetilde{1}-\widetilde{k}}{2}\} = \max\{\widetilde{\lambda}\left(gw^{-1}\right), \frac{\widetilde{1}-\widetilde{k}}{2}\}$.

Since $\widetilde{\lambda}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup, so therefore we get

$$\begin{aligned}
\overleftarrow{\lambda}_{xz}(g) &= \max\{\widetilde{\lambda}\left(g(xz)^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \max\{\widetilde{\lambda}\left(gz^{-1}x^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \max\{\widetilde{\lambda}\left(gz^{-1}y^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\lambda}\left(y^{-1}gz^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\lambda}\left(y^{-1}gz^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\lambda}\left(y^{-1}gw^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\lambda}\left(gw^{-1}y^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\lambda}\left(gw^{-1}y^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \widetilde{\lambda}\left(g(yw)^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} = \overleftarrow{\lambda}_{yw}(g)
\end{aligned}$$

So

$$\overleftarrow{\lambda}_{xz}(g) \leq \overleftarrow{\lambda}_{yw}(g) \text{ for all } g \in G.$$

Similarly, we get

$$\overleftarrow{\lambda}_{yw}(g) \leq \overleftarrow{\lambda}_{xz}(g) \text{ for all } g \in G.$$

So

$$\overleftarrow{\lambda}_{xz} = \overleftarrow{\lambda}_{yw}.$$

Hence the composition is well defined. It can be easily verified that F is a group with $\overleftarrow{\lambda}_e$ is the identity element and $\overleftarrow{\lambda}_{x^{-1}}$ is the inverse element of $\overleftarrow{\lambda}_x$ for all $x \in G$. Now

$$\begin{aligned}
\max\{\widetilde{\mu}\left(\overleftarrow{\lambda}_x \cdot \overleftarrow{\lambda}_y\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} &= \max\{\widetilde{\mu}\left(\overleftarrow{\lambda}_x \cdot \overleftarrow{\lambda}_y\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \max\{\widetilde{\mu}\left(\overleftarrow{\lambda}_{xy}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \max\{\widetilde{\lambda}(xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\} \\
&= \min\{\widetilde{\mu}\left(\overleftarrow{\lambda}_x\right), \widetilde{\mu}\left(\overleftarrow{\lambda}_y\right)\}
\end{aligned}$$

$$\max\{\widetilde{\widetilde{\lambda}}\left(\overleftarrow{\lambda}_x \cdot \overleftarrow{\lambda}_y\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \geq \min\{\widetilde{\mu}\left(\overleftarrow{\lambda}_x\right), \widetilde{\mu}\left(\overleftarrow{\lambda}_y\right)\}$$

for all $\overleftarrow{\lambda}_x, \overleftarrow{\lambda}_y \in F$.

$$\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_x}\right)=\widetilde{\lambda}(x)\leq\max\{\widetilde{\lambda}\left(x^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$
$$=\max\{\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_{x^{-1}}}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$
$$\leq\max\{\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_x}^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}.$$

Thus, $\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_x}\right)\leq\max\{\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_x}^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$.

Also

$$\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_a}\right)=\widetilde{\lambda}(a)\leq\max\{\widetilde{\lambda}\left(x^{-1}ax\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$
$$=\max\{\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_{x^{-1}ax}}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$

Thus $\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_a}\right)\leq\max\{\widetilde{\mu}\left(\overleftarrow{\widetilde{\lambda}_{x^{-1}}}\cdot\overleftarrow{\widetilde{\lambda}_a}\cdot\overleftarrow{\widetilde{\lambda}_x}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$

for all $\overleftarrow{\widetilde{\lambda}_x},\overleftarrow{\widetilde{\lambda}_a}\in F$. Therefore, $\widetilde{\mu}$ is an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of F.

4.1 Theorem

For any interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of F, there exists an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G.

Proof. Let $\widetilde{\mu}$ be an interval valued $\left(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup of F and define a interval valued fuzzy subset $\widetilde{\beta}$ of G by $\widetilde{\beta}(x)=\widetilde{\mu}\left(\widetilde{\lambda}_x\right)$ for all $x\in G$. For any $x,y\in G$, we have

$$\min\{\widetilde{\beta}(x),\widetilde{\beta}(y)\}=\min\{\widetilde{\mu}\left(\widetilde{\lambda}_x\right),\widetilde{\mu}\left(\widetilde{\lambda}_y\right)\}$$
$$\leq\max\{\widetilde{\mu}\left(\widetilde{\lambda}_x\cdot\widetilde{\lambda}_y\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$
$$\leq\max\{\widetilde{\mu}\left(\widetilde{\lambda}_{xy}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$
$$=\max\{\widetilde{\beta}(xy),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$

Thus $\max\{\widetilde{\beta}(xy),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}\geq\min\{\widetilde{\beta}(x),\widetilde{\beta}(y)\}$. And

$$\widetilde{\beta}(x)=\widetilde{\mu}\left(\widetilde{\lambda}_x\right)\leq\max\{\widetilde{\mu}\left(\widetilde{\lambda}_x^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$
$$=\max\{\widetilde{\mu}\left(\widetilde{\lambda}_{x^{-1}}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$
$$\leq\max\{\widetilde{\beta}\left(x^{-1}\right),\tfrac{\widetilde{1}-\widetilde{k}}{2}\}$$

Thus $\max\{\widetilde{\beta}\left(x^{-1}\right), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\beta}(x)$. So $\widetilde{\beta}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy subgroup of G. Next assume that $\widetilde{\mu}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal of F. Then,

$$
\begin{aligned}
B(xy) &= \widetilde{\mu}\left(\widetilde{\lambda}_{xy}\right) = \widetilde{\mu}\left(\widetilde{\lambda}_x \cdot \widetilde{\lambda}_y\right) \\
&\leq \max\{\widetilde{\mu}\left(\widetilde{\lambda}_y \cdot \widetilde{\lambda}_x\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\mu}\left(\widetilde{\lambda}_{yx}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \max\{B(yx), \tfrac{\widetilde{1}-\widetilde{k}}{2}\}.
\end{aligned}
$$

Thus $\max\{\widetilde{\beta}(yx), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\beta}(xy)$. It follows that $\widetilde{\beta}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

Definition 13. [8] Let $\widetilde{\lambda}$ be an interval valued fuzzy subgroup of G and ρ a map from G into itself. The interval valued fuzzy subset $\widetilde{\lambda}^{\rho}$ in G, $\widetilde{\lambda}^{\rho} : G \to D[0,1]$ is defined by $\widetilde{\lambda}^{\rho}(x) = \widetilde{\lambda}(\rho(x))$, $\forall x \in G$.

Definition 14. An interval valued fuzzy subgroup $\widetilde{\lambda}$ of G is said to be a interval valued fuzzy characteristic subgroup of G if for all $x \in G$ and for all automorphism ρ of G,

$$
x_{\widetilde{t}} \overline{\in} \widetilde{\lambda}^{\rho} \Rightarrow x_{\widetilde{t}} \overline{\in} \vee \overline{q_{\widetilde{k}}} \widetilde{\lambda}. \tag{1}
$$

Remark. The condition 1 is equivalent to $\max\{\widetilde{\lambda}(\rho(x)), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(x)$.

Theorem 9. Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G. If ρ is an endomorphism of G, then $\widetilde{\lambda}^{\rho}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

Proof. Let $x, y \in G$. Then, we have

$$
\begin{aligned}
\min\{\widetilde{\lambda}^{\rho}(x), \widetilde{\lambda}^{\rho}(y)\} &= \min\{\widetilde{\lambda}(\rho(x)), \widetilde{\lambda}(\rho(y))\} \\
&\leq \max\{\widetilde{\lambda}(\rho(x) \cdot \rho(y)), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\lambda}(\rho(xy)), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\quad (\text{since } \rho \text{ is endomorphism}) \\
&\leq \max\{\widetilde{\lambda}^{\rho}(xy), \tfrac{\widetilde{1}-\widetilde{k}}{2}\}.
\end{aligned}
$$

So $\min\{\widetilde{\lambda}^{\rho}(x), \widetilde{\lambda}^{\rho}(y)\} \leq \max\{\widetilde{\lambda}^{\rho}(xy), \frac{\widetilde{1}-\widetilde{k}}{2}\}$ for all $x, y \in G$.

Let $x \in G$. Then, $x^{-1} \in G$, we have

$$
\begin{aligned}
\widetilde{\lambda}^{\rho}(x) = \widetilde{\lambda}(\rho(x)) &\leq \max\{\widetilde{\lambda}\left((\rho(x))^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&\leq \max\{\widetilde{\lambda}\left(\rho\left(x^{-1}\right)\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\
&= \max\{\widetilde{\lambda}^{\rho}\left(x^{-1}\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\}. \\
&\quad (\text{since } \rho \text{ is endomorphism})
\end{aligned}
$$

Thus, $\widetilde{\lambda}^{\rho}(x) \leq \max\{\widetilde{\lambda}^{\rho}\left(x^{-1}\right), \frac{\widetilde{1}-\widetilde{k}}{2}\}$ for all $x \in G$.

Since $\rho(x) \in G$ for any $x \in G$, so we have

$$\begin{aligned}\widetilde{\lambda}^{\rho}(x) &= \widetilde{\lambda}(\rho(x)) \leq \max\{\widetilde{\lambda}\left([\rho(y)]^{-1}\rho(x)\rho(y)\right), \\ &\quad \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &\leq \max\{\widetilde{\lambda}\left(\rho\left(y^{-1}\right)\rho(x)\rho(y)\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &\leq \max\{\widetilde{\lambda}\left(\rho\left(y^{-1}xy\right)\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &\leq \max\{\widetilde{\lambda}^{\rho}\left(y^{-1}xy\right), \tfrac{\widetilde{1}-\widetilde{k}}{2}\}.\end{aligned}$$

So, $\widetilde{\lambda}^{\rho}(x) \leq \max\{\widetilde{\lambda}^{\rho}\left(y^{-1}xy\right), \frac{\widetilde{1}-\widetilde{k}}{2}\}$ for every $y \in G$.

Hence $\widetilde{\lambda}^{\rho}$ is a interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G.

If we take $\widetilde{k} = \widetilde{0}$, then in view of above theorem, we have the following corollary.

Corollary 4. Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy normal subgroup of G. If ρ is a endomorphism of G, then $\widetilde{\lambda}^{\rho}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy normal subgroup of G.

Proof. Similar as in previous theorem.

Theorem 10. If $\widetilde{\lambda}$ is an interval valued fuzzy characteristic subgroup of G, then $\widetilde{\lambda}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G.

Proof. Let $\widetilde{\lambda}$ be an interval valued fuzzy characteristic subgroup of G. Let ρ be the automorphism of G defined by $\rho(g) = x^{-1}gx, \forall g \in G$ that is

$$\rho(xg) = x^{-1}(xg)x = \left(x^{-1}x\right)gx = gx \tag{2}$$

Let $x, y \in G$. Then,

$$\begin{aligned}\widetilde{\lambda}(xy) &\leq \max\{\widetilde{\lambda}(\rho(xy)), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \\ &= \max\{\widetilde{\lambda}(yx), \tfrac{\widetilde{1}-\widetilde{k}}{2}\} \text{ (by using 2)}\end{aligned}$$

Thus, $\widetilde{\lambda}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G.

If we take $\widetilde{k} = \widetilde{0}$, then in view of above theorem.

Corollary 5. If $\widetilde{\lambda}$ is an interval valued fuzzy characteristic subgroup of G, then $\widetilde{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy normal subgroup of G.

Proof. Similar as in previous Theorem.

Definition 15. For an interval valued fuzzy subgroup $\widetilde{\lambda}$ of G the normalizer of $\widetilde{\lambda}$, denoted by $N\left(\widetilde{\lambda}\right)$ and is defined by $N\left(\widetilde{\lambda}\right) = \{y \in G; \max\{\widetilde{\lambda}(y^{-1}xy), \frac{\widetilde{1}-\widetilde{k}}{2}\} \geq \widetilde{\lambda}(x) : \forall\, x \in G\}$.

Theorem 11. Let $\widetilde{\lambda}$ be an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy subgroup of G. Then, (i) $N\left(\widetilde{\lambda}\right)$ is a subgroup of G, (ii) $\widetilde{\lambda}$ is an interval valued $\left(\overline{\in}, \overline{\in} \vee \overline{q_{\widetilde{k}}}\right)$-fuzzy normal subgroup of G if and only if $N\left(\widetilde{\lambda}\right) = G$.

Proof. Let $g,h\in N\left(\widetilde{\lambda}\right)$ and $x\in G$. Then, we have

$$\begin{aligned}\widetilde{\lambda}(x) &\leq \max\{\widetilde{\lambda}(g^{-1}xg),\tfrac{\widetilde{1-k}}{2}\}\\ &\leq \max\{\max\left\{\begin{array}{c}\widetilde{\lambda}(h^{-1}(g^{-1}xg)h),\\ \frac{\widetilde{1-k}}{2}\end{array}\right\},\tfrac{\widetilde{1-k}}{2}\}\\ &\leq \max\{\widetilde{\lambda}(h^{-1g^{-1}}xgh),\tfrac{\widetilde{1-k}}{2}\}\\ &= \max\{\widetilde{\lambda}((gh)^{-1}xgh),\tfrac{\widetilde{1-k}}{2}\}\end{aligned}$$

Thus, $\max\{\widetilde{\lambda}((gh)^{-1}xgh),\frac{\widetilde{1-k}}{2}\}\geq\widetilde{\lambda}(x)$. Hence, $gh\in N\left(\widetilde{\lambda}\right)$. Also if $g\in N\left(\widetilde{\lambda}\right)$, then obviously $g^{-1}\in N\left(\widetilde{\lambda}\right)$. Therefore, $N\left(\widetilde{\lambda}\right)$ is a subgroup of G.

Now for (ii) Let $\widetilde{\lambda}$ be an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}})$fuzzy normal. Let $y\in G$. Then, $\forall\ x\in G$ we have

$$\widetilde{\lambda}(x)\leq\max\{\widetilde{\lambda}(y^{-1}xy),\tfrac{\widetilde{1-k}}{2}\}$$

So $y\in N\left(\widetilde{\lambda}\right)$. Therefore $G\subseteq N\left(\widetilde{\lambda}\right)$ implies $N\left(\widetilde{\lambda}\right)=G$. Conversely, if $N\left(\widetilde{\lambda}\right)=G$, then from the definition of $N\left(\widetilde{\lambda}\right)$, $\widetilde{\lambda}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q_{\widetilde{k}}})$-fuzzy normal subgroup of G.

5 Acknowledgment

The authors would like to show his sincere thanks to the Pakistan Science Foundation, Islamabad Pakistan for their support. This research work supported by Pakistan Science Foundation grant No PSF/P andD/TG-II(1343)/14.

References

[1] S. Abdullah, A new type of interval valued fuzzy subgroups of groups, Submitted.

[2] S. Abdullah, M.Aslam, T. Ahmad and M. Naeem, A new typ fuzzy normal subgroups and fuzzy cosets, Journal of Intelligent and Fuzzy Systems, 2012, 10.3233/IFS-2012-0612.

[3] M. Aslam, S. Abdullah and N. Amin, *Characterizations of gamma LA-semigroups by generalized fuzzy gamma ideals*, International J. of Mathematics and Statistics, 12 (2012): 29-50.

[4] M. Akram, K.H. Dar, , K.P. Shum, Interval-valued (α,β)-fuzzy K-algebras, Applied Soft Computing, 11, 1, (2011), 1213-1222.

[5] S. K. Bhakat, $(\in,\in\vee q)$-*Level subset*, Fuzzy Sets and System 103 (1999) 529-533.

[6] S. K. Bhakat, $(\in,\in\vee q)$-*fuzzy normal, quasinormal and maximal subgroups*, Fuzzy Sets and System 112 (2000) 299-312.

[7] S. K. Bhakat, P. Das, $(\in,\in\vee q)$-*fuzzy subgroup*, Fuzzy Sets and System 80 (1996) 359-368.

[8] S. K. Bhakat, P. Das, *On the definition of a fuzzy subgroup*, Fuzzy Sets and System 51 (1992) 235-241.

[9] R. Biswas, Rosenfeld's fuzzy subgroups with interval-valued membership functions, Fuzzy Sets and Systems, 63, 1, (1994), 87-90.

[10] P. S. Das, *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. 85 (1981) 264-269.

[11] B. Davvaz, Interval-valued fuzzy subhypergroups, Korean Journal of Computational and Applied Mathematics, 6, 1, (1999), 197-202.

[12] Y. B. Jun, *Generalizations of* $(\in,\in\vee q)$-*fuzzy subalgebras in BCK/BCI-algebras*, Comput. Math. Appl. , 58 (2009): 1383 1390

[13] Y. B. Jun, M. S. Kang, and C. H. Park, *Fuzzy subgroups based on fuzzy points*, Commun. Korean Math. Soc. 26 (2011), No. 3, pp. 349-371.

[14] K.B. Latha, D.R. P. Williams and E.Chandrasekar, Interval valued (α,β)-fuzzy subgroups, Mamthematica, Tome 52, (75), 2, (2010), 177-184

[15] W.-J. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets and Systems 8 (1982), no. 2, 133–139.

[16] Mukherjee and P. Bhattacharya, *Fuzzy normal subgroups and fuzzy cosets*, Inform. Sci. 34 (1984), no. 3, 225–239.

[17] V. Murali, *Fuzzy points of equivalent fuzzy subsets*, Inform. Sci. 158 (2004) 277-288.

[18] P. M. Pu, Y.M. Liu, *Fuzzy topology I, neighbourhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. 76 (1980) 571-599.

[19] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. 35 (1971) 512-517.

[20] J. S. Rose, A Course on Group Theory, Cambridge University Press, 1978.

[21] M. Shabir, Y.B. Jun, Y. Nawaz, *Characterizations of regular semigroups by (α,β)-fuzzy ideals*, Comput. Math. Appl. , 59 (2010) 161-175.

[22] M. Shabir, Y.B. Jun and Y. Nawaz, *Semigroups characterized by $(\in,\in\vee q_k)$-fuzzy ideals*, Comput. Math. Appl. , 60 (2010) 1473-1493.

[23] X. Yuan, C. Zhang, and Y. Ren, *Generalized fuzzy groups and many-valued implications*, Fuzzy Sets and Systems 138 (2003), no. 1, 205–211.

[24] L.A. Zadeh, *Fuzzy sets*, Inform. and Control 8 (1965) 338-353.

[25] L.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Information Sciences, 8, (1975), 199-249.

[26] J. Zhan, B. Davvaz and K.P. Shum, A new view of fuzzy hypernear-rings, Information Sciences, 178, (2008), 425-438.

[27] J. Zhan, B. Davvaz and K.P. Shum, A new view of fuzzy hyperquasigroups, Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology, 20(4, 5), 147-157.