

# Analytical solutions of Kolmogorov–Petrovskii–Piskunov equation

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## Abstract

In the current study, analytical solutions are constructed by applying  $(1/G')$ -expansion method to the Kolmogorov–Petrovskii–Piskunov (KPP) equation. Hyperbolic type exact solutions of the KPP equation are presented with the successfully applied method. 3D, 2D and contour graphics are presented by giving special values to the parameters in the solutions obtained. This article explores the applicability and effectiveness of this method on nonlinear evolution equations (NLEEs).

**Keywords:** Kolmogorov–Petrovskii–Piskunov equation,  $(1/G')$ -expansion method, traveling wave solutions, exact solution.

## Kolmogorov – Petrovskii – Piskunov denkleminin analitik çözümleri

### Öz

Bu çalışmada, Kolmogorov – Petrovskii – Piskunov (KPP) denkleminin analitik çözümleri  $(1/G')$ -açılım yöntemi uygulanarak elde edilmiştir. Başarılı bir şekilde uygulanan yöntem ile KPP denkleminin hiperbolik tipte tam çözümleri sunulmuştur. Elde edilen çözümlerdeki parametrelere özel değerler verilerek 3 boyutlu, 2 boyutlu ve kontur grafikleri sunulmuştur. Bu makalede, bu yöntemin doğrusal olmayan evrim denklemleri (NLEE'ler) üzerindeki uygulanabilirliği ve etkinliği araştırılmaktadır.

**Anahtar kelimeler:** Kolmogorov–Petrovskii–Piskunov denklemi,  $(1/G')$ -açılım yöntemi, yürüyen dalga çözümleri, tam çözüm.

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## 1. Introduction

To search the exact solutions of NLEEs has been hot and an important topic in mathematics physics for long years. There are various methods for obtain exact solutions of NLEEs such as sumudu transform method [1], the homotopy perturbation method [2], the Multistage Variational Iteration Method [3],  $(G'/G)$ -expansion method [4,5], extended sinh-Gordon equation expansion method [6,7], sub equation method [8],  $(1/G')$ -expansion method [9-11], the Clarkson–Kruskal (CK) direct method [12], the modified Kudryashov method [13], adomian decomposition methods [14-16],  $(G'/G, 1/G)$ -expansion method [17], first integral method [18], collocation method [19], new sub equation method [20], residual power series method [21], homogeneous balance method [22] and so on [28-37].

Consider the KPP equation [23]

$$u_t - u_{xx} + \beta u + \nu u^2 + \delta u^3 = 0, \quad (1)$$

where  $\nu$ ,  $\beta$  and  $\delta$  are real numbers.

This equation has significant place in physics, it also includes the Fisher, Burgers-Huxley, Huxley, Fitzhugh-Nagumo and Chaffee–Infante equations. Studies have been conducted by many scientists with KPP equation. Some of studies are as follows: analytical solutions of KPP equation obtained using modified simple equation method [24], in order to obtain the solutions of KPP equation obtained, HAM was applied [25], the existence and uniqueness of the solutions of the KPP equation studied [26], new exact solutions of the KPP equation are attained using first integral method [27].

In this work, we consider the KPP equation. We have been attained exact solutions for KPP equation using  $(1/G')$ -expansion method.

## 2. $(1/G')$ -expansion method

$(1/G')$ -expansion method was first presented as a Phd. thesis by Yokus, the author of this article, in 2011, [38]. This method is inspired by the  $(G'/G)$ -expansion method. The  $(G'/G, 1/G)$ -expansion method was brought to literature by another researcher, inspired by the  $(1/G')$  and  $(G'/G)$ -expansion methods. Recently, we see that a lot of work has been done with the  $(1/G')$ -expansion method [9,10,11,39,40].

We get general form of NLEEs

$$\sigma \left( u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots \right) = 0. \quad (2)$$

Here, let  $u = u(x, t) = U(\xi)$ ,  $\xi = x - wt$ ,  $w \neq 0$ , where  $w$  is a constant and the speed of the wave. After, we can be converted into following nonlinear ODE for  $U(\xi)$ :

$$\rho(U, U', U'', U''', \dots) = 0. \tag{3}$$

The solution of Eq. (4) is assumed to have the form

$$U(\xi) = a_0 + \sum_{i=1}^n a_i \left(\frac{1}{G'}\right)^i, \tag{4}$$

where  $a_i$ , ( $i = 0, 1, \dots, n$ ) are constants,  $n$  is an integer that we will calculate with the balancing principle and  $G = G(\xi)$  provides the following second order IODE

$$G'' + \lambda G' + \mu = 0, \tag{5}$$

where  $\lambda$  and  $\mu$  are constants to be determined after,

$$\frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + A \cosh(\xi\lambda) - A \sinh(\xi\lambda)}, \tag{6}$$

where  $A$  is integral constant. If the desired derivatives of the Eq. (4) are calculated and substituting in the Eq. (3), a polynomial with the argument  $(1/G')$  is attained. An algebraic equation system is created by equalizing the coefficients of this polynomial to zero. The equation is solved using package program and put into place in the default Eq. (3) solution function. Lastly, the solutions of Eq. (1) are found.

### 3. Solutions of KPP equation

The traveling wave transmutation  $\xi = x - wt$ , allows us to convert Eq. (1) into an ODE for  $u = U(\xi)$ ,

$$-U'' - wU' + \beta U + \nu U^2 + \delta U^3 = 0. \tag{7}$$

Here we consider the highest order linear term in the Eq. (7) and the nonlinear term. These terms are  $U''$  and  $U^3$ . Here, when the derivative is taken twice in Eq. (4), it is written as a polynomial bound to  $\frac{1}{G'}$ , and the degree of this polynomial becomes  $n + 2$ . Similarly, when the third power of the Eq. (4) is taken, it is written as a polynomial connected to  $\frac{1}{G'}$ , and the degree of this polynomial is  $3n$ . When these degrees are equalized according to the homogeneous balance principle  $n = 1$  is obtained and the following situation is presented,

$$U(\xi) = a_0 + a_1 \left(\frac{1}{G'}\right), \quad a_1 \neq 0. \tag{8}$$

Replacing Eq. (8) into Eq. (7) and the coefficients of the algebraic Eq. (1) are equal to zero, can attain the following algebraic equation systems

$$\begin{aligned}
 \text{Const} & : \beta a_0 + \nu a_0^2 + \delta a_0^3 = 0, \\
 \left(\frac{1}{G'[\xi]}\right)^1 & : \beta a_1 - w\lambda a_1 - \lambda^2 a_1 + 2\nu a_0 a_1 + 3\delta a_0^2 a_1 = 0, \\
 \left(\frac{1}{G'[\xi]}\right)^2 & : -w\mu a_1 - 3\lambda\mu a_1 + \nu a_1^2 + 3\delta a_0 a_1^2 = 0, \\
 \left(\frac{1}{G'[\xi]}\right)^3 & : -2\mu^2 a_1 + \delta a_1^3 = 0.
 \end{aligned} \tag{9}$$

**Case1.**

$$w = \frac{-\beta + \lambda^2}{\lambda}, \quad a_0 = \frac{\sqrt{2}\lambda}{\sqrt{\delta}}, \quad a_1 = \frac{\frac{\sqrt{2}\beta\mu}{\sqrt{\delta}} - \frac{4\sqrt{2}\lambda^2\mu}{\sqrt{\delta}}}{\beta - 4\lambda^2}, \quad \nu = \frac{-\sqrt{2}\beta\sqrt{\delta} - 2\sqrt{2}\sqrt{\delta}\lambda^2}{2\lambda}, \tag{10}$$

replacing values Eq. (10) into Eq. (8) and we have the following exact solutions for Eq. (1):

$$u_1(x,t) = \frac{\sqrt{2}\lambda}{\sqrt{\delta}} + \frac{\frac{\sqrt{2}\beta\mu}{\sqrt{\delta}} - \frac{4\sqrt{2}\lambda^2\mu}{\sqrt{\delta}}}{(\beta - 4\lambda^2) \left( \begin{array}{l} -\frac{\mu}{\lambda} + A \cosh \left[ \lambda \left( x - \frac{t(-\beta + \lambda^2)}{\lambda} \right) \right] \\ -A \sinh \left[ \lambda \left( x - \frac{t(-\beta + \lambda^2)}{\lambda} \right) \right] \end{array} \right)}. \tag{11}$$

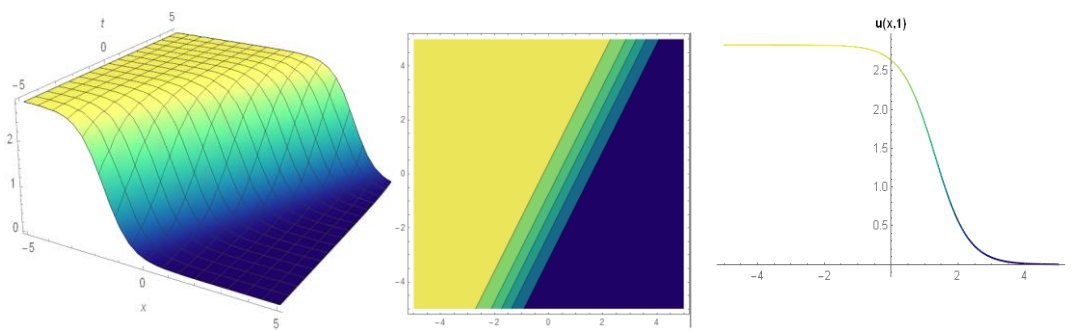


Figure1. 3D, contour and 2D graphs respectively for  $A = 5, \lambda = 2, \mu = -2, \beta = 3, \delta = 1$  values of Eq. (11).

**Case 2.**

$$w = -\frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta \lambda}, \quad a_0 = -\sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}},$$

$$a_1 = \frac{1}{\beta^2 \lambda - 4\beta \lambda^3} \left( \begin{aligned} & -\beta^2 \sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}} \mu - 2\beta \lambda^2 \sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}} \mu \\ & - 8\lambda^4 \sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}} \mu + 3\beta \delta \left( \frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2} \right)^{3/2} \mu \\ & + 8\delta \lambda^2 \left( \frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2} \right)^{3/2} \mu - 2\delta^2 \left( \frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2} \right)^{5/2} \mu \end{aligned} \right),$$

$$v = \frac{1}{2\beta \lambda^2} \left( \begin{aligned} & \beta^2 \delta \sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}} + 6\beta \delta \lambda^2 \sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}} \\ & + 4\delta \lambda^4 \sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}} - 2\beta \delta^2 \left( \frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2} \right)^{3/2} \\ & - 4\delta^2 \lambda^2 \left( \frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2} \right)^{3/2} + \delta^3 \left( \frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2} \right)^{5/2} \end{aligned} \right). \quad (12)$$

Replacing values Eq. (12) into Eq. (8) and we have the following exact solutions for Eq. (1)

$$u_2(x,t) = -\sqrt{\frac{\beta}{\delta} + \frac{\lambda^2}{\delta} - \frac{\sqrt{\delta^2 \lambda^2 (2\beta + \lambda^2)}}{\delta^2}} + \frac{a_1}{-\frac{\mu}{\lambda} + A \cosh[(-tw+x)\lambda] - A \sinh[(-tw+x)\lambda]}. \quad (13)$$

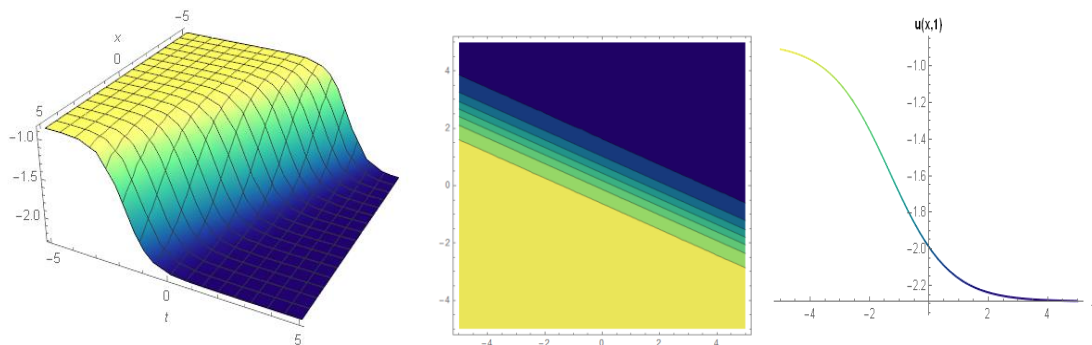


Figure2. 3D, contour and 2D graphs respectively for  $A = 5, \lambda = 2, \mu = -2, \delta = 1, a_1 = 1, w = 1, a_0 = 1, \beta = 2, \delta = 1$  values of Eq. (13).

**Case 3.**

$$w = \frac{\beta - \lambda^2}{\lambda}, \quad a_0 = 0, \quad a_1 = -\frac{\sqrt{2}\mu}{\sqrt{\delta}}, \quad v = \frac{-\sqrt{2}\beta\sqrt{\delta} - 2\sqrt{2}\sqrt{\delta}\lambda^2}{2\lambda}, \quad (14)$$

substituting Eq. (14) into Eq. (8), the following solution is obtained

$$u_3(x,t) = -\frac{\sqrt{2}\mu}{\sqrt{\delta} \left( -\frac{\mu}{\lambda} + A \cosh \left[ \lambda \left( x - \frac{t(\beta - \lambda^2)}{\lambda} \right) \right] - A \sinh \left[ \lambda \left( x - \frac{t(\beta - \lambda^2)}{\lambda} \right) \right] \right)}. \quad (15)$$

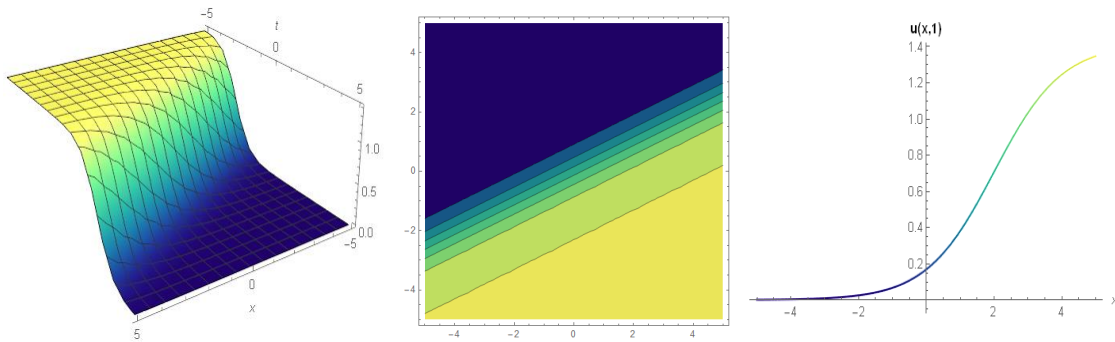


Figure3. 3D, contour and 2D graphs respectively for  $A = 5, \lambda = 1, \mu = -5, \beta = 3, \delta = 1$  values of Eq. (15).

**4. Conclusion**

In this letter, we have been obtained traveling wave solutions for the KPP equation with the help of  $(1/G')$ -expansion method. Hyperbolic type traveling wave solutions of KPP equation are presented with this powerful and reliable method. Traveling wave solutions are known to play an important role in many physical phenomena. We consider that the constants in the traveling wave solutions presented in this study will be much more valued after they gain physical meaning. Different values for the constants found solutions for 3D, 2D and contour graphs are presented. Computer technology was utilized in the construction of these solutions. This method is easy to implement, reliable and efficient for finding analytical solutions nPDEs.

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