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Unrestricted Gibonacci Hybrid Numbers

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Abstract. Gibonacci sequence is a generalization of Fibonacci sequence and hybrid numbers are a mixture of reals, complex, dual and hyperbolic numbers. In this study, we introduce unrestricted Gibonacci hybrid numbers. After giving the definition of unrestricted gibonacci hybrid numbers, we give Binet–like formula for this kind of numbers. We also obtain some identities for unrestricted Gibonacci hybrid numbers.

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1. Introduction

Fibonacci and Lucas sequences may be the most famous sequences among integer sequences. Fibonacci numbers satisfy

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. Lucas numbers satisfy the same recurrence relations, namely

$$L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2$$

except the initial conditions $L_0 = 2$ and $L_1 = 1$. Generating functions for the Fibonacci sequences F_n and L_n are

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \text{ and } \sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}$$

respectively. Binet formulas for the Fibonacci and Lucas numbers are, respectively

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. The positive root α is the famous "golden ratio". There are many generalizations for Fibonacci and Lucas sequences. One of them is the gibonacci sequence. Gibonacci sequence is defined with the recurrence relation

$$G_n = G_{n-1} + G_{n-2} \text{ for } n \geq 3 \tag{1.1}$$

with the initial conditions $G_1 = a$ and $G_2 = b$ where a and b are any real numbers. Koshy [3] gave the following connection between Fibonacci and gibonacci numbers

$$G_n = aF_{n-2} + bF_{n-1} \tag{1.2}$$

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and the Binet – like formula for the gibonacci numbers

$$G_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta} \tag{1.3}$$

where $c = a + (a - b)\beta$ and $d = a + (a - b)\alpha$. The detailed information on gibonacci numbers can be found in [3].

Hybrid numbers are introduced by Ozdemir [4]. These numbers are a mixture of reals, complex, dual and hyperbolic numbers. The set of all hybrid numbers is

$$\mathbb{K} = \left\{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\right\}.$$

Multiplication rule in $\mathbb{K}$ can be defined by using the multiplications in the basis $\{1, i, \varepsilon, h\}$ given in the following table.

Table 1. Multiplication rules of the elements of $\{1, i, \varepsilon, h\}$

$\cdot$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	$h-1$	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

Let $k_1 = a_1 + b_1 i + c_1\varepsilon + d_1 h$ and $k_2 = a_2 + b_2 i + c_2\varepsilon + d_2 h$ are two any hybrid numbers. Then by using Table 1, we obtain product of k_1 and k_2 as follows

$$\begin{aligned} k_1k_2 = a_1a_2 - b_1b_2 + b_1c_2 + b_2c_1 + i(a_1b_2 + a_2b_1 + b_1d_2 - b_2d_1) + \varepsilon(a_1c_2 + a_2c_1 + b_1d_2 - b_2d_1 - c_1d_2 \\ + c_2d_1) + h(a_1d_2 + a_2d_1 - b_1c_2 + b_2c_1). \end{aligned}$$

Similarly the sum of k_1 and k_2 is

$$k_1 + k_2 = a_1 + a_2 + i(b_1 + b_2) + \varepsilon(c_1 + c_2) + h(d_1 + d_2).$$

The conjugate of a hybrid number $k = a + bi + c\varepsilon + dh$ is

$$\overline{k} = a - bi - c\varepsilon - h$$

and the norm of k is

$$N(k) = \sqrt{|k\overline{k}|} = \sqrt{|a^2 + (b - c)^2 - c^2 - d^2|}.$$

There are some studies on hybrid numbers whose coefficients are integers sequences. For example we can refer to Horadam hybrid numbers [5, 6], bi-periodic Horadam hybrid numbers [10], Jacobsthal and Jacobsthal–Lucas hybrid numbers [8], Pell and Pell–Lucas hybrid numbers [7], k–Pell hybrid numbers [1], Fibonacci and Lucas hybrid numbers [2], tribonacci and tribonacci–Lucas hybrid numbers [9]. In all of these studies, authors choose the coefficients of elements of the set $\{1, i, \varepsilon, h\}$ successively. We pick arbitrary Gibonacci numbers for the coefficients of elements of the set $\{1, i, \varepsilon, h\}$. For this reason, we call these hybrid numbers "unrestricted gibonacci hybrid numbers".

2. Definition and Binet–Like Formula

We define unrestricted gibonacci hybrid numbers as follows:

Definition 2.1. For any integers p, q, r and non–negative integer n, the nth unrestricted gibonacci hybrid number is

$$\mathrm{G}_n^{(p,q,r)} = G_n + G_{n+p}i + G_{n+q}\varepsilon + G_{n+r}h.$$

If we take $(a, b) \to (1, 1)$, the unrestricted gibonacci hybrid numbers reduce to the unrestricted Fibonacci hybrid numbers

$$\mathrm{F}_n^{(p,q,r)} = F_n + F_{n+p}i + F_{n+q}\varepsilon + F_{n+r}h.$$

If we take $(a, b) \to (1, 3)$, the unrestricted gibonacci hybrid numbers reduce to the unrestricted Lucas hybrid numbers

$$\mathrm{L}_n^{(p,q,r)} = L_n + L_{n+p}i + L_{n+q}\varepsilon + L_{n+r}h.$$

From Definition 2.1 and recurrence relation (1.1), we obtain the following recurrece relation easily

$$\mathrm{G}_n^{(p,q,r)} = \mathrm{G}_{n-1}^{(p,q,r)} + \mathrm{G}_{n-2}^{(p,q,r)} \quad \text{for } n \geq 3.$$

For any integer n, by using the well–known identity $F_{-n} = (-1)^{n+1} F_n$ and Eq.(1.2), we obtain

$$\mathrm{G}_{-n}^{(p,q,r)} = (-1)^n \left[-a\mathrm{F}_{n+2}^{(-p,-q,-r)} + b\mathrm{F}_{n+1}^{(-p,-q,-r)}\right].$$

Binet–like formula for the unrestricted gibonacci hybrid numbers is given in the following theorem.

Theorem 2.2. *For any integers p, q, r and m, the mth unrestricted gibonacci hybrid number is*

$$\mathrm{G}_m^{(p,q,r)} = \frac{c\overline{\alpha}\alpha^m - d\overline{\beta}\beta^m}{\alpha - \beta}$$

where $\overline{\alpha} = 1 + \alpha^p i + \alpha^q \varepsilon + \alpha^r h$ and $\overline{\beta} = 1 + \beta^p i + \beta^q \varepsilon + \beta^r h$.

Proof. From the definition of unrestricted gibonacci hybrid numbers and the Binet-like formula for the gibonacci numbers in Eq.(1.3), we have

$$\begin{aligned}
\mathrm{G}_m^{(p,q,r)} &= G_m + G_{m+p} i + G_{m+q}\varepsilon + G_{m+r} h \\
&= \frac{1}{\alpha - \beta}\left[c\alpha^m - d\beta^m + (c\alpha^{m+p} - d\beta^{m+p})i + (c\alpha^{m+q} - d\beta^{m+q})\varepsilon + (c\alpha^{m+r} - d\beta^{m+r})h\right] \\
&= \frac{1}{\alpha - \beta}\left[c\alpha^m(1 + \alpha^p i + \alpha^q \varepsilon + \alpha^r h) - d\beta^m(1 + \beta^p i + \beta^q \varepsilon + \beta^r h)\right].
\end{aligned}$$

The last equation gives the lemma. □

We need the following lemma together with the Binet-like formula given in the last theorem.

Lemma 2.3. *For any integers p, q and r, we have*

$$\overline{\alpha}\overline{\beta} = U + \sqrt{5}V$$

and

$$\overline{\beta}\overline{\alpha} = U - \sqrt{5}V$$

where

$$U = L_0^{(p,q,r)} - 1 - (-1)^p + (-1)^q L_{p-q} + (-1)^r$$

and

$$V = (-1)^r F_{p-r}(i + \varepsilon) + (-1)^q F_{r-q}\varepsilon + (-1)^p F_{q-p} h.$$

Proof. From the definitions of $\overline{\alpha}$ and $\overline{\beta}$, we obtain

$$\begin{aligned}
\overline{\alpha}\overline{\beta} &= (1 + \alpha^p i + \alpha^q \varepsilon + \alpha^r h)(1 + \beta^p i + \beta^q \varepsilon + \beta^r h) \\
&= 1 - (\alpha\beta)^p + \alpha^p\beta^q + \alpha^q\beta^p + (\alpha\beta)^r + (\alpha^p + \beta^p + \alpha^p\beta^r - \alpha^r\beta^p)i + (\alpha^q + \beta^q + \alpha^p\beta^r - \alpha^r\beta^p \\
&\qquad + \alpha^r\beta^q - \alpha^q\beta^r)\varepsilon + (\alpha^r + \beta^r + \alpha^q\beta^p - \alpha^p\beta^q)h \\
&= 1 - (-1)^p + (-1)^q L_{p-q} + (-1)^r + \left[L_p + (-1)^r(\alpha^{p-r} - \beta^{p-r})\right] i \\
&\qquad + \left[L_q + (-1)^r(\alpha^{p-r} - \beta^{p-r}) + (-1)^q(\alpha^{r-q} - \beta^{r-q})\right]\varepsilon + \left[L_r + (-1)^p(\alpha^{q-p} - \beta^{q-p})\right] h \\
&= L_0^{(p,q,r)} - 1 - (-1)^p + (-1)^q L_{p-q} + (-1)^r + \sqrt{5}\left[(-1)^r F_{p-r}(i + \varepsilon) + (-1)^q F_{r-q}\varepsilon + (-1)^p F_{q-p} h\right].
\end{aligned}$$

The last equation gives the first identity in lemma. The second identity can be obtained in a similar way. □

3. Some Rresults for the Unrestricted Gibonacci Hybrid Numbers

In this section, we give some generalizations for the well-known identities by using Theorem 2.2 and Lemma 2.3. We start with Vajda's identity.

Theorem 3.1. *For any integers p, q, r, x, y and z, we have*

$$G_{x+y}^{(p,q,r)} G_{x+z}^{(p,q,r)} - G_x^{(p,q,r)} G_{x+y+z}^{(p,q,r)} = (a^2 + ab - b^2)(-1)^x F_y \left[U F_z - V L_z\right]$$

where F_n and L_n are the classical nth Fibonacci and Lucas numbers.

Proof. From the Binet-like formula for the unrestricted gibonacci hybrid numbers in Theorem 2.2, we have

$$
\begin{aligned}
&G_{x+y}^{(p,q,r)}G_{x+z}^{(p,q,r)} - G_{x}^{(p,q,r)}G_{x+y+z}^{(p,q,r)}\\
&= \frac{1}{5}\Big[\Big(c\overline{\alpha}\alpha^{x+y} - d\overline{\beta}\beta^{x+y}\Big)\Big(c\overline{\alpha}\alpha^{x+z} - d\overline{\beta}\beta^{x+z}\Big) - \Big(c\overline{\alpha}\alpha^{x} - d\overline{\beta}\beta^{x}\Big)\Big(c\overline{\alpha}\alpha^{x+y+z} - d\overline{\beta}\beta^{x+y+z}\Big)\Big]\\
&= \frac{1}{5}\Big[-cd\overline{\alpha}\overline{\beta}\alpha^{x+y}\beta^{x+z} - cd\overline{\beta}\overline{\alpha}\alpha^{x+z}\beta^{x+y} + cd\overline{\alpha}\overline{\beta}\alpha^{x}\beta^{x+y+z} + cd\overline{\beta}\overline{\alpha}\alpha^{x+y+z}\beta^{x}\Big]\\
&= \frac{(-1)^x}{5}cd\Big[-\overline{\alpha}\overline{\beta}\alpha^{y}\beta^{z} - \overline{\beta}\overline{\alpha}\alpha^{z}\beta^{y} + \overline{\alpha}\overline{\beta}\beta^{y+z} + \overline{\beta}\overline{\alpha}\alpha^{y+z}\Big]\\
&= \frac{(-1)^x}{5}cd\Big[(U+\sqrt{5}V)\beta^{z}(\beta^{y}-\alpha^{y}) + (U-\sqrt{5}V)\alpha^{z}(\alpha^{y}-\beta^{y})\Big]\\
&= \frac{(-1)^x}{5}cd\Big[(\sqrt{5}U+V)\beta^{z}(-F_y) + (\sqrt{5}U-V)\alpha^{z}(F_y)\Big]\\
&= \frac{(-1)^x}{5}cd\Big[\sqrt{5}UF_y(\alpha^{z}-\beta^{z}) - VF_y(\alpha^{z}+\beta^{z})\Big].
\end{aligned}
$$

We obtain the Vajda's identity given in the theorem from the last equation. □

If we take $(a, b) \to (1, 1)$ and $(a, b) \to (1, 3)$, then we obtain Vajda's identity for unrestricted Fibonacci and unrestricted Lucas hybrid numbers

$$F_{x+y}^{(p,q,r)}F_{x+z}^{(p,q,r)} - F_{x}^{(p,q,r)}F_{x+y+z}^{(p,q,r)} = (-1)^x F_y\Big[UF_z - VL_z\Big]$$

and

$$L_{x+y}^{(p,q,r)}L_{x+z}^{(p,q,r)} - L_{x}^{(p,q,r)}L_{x+y+z}^{(p,q,r)} = 5(-1)^{x+1} F_y\Big[UF_z - VL_z\Big]$$

respectively. If we set $(p, q, r) = (1, 2, 3)$, we have $U = i + 3\varepsilon + 4h$ and $V = i + 2\varepsilon - h$. So the last two identities reduce to Vajda's identities for the Fibonacci and Lucas hybrid numbers

$$F_{x+y}^{(1,2,3)}F_{x+z}^{(1,2,3)} - F_{x}^{(1,2,3)}F_{x+y+z}^{(1,2,3)} = (-1)^x F_y\Big[(i+3\varepsilon+4h)F_z - (i+2\varepsilon-h)L_z\Big]$$

and

$$L_{x+y}^{(1,2,3)}L_{x+z}^{(1,2,3)} - L_{x}^{(1,2,3)}L_{x+y+z}^{(1,2,3)} = 5(-1)^{x+1} F_y\Big[(i+3\varepsilon+4h)F_z - (i+2\varepsilon-h)L_z\Big]$$

respectively. If we take $z = -y$ in Vajda's identity and use the well-known identitites $F_{-n} = (-1)^{n+1}F_n$, $L_{-n} = (-1)^n L_n$ and $F_nL_n = F_{2n}$, we obtain the Catalan's identity for the unrestricted gibonacci hybrid numbers given in the following theorem.

Corollary 3.2. *For any integers p, q, r, x and y, we have*

$$G_{x+y}^{(p,q,r)}G_{x-y}^{(p,q,r)} - \Big[G_{x}^{(p,q,r)}\Big]^2 = (a^2+ab-b^2)(-1)^{x+y+1}\Big[UF_y^2 - VF_{2y}\Big].$$

Setting $(a, b) \to (1, 1)$ and $(a, b) \to (1, 3)$ gives Catalan's identities for the unrestricted Fibonacci and Lucas hybrid numbers

$$F_{x+y}^{(p,q,r)}F_{x-y}^{(p,q,r)} - \Big[F_{x}^{(p,q,r)}\Big]^2 = (-1)^{x+y+1}\Big[UF_y^2 - VF_{2y}\Big]$$

and

$$L_{x+y}^{(p,q,r)}L_{x-y}^{(p,q,r)} - \Big[L_{x}^{(p,q,r)}\Big]^2 = 5(-1)^{x+y}\Big[UF_y^2 - VF_{2y}\Big]$$

respectively. Setting $(p, q, r) \to (1, 2, 3)$, we obtain Catalan's identities for the Fibonacci and Lucas hybrid numbers

$$F_{x+y}^{(1,2,3)}F_{x-y}^{(1,2,3)} - \Big[F_{x}^{(1,2,3)}\Big]^2 = (-1)^{x+y+1}\Big[(i+3\varepsilon+4h)F_y^2 - (i+2\varepsilon-h)F_{2y}\Big]$$

and

$$L_{x+y}^{(1,2,3)}L_{x-y}^{(1,2,3)} - \Big[L_{x}^{(1,2,3)}\Big]^2 = 5(-1)^{x+y}\Big[(i+3\varepsilon+4h)F_y^2 - (i+2\varepsilon-h)F_{2y}\Big]$$

respectively. If we set $y = 1$ in the Catalan's identity, we get Cassini's identity for the unrestricted gibonacci hybrid numbers given in the following theorem.

Corollary 3.3. *For any integers p, q, r and x, we have*

$$G_{x+1}^{(p,q,r)}G_{x-1}^{(p,q,r)} - \Big[G_{x}^{(p,q,r)}\Big]^2 = (a^2+ab-b^2)(-1)^{x}\Big[U - V\Big].$$

Similar to the Vajda's and Catalan's identities, Cassini's identities for the unrestricted Fibonacci and Lucas hybrid numbers are

$$F_{x+1}^{(p,q,r)}F_{x-1}^{(p,q,r)}-\left[F_x^{(p,q,r)}\right]^2=(-1)^x\left[U-V\right]$$

and

$$L_{x+1}^{(p,q,r)}L_{x-1}^{(p,q,r)}-\left[L_x^{(p,q,r)}\right]^2=5(-1)^{x+1}\left[U-V\right]$$

respectively. Cassini's identities for Fibonacci and Lucas hybrid numbers are

$$F_{x+1}^{(1,2,3)}F_{x-1}^{(1,2,3)}-\left[F_x^{(1,2,3)}\right]^2=(-1)^x\left[2i+5\varepsilon+3h\right]$$

and

$$L_{x+1}^{(1,2,3)}L_{x-1}^{(1,2,3)}-\left[L_x^{(1,2,3)}\right]^2=(-1)^{x+1}\left[10i+25\varepsilon+15h\right]$$

respectively. The d'Ocagne's identity for the unrestricted gibonacci hybrid numbers is another important identity which is given in the next theorem.

Theorem 3.4. *For any integers p,q,r,x and y, we have*

$$G_x^{(p,q,r)}G_{y+1}^{(p,q,r)}-G_{x+1}^{(p,q,r)}G_y^{(p,q,r)}=(a^2+ab-b^2)(-1)^y\left[UF_{x-y}+VL_{x-y}\right].$$

Proof. From the Binet-like formula for the unrestricted gibonacci hybrid numbers in Theorem 2.2, we have

$$\begin{aligned}
&G_x^{(p,q,r)}G_{y+1}^{(p,q,r)}-G_{x+1}^{(p,q,r)}G_y^{(p,q,r)}\\
&=\frac{1}{5}\Big[\left(c\overline{\alpha}\alpha^x-d\overline{\beta}\beta^x\right)\left(c\overline{\alpha}\alpha^{y+1}-d\overline{\beta}\beta^{y+1}\right)-\left(c\overline{\alpha}\alpha^{x+1}-d\overline{\beta}\beta^{x+1}\right)\left(c\overline{\alpha}\alpha^y-d\overline{\beta}\beta^y\right)\Big]\\
&=\frac{cd}{5}\Big[-\overline{\alpha}\overline{\beta}\alpha^x\beta^{y+1}-\overline{\beta}\overline{\alpha}\alpha^{x+1}\beta^y+\overline{\alpha}\overline{\beta}\alpha^{x+1}\beta^y+\overline{\beta}\overline{\alpha}\alpha^x\beta^{y+1}\Big]\\
&=\frac{(-1)^y}{5}cd\Big[(U+\sqrt{5}V)\alpha^{x-y}(\alpha-\beta)-(U-\sqrt{5}V)\beta^{x-y}(\alpha-\beta)\Big]\\
&=\frac{(-1)^y}{5}cd\Big[\sqrt{5}U(\alpha^{x-y}-\beta^{x-y})-5V(\alpha^{x-y}+\beta^{x-y})\Big]\\
&=\frac{(-1)^y}{5}cd\Big[UF_{x-y}+VL_{x-y}\Big].
\end{aligned}$$

The last equation ends the proof. □

We can obtain d'Ocagne's identities for the unrestricted Fibonacci and Lucas hybrid numbers similar to the identities mentioned above. d'Ocagne's identities for the unrestricted Fibonacci hybrid numbers is

$$F_x^{(p,q,r)}F_{y+1}^{(p,q,r)}-F_{x+1}^{(p,q,r)}F_y^{(p,q,r)}=(-1)^y\left[UF_{x-y}+VL_{x-y}\right]$$

and d'Ocagne's identities for the unrestricted Lucas hybrid numbers is

$$L_x^{(p,q,r)}L_{y+1}^{(p,q,r)}-L_{x+1}^{(p,q,r)}L_y^{(p,q,r)}=5(-1)^{y+1}\left[UF_{x-y}+VL_{x-y}\right].$$

d'Ocagne's identities for the Fibonacci and Lucas hybrid numbers are

$$F_x^{(1,2,3)}F_{y+1}^{(1,2,3)}-F_{x+1}^{(1,2,3)}F_y^{(1,2,3)}=(-1)^y\left[(i+3\varepsilon+4h)F_{x-y}+(i+2\varepsilon-h)L_{x-y}\right]$$

and

$$L_x^{(1,2,3)}L_{y+1}^{(1,2,3)}-L_{x+1}^{(1,2,3)}L_y^{(1,2,3)}=5(-1)^{y+1}\left[(i+3\varepsilon+4h)F_{x-y}+(i+2\varepsilon-h)L_{x-y}\right]$$

respectively. We give an interesting result for the unrestricted gibonacci hybrid numbers in the next theorem.

Theorem 3.5. *For any integers p,q,r,x and y, we have*

$$G_x^{(p,q,r)}G_y^{(p,q,r)}-G_y^{(p,q,r)}G_x^{(p,q,r)}=2(-1)^x(a^2+ab-b^2)VF_{y-x}.$$

Proof. From the Binet-like formula for the unrestricted gibonacci hybrid numbers in Theorem 2.2, we obtain

$$\begin{aligned}
& G_x^{(p,q,r)} G_y^{(p,q,r)} - G_y^{(p,q,r)} G_x^{(p,q,r)} \\
&= \frac{1}{5}\Big[\Big(c\overline{\alpha}\alpha^x - d\overline{\beta}\beta^x\Big)\Big(c\overline{\alpha}\alpha^y - d\overline{\beta}\beta^y\Big) - \Big(c\overline{\alpha}\alpha^y - d\overline{\beta}\beta^y\Big)\Big(c\overline{\alpha}\alpha^x - d\overline{\beta}\beta^x\Big) \Big] \\
&= \frac{cd}{5}\Big[-\overline{\alpha}\overline{\beta}\alpha^x\beta^y - \overline{\beta}\overline{\alpha}\alpha^y\beta^x + \overline{\alpha}\overline{\beta}\alpha^y\beta^x + \overline{\beta}\overline{\alpha}\alpha^x\beta^y \Big] \\
&= \frac{cd}{5}\Big[(-\overline{\alpha}\overline{\beta} + \overline{\beta}\overline{\alpha})\alpha^x\beta^y + (\overline{\alpha}\overline{\beta} - \overline{\beta}\overline{\alpha})\alpha^y\beta^x \Big] \\
&= \frac{2\sqrt{5}}{5} cdV\Big[-\alpha^x\beta^y + \alpha^y\beta^x \Big] \\
&= \frac{2(-1)^x}{\sqrt{5}} cdV\Big[\alpha^{y-x} - \beta^{y-x} \Big].
\end{aligned}$$

The last equation proves the theorem. □

4. Conclusion

Fibonacci sequence can be the most famous sequences among integer sequence because of its splendid properties. There are two ways to generalize Fibonacci sequence. The first is to keep initial conditions by changing the recursive equation while the second is to change initial conditions by keeping the recursive equation. Gibonacci sequence is an example of the second one. There is a huge interest in hyper-complex numbers whose coefficients are elemets of an integer sequence in these days. The starting point could be the Fibonacci and Lucas quaternions. Following these quaternions many hyper complex numbers have been defined and studied. We examine unrestricted gibonacci hybrid numbers in the current study. Hybrid numbers are a mixture of reals, complex, dual and hyperbolic numbers. The reason of use the word "unrestricted" is that coefficients of an unrestricted gibonacci numbers can be selected randomly. Binet formula, Vajda's identity, Catalan's identity, Cassini's identity and some other identities for the unrestricted gibonacci numbers can be found in the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

References

[1] Catarino, P., *On k-Pell hybrid numbers*, Journal of Discrete Mathematical Sciences and Cryptography, **22(1)** (2019), 83–89.

[2] Kizilates, C., *A new generalization of Fibonacci hybrid and Lucas hybrid numbers*, Chaos, Solitons and Fractal, **130** (2020), 109449.

[3] Koshy, T., Fibonacci and Lucas numbers with applications, John Wiley and Sons, New Jersey, 2018.

[4] Ozdemir, M., *Introduction to hybrid numbers*, Advances in Applied Clifford Algebras, **28(1)** (2018), 11–43.

[5] Senturk, T. D., Bilgici, G., Dasdemir, A., Unal, Z., *A study on Horadam hybrid numbers.*, Turkish Journal of Mathematics, **44(4)** (2020), 1212–1221.

[6] Szynal–Liana, A., *The Horadam hybrid numbers*, Discussiones Mathematicae-General Algebra and Applications, **38(1)** (2018), 91–98.

[7] Szynal–Liana, A., Wloch, I., *On Pell and Pell-Lucas hybrid numbers*, Comment. Math. Prace Mat., **58** (2018), 11-17.

[8] Szynal–Liana, A., Wloch, I., *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Annales Mathematicae Silesianae, **33(1)** (2019), 276-283.

[9] Tasyurdu, Y., *Tribonacci and tribonacci-Lucas hybrid numbers*, International Journal of Contemporary Mathematical Sciences, **14(4)** (2019), 245-254.

[10] Tan, E., Ait-Amrane, N.R. *On a new generalization of Fibonacci hybrid numbers*, arXiv:2006.09727.