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Stability in first order delay integro-differential equations

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Abstract

In this study, some results are given concerning the behavior of the solutions for linear delay integro-differential equations. These results are obtained by the use of two distinct real roots of the corresponding characteristic equation.

Keywords: *Integro-differential equation, stability, delay.*

Birinci mertebeden gecikmeli integro-diferansiyel denklemlerde kararlılık

Öz

Bu çalışmada, doğrusal gecikmeli integro-diferansiyel denklemler için çözümlerin davranışı ile ilgili bazı sonuçlar verilmiştir. Bu sonuçlar, karşılık gelen karakteristik denklemin iki ayrı reel kökünün kullanılmasıyla elde edilmiştir.

Anahtar kelimeler: *İntegro-diferansiyel denklem, kararlılık, gecikme.*

1. Introduction

Consider initial value problem for first order delay integro-differential equation

$$x'(t) = ax(t) + bx(t-\tau) + c\int_{t-\tau}^{t} x(s)ds\ , \quad t \geq 0 \tag{1}$$
$$x(t) = \phi(t)\ , \quad -\tau \leq t \leq 0\ , \tag{2}$$

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where a, b and c are real numbers, τ is a positive real number and $\phi(t)$ is a given continuous initial function on the interval $[-\tau, 0]$.

In this paper, our aim is to create a new result for the solution of equation (1). Similar results to the solutions of first order linear delay integro-differential equations were obtained by the authors in [1-6]. Our work in this article is mainly motivated by the results of Philos and Purnaras in [7-9]. Since the first systematic study was carried out by Volterra [6], this type of equations have been investigated in various fields, such as mathematical biology and control theory (see, e.g., [10-12]). For the basis theory of integral equations, we choose to refer to the books by Burton [13] and Corduneanu [14].

This paper is concerned with the asymptotic behavior and stability of scalar first order linear delay integro-differential equations. A basic asymptotic criterion is established. Furthermore, a useful estimate of the solutions and a stability criterion are obtained. The results were obtained using a real root of the corresponding characteristic equation. The techniques which used to obtain the results are a combination of the methods used in [4, 7-9, and 15].

By a "solution" of the first order delay integro-differential equation (1), we mean a continuous real-valued function x defined on the interval $[-\tau, \infty)$ and satisfies (1) for all $t \geq 0$. It is known that (see, for example, [10]), for any given initial function ϕ, there exists a unique solution of the initial value problem (1)-(2) or, more briefly, the solution of (1)-(2).

Together with the first order delay integro-differential equation (1), we associate the following equation

$$\lambda = a + be^{-\lambda\tau} + c\int_0^{\tau} e^{-\lambda s} ds \tag{3}$$

which will be called the *characteristic equation* of (1). The equation (3) is obtained from (1) by looking for solutions in the form $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$, where λ is a root of the equation (3).

The paper is organized as follows. A known (see Yeniçerioğlu and Yalçınbaş [15]) useful exponential estimate for the solutions of the first order delay integro-differential equation (1) is presented in Section 2. Section 3 is devoted to two lemma which concerns the real roots of the characteristic equation (3). The main result of the paper will be given in Section 4.

2. A known asymptotic result

In this section, we present a useful exponential prediction for the solutions of equation (1), which is closely related to the main outcome of this article. This exponential estimate for the solutions and also stability criteria have been obtained by Yeniçerioğlu and Yalçınbaş [15].

Theorem 2.1. *Assume that*

$$be^{-\gamma\tau} + c\gamma^{-1}(1 - e^{-\gamma\tau}) > \gamma - a$$

and

$$|b|\tau e^{-\gamma\tau} + |c|\gamma^{-1}[\gamma^{-1}(1 - e^{-\gamma\tau}) - \tau e^{-\gamma\tau}] \le 1.$$

Let λ_0 be real root of the characteristic equation (3) in the interval (γ, ∞). Then, for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution of (1)-(2) satisfies

$$\left| e^{-\lambda_0 t} x(t) - \frac{L(\phi)}{1 + \beta_{\lambda_0}} \right| \le M(\phi)\mu_{\lambda_0} \,, \quad t \ge 0 \,,$$

where

$$\beta_{\lambda_0} = b\tau e^{-\lambda_0 \tau} + c{\lambda_0}^{-1}\left[{\lambda_0}^{-1}\left(1 - e^{-\lambda_0\tau}\right) - \tau e^{-\lambda_0\tau}\right],$$

$$\mu_{\lambda_0} = |b|\tau e^{-\lambda_0 \tau} + |c|{\lambda_0}^{-1}\left[{\lambda_0}^{-1}\left(1 - e^{-\lambda_0\tau}\right) - \tau e^{-\lambda_0\tau}\right],$$

$$L(\phi) = \phi(0) + be^{-\lambda_0\tau} \int_{-\tau}^{0} e^{-\lambda_0 r} \phi(r)dr + c \int_{0}^{\tau} e^{-\lambda_0 \tau} \left[\int_{-s}^{0} e^{-\lambda_0 r} \phi(r)dr \right] ds$$

and

$$M(\phi) = \max_{-\tau \le t \le 0} \left| e^{-\lambda_0 t}\phi(t) - \frac{L(\phi)}{1 + \beta_{\lambda_0}} \right|.$$

3. Two lemma

Here, we give two lemma about the real roots of the characteristic equation (3).

Lemma 3.1. *Let λ_0 be real root of* (3) *and let β_{λ_0} be defined as in Theorem 2.1. Assume that*

$$b < 0 \quad and \quad c \le 0 \,. \tag{4}$$

Then $1 + \beta_{\lambda_0} > 0$ if (3) *has another real root less than λ_0, and $1 + \beta_{\lambda_0} < 0$ if* (3) *has another real root greater than λ_0.*

Proof of Lemma 3.1. Let $F(\lambda)$ denote the characteristic function of (3), i.e.,

$$F(\lambda) = \lambda - a - be^{-\lambda\tau} - c\int_0^\tau e^{-\lambda s} ds \tag{5}$$

for $\lambda \in \mathbb{R}$. We obtain immediately

$$F'(\lambda) = 1 + b\tau e^{-\lambda\tau} + c\int_0^\tau s e^{-\lambda s} ds \tag{6}$$

for $\lambda \in \mathbb{R}$. Furthermore,

$$F''(\lambda) = -b\tau^2 e^{-\lambda\tau} - c\int_0^\tau s^2 e^{-\lambda s} ds$$

for $\lambda \in \mathbb{R}$. That is, considering (4), we conclude that

$$F''(\lambda) > 0 \quad \text{for } \lambda \in \mathbb{R}. \tag{7}$$

Now, assume that (3) has another real root λ_1 with $\lambda_1 < \lambda_0$ (respectively, $\lambda_1 > \lambda_0$). From the definition of the function F by (5) it follows that $F(\lambda_1) = F(\lambda_0) = 0$, and consequently Rolle's Theorem guarantees the existence of a point α with $\lambda_1 < \alpha < \lambda_0$ (resp., $\lambda_1 > \alpha > \lambda_0$) such that $F'(\alpha) = 0$. But, (7) implies that F' is positive on (α,∞) (resp., F' is negative on (-∞,α)). Thus we must have $F'(\lambda_0) > 0$ (resp., $F'(\lambda_0) < 0$). The proof of Lemma 3.1 can be completed, by observing that

$$F'(\lambda_0) = 1 + \beta_{\lambda_0}.$$

Lemma 3.2. *Assume that statement* (4) *is true. Then we have:*

a) *In the interval* $[a, \infty)$ *, the equation* (3) *has no roots.*

b) *Suppose that*

$$-be^{-\left(a-\frac{1}{\tau}\right)\tau} - c\int_0^\tau e^{-\left(a-\frac{1}{\tau}\right)s} ds < \frac{1}{\tau}. \tag{8}$$

Then:

(i) $\lambda = a - \frac{1}{\tau}$ *is not a root of equation* (3).

(ii) *In the interval* $\left(a - \frac{1}{\tau}, a\right)$, (3) *has a unique root.*

(iii) *In the interval* $\left(-\infty, a - \frac{1}{\tau}\right)$, (3) *has a unique root.*

Proof of Lemma 3.2.

a) Let $\hat{\lambda}$ be real root of (3). Using (4), we can immediately see that

$$be^{-\hat{\lambda}\tau} + c\int_0^\tau e^{-\hat{\lambda}s} ds < 0.$$

Hence, from (3) it follows that $\hat{\lambda} - a < 0$, i.e., $\hat{\lambda} < a$. We have thus proved that every real root of (3) is always less than a.

b) Consider the real-valued function F defined by (5). As in the proof of Lemma 3.1, we see that (7) holds and consequently

$$F \text{ is convex on } \mathbb{R}. \tag{9}$$

Next, we observe that, as in the proof of Lemma 3.2, assumption (8) means that

$$F\left(a - \frac{1}{\tau}\right) = -\frac{1}{\tau} - be^{-\left(a-\frac{1}{\tau}\right)\tau} - c\int_0^\tau e^{-\left(a-\frac{1}{\tau}\right)s} ds < -\frac{1}{\tau} + \frac{1}{\tau} = 0. \tag{10}$$

Inequality (10) implies, in particular, that $\lambda = a - \frac{1}{\tau}$ is not a root of (3). From (5) we obtain

$$F(a) = -be^{-a\tau} - c\int_0^\tau e^{-as}ds\,.$$

So, by using (4), we conclude that

$$F(a) > 0\,. \tag{11}$$

Furthermore, from (5) we get

$$F(\lambda) \geq \lambda - a - be^{-\lambda\tau} \quad \text{for } \lambda \in \mathbb{R}\,.$$

Using this inequality, it is not difficult to show that

$$F(-\infty) = \infty\,. \tag{12}$$

Using (9), (10) and (11), it follows that in the interval $\left(a - \frac{1}{\tau}\,, a\right)$, the equation (3) has a unique real root. Furthermore, (9), (10) and (12) guarantee that equation (3) has a unique real root in the interval $\left(-\infty\,, a - \frac{1}{\tau}\right)$. Proof of Lemma 3.2 is complete.

4. The main result

Theorem 4.1. *Let λ_0 be real root of the equation* (3), *and let β_{λ_0} and $L(\phi)$ be defined as in Theorem 2.1. Suppose that statement* (4) *is true. Also, let λ_1 be real root of* (3) *with $\lambda_1 \neq \lambda_0$ (Note that, Lemma 3.1. guarantees that $1 + \beta_{\lambda_0} \neq 0$). Then the solution of (1)-(2) satisfies*

$$C_1(\lambda_0, \lambda_1; \phi) \leq e^{-\lambda_1 t}\left[e^{-\lambda_0 t}x(t) - \frac{L(\phi)}{1+\beta_{\lambda_0}}\right] \leq C_2(\lambda_0, \lambda_1; \phi) \tag{13}$$

for all $t \geq 0$, where

$$C_1(\lambda_0, \lambda_1; \phi) = \min_{-\tau \leq t \leq 0}\left\{e^{-\lambda_1 t}\left[e^{-\lambda_0 t}\phi(t) - \frac{L(\phi)}{1+\beta_{\lambda_0}}\right]\right\} \tag{14}$$

and

$$C_2(\lambda_0, \lambda_1; \phi) = \max_{-\tau \leq t \leq 0}\left\{e^{-\lambda_1 t}\left[e^{-\lambda_0 t}\phi(t) - \frac{L(\phi)}{1+\beta_{\lambda_0}}\right]\right\}. \tag{15}$$

Equivalently to the (13) inequalities, we see that it can be written as follows

$$C_1(\lambda_0, \lambda_1; \phi)e^{\lambda_1 t} \leq \left[e^{-\lambda_0 t}x(t) - \frac{L(\phi)}{1 + \beta_{\lambda_0}}\right] \leq C_2(\lambda_0, \lambda_1; \phi)e^{\lambda_1 t} \quad , \quad t \geq 0\,.$$

Hence, if $\lambda_1 < 0$, then the solution of (1)-(2) satisfies

$$\lim_{t\to\infty} e^{-\lambda_0 t}x(t) = \frac{L(\phi)}{1+\beta_{\lambda_0}}.$$

Also, we observe that (13) is equivalent to

$$e^{\lambda_0 t}\left[C_1(\lambda_0,\lambda_1;\phi)e^{\lambda_1 t} + \frac{L(\phi)}{1+\beta_{\lambda_0}}\right] \le x(t) \le e^{\lambda_0 t}\left[C_2(\lambda_0,\lambda_1;\phi)e^{\lambda_1 t} + \frac{L(\phi)}{1+\beta_{\lambda_0}}\right]$$

for all $t \ge 0$.

Proof of Theorem 4.1. Consider an arbitrary initial function $\phi \in C([-\tau,0],\mathbb{R})$ and let x be the solution of (1)–(2). Define

$$y(t) = e^{-\lambda_0 t}x(t) \quad \text{for} \quad t \ge -\tau$$

and next, set

$$z(t) = y(t) - \frac{L(\phi)}{1+\beta_{\lambda_0}} \quad \text{for} \quad t \ge -\tau\,.$$

As shown by Yeniçerioğlu and Yalçınbaş [15], the fact that y satisfies (1) for $t \ge 0$ is equivalent to the fact that w satisfies

$$z(t) = -be^{-\lambda_0 \tau}\int_{t-\tau}^{t} z(s)ds - c\int_0^{\tau} e^{-\lambda_0 s}\left\{\int_{t-s}^{t} z(u)du\right\}ds \tag{16}$$

for $t \ge 0$. Also, due to the y and z transformations, the following initial condition is obtained using the (2) initial condition.

$$z(t) = e^{-\lambda_0 t}\phi(t) - \frac{L(\phi)}{1+\beta_{\lambda_0}} \quad \text{for} \quad t \in [-\tau,0]. \tag{17}$$

Now, we define

$$h(t) = e^{(\lambda_0-\lambda_1)\,t}z(t) \quad \text{for} \quad t \ge -\tau\,. \tag{18}$$

Because of the y and z transformations, the following expression is obtained for the function h:

$$h(t) = e^{-\lambda_1 t}\left[x(t) - e^{\lambda_0 t}\frac{L(\phi)}{1+\beta_{\lambda_0}}\right] \quad \text{for } t \ge -\tau\,. \tag{19}$$

Moreover, by using the h function, (16) can be written as equivalent

$$h(t) = -be^{-\lambda_0 \tau}\int_{-\tau}^{0} e^{(\lambda_1-\lambda_0)\,s}h(s+t)ds - c\int_0^{\tau} e^{-\lambda_0 s}\left\{\int_{-s}^{0} e^{(\lambda_1-\lambda_0)\,u}h(u+t)du\right\}ds \tag{20}$$

for $t \ge 0$ and (17) becomes

$$h(t) = e^{-\lambda_1 t}\left[\phi(t) - e^{\lambda_0 t}\frac{L(\phi)}{1+\beta_{\lambda_0}}\right] \quad \text{for} \quad t \in [-\tau, 0]. \tag{21}$$

As solution x satisfies the initial condition (2), we can use the function h as well as the definitions of $C_1(\lambda_0, \lambda_1; \phi)$ and $C_2(\lambda_0, \lambda_1; \phi)$ by (14) and (15), respectively, to see that

$$C_1(\lambda_0, \lambda_1; \phi) = \min_{-\tau \le t \le 0} h(t) \quad \text{and} \quad C_2(\lambda_0, \lambda_1; \phi) = \max_{-\tau \le t \le 0} h(t). \tag{22}$$

Considering (19) and (22), the double inequality (13) can be written equivalent to

$$\min_{-\tau \le s \le 0} h(s) \le h(t) \le \max_{-\tau \le s \le 0} h(s) \qquad \text{for all } t \ge 0. \tag{23}$$

We need to prove that (23) inequalities are met. We will use the fact that h satisfies (20) for all $t \ge 0$ to show that (23) is valid. We just need to prove the following inequality

$$h(t) \ge \min_{-\tau \le s \le 0} h(s) \qquad \text{for every } t \ge 0\,. \tag{24}$$

The proof of the inequality

$$h(t) \le \max_{-\tau \le s \le 0} h(s) \quad \text{for every } t \ge 0$$

can be obtained in a similar manner and is therefore omitted. We will obtain (24) for the rest of the proof. To do this, we are considering an arbitrary real number A with $A < \min_{-\tau \le s \le 0} h(s)$, i.e., with

$$h(t) > A \quad \text{for} \quad -\tau \le t \le 0. \tag{25}$$

We will show that

$$h(t) > A \quad \text{for all } t \ge 0. \tag{26}$$

For this purpose, suppose that (26) is not provided. Then, due to (25), there is a point $t_0 > 0$ such that

$$h(t) > A \quad \text{for } -\tau \le t < t_0\,, \quad \text{and} \quad h(t_0) = A.$$

Thus, by using (3), from (20) we obtain

$$\begin{aligned} A &= h(t_0) \\ &= -be^{-\lambda_0\tau}\int_{-\tau}^{0} e^{(\lambda_1-\lambda_0)\,s} h(s+t_0)ds - c\int_{0}^{\tau} e^{-\lambda_0 s}\left\{\int_{-s}^{0} e^{(\lambda_1-\lambda_0)\,u} h(u+t_0)du\right\}ds \\ &> A\left(-be^{-\lambda_0\tau}\int_{-\tau}^{0} e^{(\lambda_1-\lambda_0)s}ds - c\int_{0}^{\tau} e^{-\lambda_0 s}\left\{\int_{-s}^{0} e^{(\lambda_1-\lambda_0)\,u}du\right\}ds\right. \end{aligned}$$

$$= \frac{A}{\lambda_1 - \lambda_0}\left(-be^{-\lambda_0\tau}\left[1 - e^{-(\lambda_1-\lambda_0)\tau}\right] - c\int_0^{\tau} e^{-\lambda_0 s}\left[1 - e^{-(\lambda_1-\lambda_0)s}\right] ds\right)$$
$$= \frac{A}{\lambda_1 - \lambda_0}\left(-b\left[e^{-\lambda_0\tau} - e^{-\lambda_1\tau}\right] - c\int_0^{\tau} \left[e^{-\lambda_0 s} - e^{-\lambda_1 s}\right] ds\right)$$
$$= \frac{A}{\lambda_1 - \lambda_0}\left(-b\left[e^{-\lambda_0\tau} - e^{-\lambda_1\tau}\right] - c\left[\lambda_1^{-1}\left(e^{-\lambda_1\tau} - 1\right) - \lambda_0^{-1}\left(e^{-\lambda_0\tau} - 1\right)\right]\right]$$
$$= \frac{A}{\lambda_1 - \lambda_0}\left(-be^{-\lambda_0\tau} - c\left[-\lambda_0^{-1}\left(e^{-\lambda_0\tau} - 1\right)\right] + be^{-\lambda_1\tau} + c\left[-\lambda_1^{-1}\left(e^{-\lambda_1\tau} - 1\right)\right]\right)$$
$$= \frac{A}{\lambda_1 - \lambda_0}\left(-be^{-\lambda_0\tau} - c\int_0^{\tau} e^{-\lambda_0 s} ds + be^{-\lambda_1\tau} + c\int_0^{\tau} e^{-\lambda_1 s} ds\right)$$
$$= \frac{A}{\lambda_1 - \lambda_0}\left(a - \lambda_0 + \lambda_1 - a\right) = A\,.$$

So we came to a contradiction and therefore (26) is correct. Since (26) is satisfied for all real number A with $A < \min_{-\tau \le s \le 0} h(s)$, it follows that (24) is always performed. The proof of the Theorem 4.1 is complete.

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